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One century of Minkowski's paper: reconstruction from integral geometry data

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ABSTRACT

This expository article contains a introduction to some analytic methods of image reconstruction from data of its projections like integrals over lines, planes etc.

1 Introduction

One hundred years ago Hermann Minkowski [1] has started the problem: to reconstruct an even function f on the sphere S^2 from knowledge of integrals

$$Mf(C) = \frac{1}{2}\int_C f ds \tag{1}$$

over big circles C. Here ds is the linear element in the Euclidean sphere S^2. Minkowski has proved the uniqueness theorem: vanishing of all integrals of an even continuous function f implies $f \equiv 0$. (It is not, of course, true for odd functions, since all the integrals vanish.) After Minkowski's death, Paul Funk, [2], 1916 has found an explicit reconstruction formula for f from data of integrals (1). Funk took the average of integrals (1) with respect to an actions of the rotation group S^1 on S^2. Considering Mf as a function on the sphere, he showed that the average of Mf is related to the average of f by the Abel transform. A reconstruction of f was done by inversion of the Abel transform. This gives, in modern terms, an inversion for the operator M (Minkowski-Funk transform). Johann Radon studied the similar problem for Euclidean plane E (encouraged, apparently, by W.Blaschke): to reconstruct a function f from knowledge of integrals

$$Rf(L) = \int_L f ds \tag{2}$$

for all lines $L \subset E$. In 1917 Radon (see [5]) found a solution by applying Funk's method. These papers did not get a development since many years and were almost forgotten [1] (in spite of the term "Integralgeometrie" survived since then). The focus of interest (and mode) in analysis moved then to the side of more abstract theories.

The situation did not changed until the last three decades when the interest to reconstruction problems like Minkowski's and Radon's ones grew tremendously stimulated by the spectrum of new high-tech methods of image reconstruction. These are first at all various kinds of tomography: X-ray, gamma, NMR, electron, positron, ultrasound, thermoacoustic, seismic tomography, synthetic radar imaging and others.

The objective of this paper is to give an introduction to the current theory of integral transforms like (1), (2). We focus mostly on analytic methods of reconstruction. The problems arising in adaptation of analytic methods to processing of real numeric data are serious enough for special discussions.

On the other hand, the beauty and richness of the theory make it very attractive for pure mathematicians.

This article is not a regular survey of the topic; the reference list contains the items that can help the reader to enter the subject.

2 Radon transform and inversion

2.1 Basic formulae

Let E be an Euclidean space with the interior product $(x, y) \mapsto x \cdot y$. We write the Fourier integral for a density $\phi = f dV \in L_1(E)$ in the form

$$F_{x \to \xi}(\phi)(\xi) = \int_E \exp(-2\pi \imath \xi \cdot x) f(x) dV, \ \xi \in E^* \qquad (3)$$

where dV is the Euclidean volume element. The Fourier image $\hat{f} = F(f dV)$ is a function on the dual Euclidean space E^* (which is identified sometimes with E.)

Let $n = \dim E$; denote by $A_{n-1}(E)$ the set of all affine hyperplanes in E. Take a hyperplane H, choose a unit orthogonal vector ω to H and denote by p the distance from the origin to H in the direction ω, i.e. $\omega \cdot x = p$ is the equation of $H = H(\omega, p)$. At the same time $-\omega \cdot x = -p$ is another equation of the same hyperplane $H(-\omega, -p) = H(\omega, p)$. Thus we have two-fold covering $S^{n-1} \times \mathbb{R} \to A_{n-1}(E)$ where S^{n-1} is the unit sphere in E. The topological space $A_{n-1}(E)$ is homeomorphic to the projective space of dimension n without one point. This point corresponds to the improper hyperplane in the projective closure of E. Let dS be the surface element on submanifolds in E. The Radon transform of an integrable function f in E is defined as follows

$$Rf(\omega, p) = \int_{H(\omega,p)} f dS$$

[1]The reconstruction formulae of Funk and Radon have attracted no interest and almost forgotten before the tomography era came. In the ten pages obituary written by P.Funk in 1958 to Radon's death there is no commentary on the paper of 1917.

This integral is defined for any ω and for almost all $p \in \mathbb{R}$.

Proposition 1 *For an arbitrary $f \in L_1(E)$ the equation holds*

$$F_{x \to \xi}(f)(\sigma\omega) = F_{p \to \sigma} Rf(\omega, p)$$

This fact is called "slice theorem". The inversion of the Radon transform can be implemented by inverting of the Fourier transform: $f = F^{-1}_{\sigma\omega \to x}(F_{p \to \sigma} Rf(\omega, p))$, where the inverse Fourier transform looks similar to (3):

$$F^*(\phi)(x) = \int_{E^*} \exp(2\pi \imath x \cdot \xi)\phi(\xi) dV^* \qquad (4)$$

where dV^* is the volume density in E^*. Simplifying the composition of two Fourier transforms yields the following reconstruction formulae:

Theorem 2 *For an arbitrary function f in E such that $D^i f \in L_1(E)$ for $|i| \leq n$ that satisfies the above conditions and $g \doteq Rf$ we have*

$$f(x) = \frac{(-1)^{(n-2)/2}}{2(2\pi)^{n-1}} \int_{S^{n-1}} \mathbf{H}\partial_p^{n-1} g(\omega, \omega \cdot x) dS \qquad (5)$$

for even n and

$$f(x) = \frac{(-1)^{(n-1)/2}}{2(2\pi)^{n-1}} \int_{S^{n-1}} \partial_p^{n-1} g(\omega, \omega \cdot x) dS \qquad (6)$$

for odd n.

Here $\mathbf{H}$ stands for the Hilbert transform with respect to p-variable. For an arbitrary Lipschitz function $a \in L_2(\mathbb{R})$ the Hilbert operator is defined by means of the principle value integration

$$\mathbf{H}a(p) \doteq \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|q| > \varepsilon} \frac{a(p-q)dq}{q}$$

It can be extended to an operator in L_2 such that $||\mathbf{H}|| = 1$. In fact, the equations (5) and (6) hold for a larger class of originals f : it is sufficient that the right side is well defined. In the case $n = 3$ the equation (6) turns to

$$f(x) = -\frac{1}{8\pi^2} \int_{S^2} g''(\omega, \omega \cdot x) dS$$

Note that the formulae (6) are *local*, i.e. for reconstruction of the value of f in a point x we only need to know the values of $\partial_p^{n-1} Rf$ for hyperplanes $H(\omega, p)$ through the point x. The formulae (5) are *nonlocal* since we need to know $R^{(n-1)} f$ for *all* hyperplanes.

2.2 Radon's formulae

For n even we do the substitution $p = \omega \cdot x + q$ in (5) and take in account that $g^{(n-1)}(\omega, p) = \partial_p^{n-1} g(\omega, p)$ is an odd function in p

$$(2\pi\imath)^n f(x) = \lim_{\varepsilon \to 0} \int_{S^{n-1}} \int_{|q| \geq \varepsilon} \frac{g^{(n-1)}(\omega, \omega \cdot x + q) dq}{q} d\omega$$
$$= \lim \int \int_{q \geq \varepsilon} \frac{g^{(n-1)}(\omega, \omega \cdot x + q) - g^{(n-1)}(\omega, -\omega \cdot x + q)}{q} dq$$

The limit exists if $g^{(n-1)}(\omega, p)$ is a Lipschitz function with respect to p. Now we change the order of integration and write the right side as follows

$$\int_0^\infty \frac{dq}{q} \left[\int_{S^{n-1}} g^{(n-1)}(\omega, \omega \cdot x + q) d\omega - \int g^{(n-1)}(\omega, -\omega \cdot x + q) d\omega \right]$$

Make the substitution $\omega \mapsto -\omega$ in the second integral and see that it gives the same quantity as the first one. Therefore we obtain

$$f(x) = \frac{\pi^{n/2}}{2^{n-2}\Gamma(n/2)} \int_0^\infty F^{(n-1)}(q) \frac{dq}{q} \tag{7}$$

where

$$F(q) \doteq \frac{1}{|S^{n-1}|} \int g(\omega \cdot x + q, \omega) d\omega$$

is the normalized back projection and $|S^{n-1}| = 2\pi^{n/2}/\Gamma(n/2)$ is the area of the unit sphere.

3 Factorable mappings

Consider a more general situation. Let X be a Riemannian manifold of dimension n with the metric tensor g and $Y = \{Y\}$ be a family of closed submanifolds of X of dimension k, $0 < k \leq n$. Take a continuous function $f : X \to \mathbb{C}$ that decreases sufficiently fast at infinity. Define the family of integrals

$$Mf(Y) = \int_Y f dV(Y), \ Y \in Y \tag{8}$$

where $dV(Y)$ is the volume element on Y induced by the metric g. We call the function $Mf|Y$ generalized Minkowski-Funk-Radon or *integral mean* transform of f. For an Euclidean space X and the family of affine subspaces, it coincides with (2). For the projective plane $X = \mathbb{P}^2$ and the family of projective lines this transform coincides with (1). Indeed, an even function f on the sphere defines a function on the quotient $S^2/\mathbb{Z}_2 = \mathbb{P}^2$, where the group $\mathbb{Z}_2$ identifies the opposite points in the sphere. The canonical metric in the projective plane is inherited from the Euclidean metric on the sphere $S^2 \subset E^3$ and the integral over a projective line equals one half of the integral over the corresponding big circle.

For these two special cases we know the explicit inversion formulae for the integral transform M. These formulae can be translated to another geometrical situation by means of the following simple arguments:

Definition. Let $\Phi : X_1 \to X_2$ be a diffeomorphism of Riemannian manifolds (it need not to be an isometry), $Y = \{Y\}$ be a smooth family of submanifolds of X_1. We say that Φ is (infinitesimally) factorable with respect to this family if for an arbitrary $Y \in Y$ and arbitrary point $x \in Y$ the equation holds for the Jacobian of Φ :

$$\frac{dV_{(2)}(\Phi(x), \Phi(Y))}{dV_{(1)}(x, Y)} = j(x) J(Y) \tag{9}$$

where $dV_{(i)}(x, Z)$ is the Riemannian volume element of a submanifold $Z \subset X_i, i = 1, 2$ at a point x. The functions $j : X \to \mathbb{R}$ and $J : Y \to \mathbb{R}$ depend on Φ only; we call this functions *jacobian factors* of the mapping Φ. An application of this property is obvious: the problem of inversion of the integral operator M for the family $\Phi(Y) = \{\Phi(Y), Y \in Y\}$ is reduced to that for the family Y by

$$M[f](\Phi(Y)) = \int_{\Phi(Y)} f dV_{(2)} = J(Y) \int_Y \Phi^*(f) j(x) dV_{(1)} = J(Y) M[\Phi^*(f) j](Y)$$

where f is a function on X_2 and $\Phi^*(f)(x) \doteq f(\Phi(x))$. If there is an inversion operator I for $M|Y$,we define the inversion operator for $M|\Phi(Y)$ as follows: $f = I(JM[\Phi^*(f)j])$. The reduction can be reverted since the mapping Φ^{-1} is also factorable. Moreover, the *transitivity* property holds: if a diffeomorphism $\Psi : X_2 \to X_3$ is factorable for the family $\Phi(Y)$ and a Riemannian space X_3, then the composition $\Psi\Phi : X_1 \to X_3$ is factorable for Y with the jacobian factors

$$j_{\Psi\Phi}(x) = j_\Psi(\Phi(x)) j_\Phi(x), \; J_{\Psi\Phi}(Y) = J_\Psi(\Phi(Y)) J_\Phi(Y)$$

Example 1. Any conformal mapping Φ possesses the property (9) for the family Y_k of all k-dimensional submanifolds with the jacobian factors $j = (\Phi^*(g_2)/g_1)^{k/2}$, $J = 1$, where $g_{1,2}$ are the metric tensors in $X_{1,2}$.

Example 2. Let D be the unit disc in Euclidean plane. The automorphism of D given by $G(z) = 2z\left(1 + |z|^2\right)^{-1}$ is factorable for the family of circle arcs $A \subset D$ that are orthogonal to the circle ∂D. The jacobian factors are

$$j_G(z) = \frac{1 - |z|^2}{\left(1 + |z|^2\right)^2}, \; J_G(A) = \frac{2\left(1 + r^2\right)^{1/2}}{r}$$

where r is the radius of an arc A. The image $G(A)$ is the chord in D that leans the arc A.

Example 3. Let E be an Euclidean space of dimension n, dS be the surface element in E. Take the projective closure $\bar{E} = E \cup H_\infty$ where $H_\infty = \mathbb{P}(E)$ is the improper projective hyperplane, and consider a projective transformation L of $\hat{E}$. It defines a diffeomorphism $L : E\backslash L^{-1}(H_\infty) \to E\backslash L(H_\infty)$. This mapping is factorable for the variety A_k of k-dimensional affine subspaces of E and arbitrary k. To show this property, we take the Euclidean space $\mathbf{E} = \mathbb{R} \dotplus E$ of dimension $n+1$ with coordinates $x_0, ..., x_n$ and consider the isometrical embedding $e : E \to \mathbf{E}, e(x) = (1, x)$. Take a linear automorphism $\mathbf{L}$ of $\mathbf{E}$ that generates L, i.e. $L = \mathrm{p}\mathbf{L}e$ where $\mathrm{p} : \mathbf{E}\backslash \{x_0 = 0\} \to E$ is the central projection. Let $p_0 : \mathbf{E} \to \mathbb{R}$ be the coordinate projection.

Proposition 3 *We have for an arbitrary* $A \in A_k$

$$\frac{dV(L(x), L(A))}{dV(x, A)} = |p_0(\mathbf{L}(e(x)))|^{-k-1} \frac{[\mathbf{L}(a^0), ..., \mathbf{L}(a^k)]}{[a^0, ..., a^k]}$$

where $a^0, ..., a^k$ *are arbitrary points in* E *that span* A.

The last condition means that $A = \{x = \sum t_i a^i, \sum t_i = 1\}$. For arbitrary points $b^0, ..., b^k \in \mathbf{E}$ the number $[b^0, ..., b^k] \geq 0$ is defined by

$$[b^0, ..., b^k]^2 = \sum_{0=i_0<i_1<...<i_k} |B_{i_0,...,i_k}|^2 \tag{10}$$

where $B_{i_0,...,i_k}$ means the minor of the matrix B formed by the rows $b^0, ..., b^k$ with the columns numbers $i_0, ..., i_k$.

3.1 Reconstruction from data of arc means

Let E_+ be a half plane in an Euclidean plane, say $E_+ = \{(x, y), y > 0\}$, and Y be the family of circle arcs in E_+ that are orthogonal to ∂E_+. All orthogonal straight lines are also included in the family Y. The arc mean transform

$$Mf(A) = \int_A f ds$$

is of interest in several applied problems; ds is the length element. The problem of inversion of M in this form is a *complete data* problem since for any point $p \in E_+$ and any tangent vector t in x there exists at least one curve $A \in Y$ through p that is orthogonal to t. This condition is no satisfied in practice, since the integral means for very long arcs are not available. We consider the same integral transform for limited data. Take the unit disc D in E and consider the half disc $D_+ = E_+ \cap D$. Let Y_D be the subset of Y consisting of arcs $A \subset D$. The problem is to reconstruct a function f in E with compact support $\mathrm{supp}\, f \subset D_+$ from the limited arc mean transform $Mf \mid Y_D$. This is a problem with *incomplete data* since for any point $x \in D_+$ the normal vector t to arcs $A \in Y_D$ runs over two vertical angles of diapason $\phi < \pi$. In fact, we have $\phi = \pi - \alpha$ where α is the angular length of the circle through x and the points ± 1. The diapason is almost complete for p close to the diameter of D_+ and is very small for p close to the circle ∂D.

3.2 The limited arc mean problem

The inversion problem for the pencil Y is reduced to the Radon transform in plane in the following three steps. Choose coordinates $(x.y)$ in E in such a way that $E_+ = \{y > 0\}$, and D is the unit disc with center in the origin.
Step 1. Introduce the complex coordinate $z = x + \imath y$ in D and apply the transform

$$z_1 = F(z) \doteq \frac{\imath - z}{\imath + z}$$

The image of E_+ is the unit disc D_K and the image of $D_+ = E_+ \cap D$ is the right half-disc. Any arc $A \in Y$ is transformed to the circular arc $F(A)$ in D_K that is orthogonal to the boundary. This transform is conformal hence factorable.
Step 2. Apply the mapping of Example 2 $w_1 = G(z_1)$. It maps D_K onto the identic disc D_B and the right half of D_K to the right half of D_B. Any arc $F(A)$ is transformed to the chord with the same ends.
Step 3. Apply the projective transform

$$(u, v) = P(u_1, v_1) \quad u = \frac{1}{u_1},\ v = \frac{v_1}{u_1}$$

The vertical diameter of D_B maps to the improper projective line and the unit circle is transformed to the hyperbola $v_1^2 + 1 = u_1^2$. The image of the disc D_B is equal to the set $W = \{v_1^2 + 1 < u_1^2\}$; the image of the right half is equal to the right connected component U of W. The image of a chord $L \subset D_B$ is a chord in the set U with the ends in the hyperbola. By the transitivity property, this mapping is factorable too. Take the composition $Q = PGF$; it follows from the previous formulae that it is factorable with the jacobian factors

$$j(z) = \frac{4y}{(1 - |z|^2)^2},\ J(A) = \sqrt{1 + \left(\frac{ab - 1}{a - b}\right)^2}$$

where a, b are the ends of A. This implies the formula

$$\int_{Q(A)} j^{-1} f ds^* = J(A) \int_A f ds$$

where ds^* is the Euclidean line element in U. The support of the function g is a compact subset of U and curve $Q(A)$ is an arbitrary finite chord of the hyperbola ∂U. Let ψ be the angle of the normal to $Q(A)$; we have $|\psi| < \pi/4$. Vice versa, an arbitrary line in U whose normal has angle in this diapason, is a finite chord.

Corollary 4 *For functions f with compact support* supp $f \subset D_+$ *the limited arc mean transform is reduced to the Radon transform with the limited angle diapason* $|\psi| < \pi/4$.

The Radon transform with incomplete data can be inverted by interpolation of entire functions of Paley-Wiener class, see Sec.6.

Remark. There are three classical models of the Lobachewski (hyperbolic) plane:

(i) Poicare's model in the half-plane E_+ with the metric $y^{-2}ds^2$, geodesics are the arcs $A \in A$;

(ii) Klein's model D_K in the disk D with the metric $\left(1-|z|^2\right)^{-2} ds^2$; geodesics are arcs orthogonal to ∂D, and

(iii) Beltrami's model D_B in D the metric is

$$g = \left(1-|z|^2\right)^{-2}\left[\left(1-y^2\right)dx^2 + 2xydxdy + \left(1-x^2\right)dy^2\right]$$

the geodesics are chords.

The mappings $F : E_+ \to D_K, G : D_K \to D_B$ are isometries between these models. These mappings are not, of course, isometries for Euclidean metrics in E_+ and D, but they are factorable for the families of hyperbolic geodesics.

3.3 More examples

Example 4. Let S_0 be the unit sphere in an Euclidean space E with the center in the origin, $E\setminus\{0\} \to S_0$ be the central projection. Take an arbitrary sphere S in $E\setminus\{0\}$. The projection defines the mapping $\pi : S \to S_0$. It is factorable for the family of k-spheres $F \cap S$ where F is an arbitrary $k+1$-subspace of E. The volume relation is

$$\frac{dV(x, F\cap S)}{dV(x_0, F\cap S_0)} = \frac{|x|^{k+1}}{|x\cdot(x-\xi)|} r_F,\ x_0 = \pi(x) \tag{11}$$

where ξ is the center of S, r_F is the radius of the sphere $F \cap S$. The jacobian factors are $j(x) = |x\cdot(x-\xi)|^{-1}|x|^{k+1}$, $J(F) = r_F$. If the origin is inside S, the dominator $x\cdot(x-\xi)$ does not vanish on S; otherwise it vanishes in each point where a ray from the origin is tangent to S.

Example 5. We can take a hyperplane $H \subset E\setminus\{0\}$ instead of the sphere S in Example 4. The central projection $\pi : H \to S_0$ is again factorable:

$$\frac{dV(x, F\cap H)}{dV(x_0, F\cap S_0)} = \frac{|x|^{k+1}}{\operatorname{dist}(F\cap H, 0)},\ x_0 = \pi(x) \tag{12}$$

This follows from (11) if we apply (11) and move ξ to infinity along the line orthogonal to H at x.

3.4 Spaces of constant curvature

There are three types of complete simply connected Riemannian manifolds of constant sectional curvature: elliptic, Euclidean and hyperbolic. An Euclidean space has zero curvature. Any straight line is a geodesic and vice versa. A elliptic space of dimension n is the real projective space $\mathbb{P}^n = S^n/\mathbb{Z}_2$ with the metric inherited from the unit sphere $S^n \subset \mathbf{E}$ where $\mathbf{E}$ is an Euclidean space of dimension $n+1$. The sectional curvature of the elliptic space is equal everywhere to 1. For a subspace $F \subset \mathbf{E}$ of dimension 2 the intersection $F \cap S^n$ is a big circle; its image in the elliptic space $\mathbb{P}^n$ is a closed geodesic curve γ. For an arbitrary subspace F the manifold $Y \doteq F \cap S^n/\mathbb{Z}_2$ is a projective subspace. It is a totally geodesic manifold of $\mathbb{P}^n$ since any two points of Y can be connected by a geodesic $\gamma \subset Y$. Vice versa, any closed totally geodesic submanifold of the projective space is equal a projective subspace.

A hyperbolic space of dimension n can be constructed in a similar way. Choose Euclidean coordinates $x_0, x_1, ..., x_n$ in $\mathbf{E}$ and consider the hyperboloid $Q \subset \mathbf{E}$ given by

$$x_0^2 = x_1^2 + ... + x_n^2 + 1,\ x_0 > 0$$

This is one fold of the two-fold quadratic hypersurface. Consider the standard Euclidean metric in $\mathbf{E}$ by means of the coordinate system $x_0, x_1, ..., x_n$ and take the induced metric in Q. This is a hyperbolic space H of sectional curvature -1. It is obviously closed and homeomorphic to a ball. For any subspace $F \subset \mathbf{E}$ of dimension 2 the intersection $F \cap Q$ is a closed geodesic curve; for an arbitrary F it is a totally geodesic submanifold of H.

Alongside of Q, we consider the Euclidean submanifold $E \doteq \{x_0 = 1\}$ and hemisphere $S_+ \doteq \{x; |x| = 1, x_0 > 0\}$ as a model of the elliptic space $\mathbb{P}^n$. The central projection π in $\mathbf{E} \backslash \{0\}$ defines the diffeomorphisms

$$Q \overset{\pi}{\to} E \overset{\pi}{\leftarrow} S_+ \tag{13}$$

We have $\pi(S_+) = E$ and the set $\pi_E(Q)$ is an open ball of radius 1. By the aforesaid, the intersection of these submanifolds with a subspace F is a totally geodesic submanifold in the elliptic, Euclidean and hyperbolic space, respectively, see Fig.1.

Example 6. The mappings (13) are factorable, namely for an arbitrary subspace $F \subset \mathbf{E}$ of dimension $k+1$ we have

$$\frac{dV_S(y, F \cap S_+)}{dV_E(x, F \cap E)} = \left(1 + d^2(F)\right)^{1/2} \left(1 + |x|^2\right)^{-(k+1)/2} \tag{14}$$

$$\frac{dV_H(y, F \cap Q)}{dV_E(x, F \cap E)} = \left(1 - d^2(F)\right)^{1/2} \left(1 - |x|^2\right)^{-(k+1)/2} \tag{15}$$

where $\pi(y) = x$ is a point in E where the volume forms are compared and $d(F) = \mathrm{dist}_E(F \cap E, 0)$. Note that (14) is equivalent to (11).

3.5 Geodesic transforms

Let $\mathbb{P}^n$ be a real projective space of dimension n, $G(\mathbb{P}^n)$ be the manifold of projective subspaces $Y \subset \mathbb{P}^n$ (which are images of big spheres $C \subset S^n$). The integral mean transform M given by (8) restricted to $G(\mathbb{P}^n)$ is called (generalized) Minkowski-Funk transform. The particular case $n = 2$ the transform (8) coincides with (1), since the function f can be consider as an even function in S^2 and (1) coincides with $M_1 f(C/\mathbb{Z}_2)$ for any big circle C. For an Euclidean space E and for a hyperbolic space H and consider the integral mean transform (8) on the manifold of totally geodesic manifolds of dimension k. In all three cases, (8) is called *geodesic mean* transform. The geodesic mean transform in Euclidean space coincides with the affine mean transform; it is called Radon transform for affine subspaces of dimension $k = n - 1$ and X-ray transform for $k = 1$.

Corollary 5 *The geodesic mean transform in hyperbolic space* H *is equivalent to the affine mean transform in the unit ball of Euclidean space E of the same dimension. The affine mean transform in E is reduced to the Minkowski-Funk transform in elliptic space $\mathbb{P}^n$.*

The inverse reduction does not hold: the Minkowski-Funk transform is reduced to the affine mean transform only for functions f in S_+ that vanish sufficiently fast on the equator, more precise, the following condition is necessary: $f(y) = o\left(y_0^{k+1}\right)$ as $y \in S_+, y_0 \to 0$.

3.6 Deduction of inversion formula for hyperbolic space

The Funk formula reads for an even function f on S^2 :

$$f(y) = -\frac{1}{\pi}\int_0^{\pi/2} \frac{dF(y,q)}{\sin q} + \frac{1}{\pi} Mf(y^\circ) \tag{16}$$

where

$$F(y,q) = \frac{1}{2\pi}\int_{d(C,y)=q} Mf(C)ds$$

is the integral mean of $Mf(C)$ over the family of the big circles whose angular distance to the point $y \in S^2$ is equal q and y° is the geodesic whose distance to y is equal $\pi/2$. The first term of (16) is thought as a improper Stieltjes integral.

Let g be an arbitrary continuous function in E such that $g = o\left(|x|^{-2}\right)$ at infinity. Define the function $f(y) \doteq (1+|x|^2)g(x)$ on S_+ where $\pi(y) = x$. It tends to zero as y approach the equator of S_+; we set $f = 0$ on the equator and extend it to S^2 as an even function. By (14) we have for an arbitrary big circle C

$$\begin{aligned} M_S f(C) &= \int_C f dV_S = (1+d^2(L))^{1/2}\int_{\pi(C)} g dV_E \\ &= (1+d^2(L))^{1/2} M_E g(L) \end{aligned} \tag{17}$$

where $L = \pi(C)$ is a straight line in E^2 and M_E is the affine integral transform in E. If we know $M_E g$, we can calculate $M_S f$ and apply (16) for the point $x = 0, y_0 = (1,0)$ taking in account that the second term vanishes:

$$f(y) = -\frac{1}{\pi}\int_0^{\pi/2} \frac{dF(y,q)}{\sin q} \tag{18}$$

where

$$F(y,q) = \frac{1}{2\pi}\int_{d_S(\gamma,y)=q} M_S f(\gamma) d\phi$$

The distance between λ and the origin in E is equal $\tan q$ if $d_S(\gamma, y_0) = q$. Therefore $1 + d^2(L) = \cos^{-2} q$ and by (17) the right side is equal to

$$\frac{1}{2\pi \cos q} \int_{d(L)=\tan q} M_E g(L) d\phi = \frac{G(x, \tan q)}{\cos q}$$

where

$$G(x, r) \doteq \frac{1}{2\pi} \int_{d(L)=r} M_E g\,(L)\, d\phi \tag{19}$$

Substitute this equation to (18) and change the variable q to $r = \tan q$:

$$f(y) = -\frac{1}{\pi} \int_0^\infty \frac{d(\sec q\, G(y, r))}{\sin q}$$

We have

$$\frac{d(\sec q\, G(y, r))}{\sin q} = \frac{dG}{r} + r dG + G dr = \frac{dG}{r} + d(rG)$$

The last term vanishes after integration over the ray $(0, \infty)$ since the product rG vanishes at the ends. The equation $rG = o\left(r^{-1}\right)$ for $r \to \infty$ follows from $g = o\left(|x|^{-2}\right)$. This yields

$$g\,(0) = f(y) = -\frac{1}{\pi} \int_0^\infty \frac{dG\,(0, r)}{r}$$

Moving the origin to an arbitrary point $x \in E$, yields

$$g\,(x) = -\frac{1}{\pi} \int_0^\infty \frac{dG\,(x, r)}{r} \tag{20}$$

This is Radon's formula, see Sec.2.2.

The same arguments applied to the projection $\pi : Q \to E$ give by means of (15) the inversion formula for the hyperbolic plane

$$g\,(x) = -\frac{1}{\pi} \int_0^\infty \frac{dG\,(x, q)}{\sinh q} \tag{21}$$

where G is again defined by (19) and q is the hyperbolic distance.

Comparing the formulae (16),(20) and (21), we see the obvious similarity. The form of the dominators: $r, \sin r, \sinh r$ shows direct impact of Euclidean, elliptical and hyperbolic geometries, respectively.

3.7 Inversion of Minkowski-Funk transform

We write an inversion formula for the Minkwoski-Funk transform M_{n-1} restricted to manifold of projective subspaces $Y \subset \mathbb{P}^n$ of dimension $n-1$. Let $\mathbb{P}^{n*}$ be the dual projective space; a points $z \in \mathbb{P}^{n*}$ defines the polar z° which is the orthogonal projective subspaces of $\mathbb{P}^n$ of dimension $n-1$. Let g be a Borel function on $\mathbb{P}^n$; the function $M_{n-1}g(z^\circ)$ is defined in $\mathbb{P}^n$. We use the model $S^n/\mathbb{Z}_2$ for the elliptic space and its dual. Take the point $y=(1,0)\in S^n\subset E^{n+1}$ and consider the family of spheres $\{z;(z,y)=\cos\phi\}, 0\leq\phi\leq\pi/2$ in S^n. The average of $M_{n-1}g$ over a sphere

$$G(s,\phi)=\frac{1}{|S^{n-1}|}\int_{(z,y)=\cos\phi}M_{n-1}g(z^\circ)\,dV$$

By (14) $G(s,\phi)=\sec\phi F(y,\tan\phi)$ where $F(r)$ is the spherical mean of data $M_E f(L)$, $f=\left(1+|x|^2\right)^{-n/2}g$ over the sphere $\{d(L)=r=\tan\phi\}$. By (7) the reconstruction is given by

$$\begin{aligned}g(s)&=c_n\int_0^\infty F^{(n-1)}(r)\frac{dr}{r}\\&=c_n\int_0^{\pi/2}\left(\cos\phi\frac{\partial}{\partial\phi}\cos\phi\right)^{n-1}G(s,\phi)\frac{d\phi}{\sin\phi}\end{aligned}$$

where n is even and $c_n=1/2^{n-2}(-\pi)^{n/2}\Gamma(n/2)$.

4 General integral mean transform

Consider the general integral mean operator (8) for a family Y of closed submanifolds $Y\subset X$. The reconstruction problem is to find the function f from data of $Mf|Y$. More complicated versions of (8) arise in applications. A weight function $w=w(x,Y)$ (known or unknown) can appear in the integral. Also the "image" f might be not a scalar function but a section of a tensor bundle, like differential form symmetric or skew symmetric. The corresponding theories are far from to be exhausted.

We focus on the simplest case where f is a scalar function and $w=1$. A closed analytic reconstruction formula is only known in few cases. If there is no such a formula one can try to apply numerical methods. An actual numerical algorithm contains usually a regularization procedure and gives a convergent result whichever the input data are. To ensure reliability of the result, the family Y (i.e. the acquisition geometry) should be big enough to guarantee existence of a continuous reconstruction operator $R: Mf|Y\mapsto f$. The condition of continuity can be specified for a family Y that has structure of a smooth manifold. The mapping R is then supposed to be continuous as an operator from the space of smooth functions f with compact support to a space of smooth functions in Y.

4.1 Completeness condition

The completeness condition gives an answer to this question.

Definition. A family Y of submanifolds of X is called *complete* in a subset $G\subset X$ if for an arbitrary $x\in G$ and arbitrary covector t at x there exists $Y\in Y$ such that $x\in Y$ and t is orthogonal to Y. This condition is almost necessary: if there exists a continuous reconstruction operator R for the class of smooth functions f supported by a compact set K in X, then the family Y is complete in the interior of the set K. On the other hand, it can shown that if a manifold Y is complete in K, then there exists, at least, a continuous parametrix for R. Here we use the term "parametrix" in a sense similar to that in the theory of pseudodifferential operators. In fact, the reconstruction is reduced to inversion of a pseudodifferential operator by means of the back projection operator;

$$M^{\sharp}g(x) = \int_{Y(x)} g(Y)\, d\sigma$$

g is a function on Y, $Y(x)$ is the family of manifolds $Y \in Y$ that contain a point x and $d\sigma$ is a measure on this family. The completeness condition means that the operator $M^{\sharp}M$ is of elliptic type.

4.2 Reconstruction from incomplete data

In the most of practical situations the set of available projections is incomplete. We discuss a special case of this result in more details.

Limited diapason. The hyperplane means of f are known for hyperplanes whose angle ϕ with x_1-axes are in the diapason $\phi \le \alpha < \pi/2$. If the function f has compact support we can use interpolation methods for band-limited functions. One of them is

Proposition 6 *A function $\phi \in L_2(\mathbb{R})$ such that* $\operatorname{supp}\hat{\phi} \subset [-1/2, 1/2]$ *can be reconstructed in $(-\delta, \delta)$ for arbitrary $\delta > 0$ as follows*

$$\phi(\zeta) = \exp\left(\pi\sqrt{\delta^2 - \zeta^2}\right) \int_{\Gamma} \frac{\sin\left(\pi\sqrt{\lambda^2 - \delta^2}\right)}{\pi|\lambda - \zeta|} \phi(\lambda) d\lambda \qquad (22)$$

where $\Gamma \doteq (-\infty, -\delta] \cup [\delta, \infty)$, $\operatorname{Re}\sqrt{\delta^2 - \zeta^2} > 0$.

Suppose that $\operatorname{supp} f$ is contained in the strip $|x_1| \le 1/2$. We have $F_{p\to\rho}Rf = \hat{f}(\rho\omega)$, hence we know the Fourier transform $\hat{f}(\xi)$ in the domain $\Gamma = \{|\xi'| \le d|\xi_1|\}, \xi' = (\xi_2, ..., \xi_n), d = \tan\alpha$ (a spherical cone around the x_1-axes). Fix ξ'-coordinates and consider the function $\phi(\zeta) \doteq \hat{f}(\zeta, \xi')$. It is band-limited and known for $|\zeta| \ge d|\xi'|$; the equation (22) can be apply for $\delta = d|\xi'|$. In the general case we replace f by $f_r(x_1, x') \doteq f(rx_1, x')$ for an appropriate r and take $\delta = dr|\xi'|$.

The incomplete data of hyperplane means of f can not dominate the energy L_2-norm of f. A weaker continuity property holds which is described by the following result. Consider the quadratic function $q(\xi) \doteq \xi_1^2 - d^{-1}|\xi|^2$. It is positive in the domain Γ and negative in the complementary set. The energy in the "audible" zone Γ can be expressed in terms of data by means of

Proposition 7 *We have*

$$\int_{q\ge 0} |\hat{f}(\xi)|^2 d\xi = (2\pi)^{(1-n)/2} \int \left|\partial_p^{n-1/2} Rf(\omega, p)\right|^2 dp d\omega$$

For n even the derivative $\partial^{(n-1)/2}$ is a pseudodifferential operator with symbol $(2\pi\rho)^{(n-1)/2}$. It the "silent" zone $\mathbb{R}^n \setminus \Gamma$ we have only a weak estimate

Corollary 8 *For any function $f \in L_2(E)$ with support in the strip $|x_1 - a| \leq r/2$ the inequality holds*

$$\int_{q<0} |\exp\left(-\pi r\sqrt{q(\xi)}\right) \hat{f}(\xi)|^2 d\xi \leq \int_{q\geq 0} |\hat{f}(\xi)|^2 d\xi \tag{23}$$

This estimate can not be much improved.

Exterior problem: to reconstruct a function f in Euclidean plane E, from knowledge of line integrals for lines $L \subset E \setminus B$ where B is the unit ball. There is no simple reconstruction formula. The solution given by A.Cormack is based on the decomposition of f and $M_1 f$ in harmonics.

No inversion problem with incomplete data can be solved by means of a explicit formula or by a stable numerical algorithm. Sometimes it can be solved by more complicated methods like infinite functional series solving of integral equation of the first kind or analytical continuation. Important question arises: which meaning has the solution in a practical situation. A partial answer can be done in geometrical terms:

Let X be the space where an unknown original function f is defined. Suppose that a compact set $K \subset X$ is known such that $\operatorname{supp} f \subset K$. We wish to reconstruct f from the mean transform Mf defined for a family Y of submanifolds of X. If no more a priori information is accessible, the energy of unknown original f is assumed to be spread uniformly over the cotangent bundle $T^*(K)$. Take a manifold $Y \in Y$ and consider the conormal bundle $N^*(Y \cap K) \subset T^*(K)$ of this curve. Denote by $N^*(Y)$ the union of sets $N^*(Y \cap K)$. This is a conic subbundle of $T^*(K)$. We call this subbundle the ***audible*** zone. It can be shown that the part of the energy of the original f inside the audible zone can be reasonably estimated by a suitable norm of Mf. The complementary part of the energy which is contained in the ***silent*** zone $T^*(K) \setminus N^*(Y)$ can be estimated with a weight. This weight is a function in the cotangent bundle that exponentially decreases, when the point moves away from the audible zone. The inequality (23) is an example of an estimate of this kind. An exponential factor of this kind is indispensable.

5 Affine integral transform

Let E be an Euclidean space of dimension n. For any integer k, $0 < k < n$ we consider the manifold $A_k(E)$ of all affine subspaces $A \subset E$. It is an algebraic manifold of dimension $(k+1)(n-k)$. Denote by M_k the restriction of the integral means transform to $A_k(E)$. Suppose we know the integral $M_k f(A)$ for all k-affine subspace A. If $k = n-1$ we can recover the function f by formulae of Sec.2. For an arbitrary $k < n$ a function f can be reconstructed from knowledge of $M_k f$ in the manifold $A_k(E)$. Take an arbitrary subspace $G \in A_{k+1}(E)$. This is again an Euclidean space and any hyperplane H in G is an element of $A_k(E)$. Therefore we can reconstruct the

function $f|G$ by means of inversion of the Radon transform in G. Hence we know the function f in E since the subspace G of dimension $k+1$ is arbitrary.
There are, of course, many other methods of reconstruction since the data of $M_k f$ is *redundant* for the case $k < n-1$ since $(k+1)(n-k) > n$. The case $k = 1$ which is important, in particular, for the X-ray inversion algorithms in three-space. The equation (24) shows the rate of redundance of the data $A_1(E)$ for $n = 3$. For practical applications is important to receive an reconstruction from as small sampling of integral data as possible.

To avoid redundance we state the reconstruction problem as follows:
Problem: to find a reconstruction formula or an algorithm $M_1 f|\Sigma \mapsto f$ for functions f supported by a compact set $K \subset E$ for a submanifold $\Sigma \subset A_1(E)$ of dimension $n = \dim E$. We call such a manifold *pencil*. The data of line integrals $M_1 f(L)$, $L \in \Sigma$ has no dimension redundancy. We need, of course, assume that the pencil Σ satisfies the completeness condition for K.

5.1 Line transform and John equation

We consider for simplicity the case $n = 3$, $k = 1$. We have $\dim A_1(E) = 4$, hence $g = M_1 f$ is a function of 4 variables in any chart of the manifold $A_1(E)$. It is far from to be arbitrary. Take the chart F that contains all straight lines L that are not parallel to the plane $x_3 = 0$ in E. Take a point $(y_1, y_2, 0) \in L$; let $v = (v_1, v_2, 1)$ be a vector parallel to L; the numbers (y_1, y_2, v_1, v_2) are coordinates of L in the chart F. This coordinates parameterize the line mean

$$g(y_1, y_2, v_1, v_2) = \int_{L(y_1, y_2, v_1, v_2)} f(x)\, ds$$

Proposition 9 *The function g satisfies the equation*

$$\frac{\partial^2 h}{\partial y_2 \partial y_3'} = \frac{\partial^2 h}{\partial y_2' \partial y_3} \tag{24}$$

where $h(y_1, y_2, v_1, v_2) \doteq \left(1 + v_1^2 + v_2^2\right)^{-1/2} g(y_1, y_2, v_1, v_2)$.

PROOF. The line L is given by parametric equations $x_1 = y_1 + tv_1$, $x_2 = y_2 + tv_2$, $x_3 = t$ and the Euclidean line density in L is equal to $ds = \sqrt{1 + v_1^2 + v_2^2}dt$ consequently

$$h(y_1, y_2, v_1, v_2) = \int f(y_1 + tv_1, y_2 + tv_2, t)dt$$

Therefore

$$\frac{\partial^2 h}{\partial y_1 \partial v_2} = \int_{-\infty}^{\infty} (1-t)t \frac{\partial^2 f}{\partial x_1 \partial x_2}(y_1 + tv_1, y_2 + tv_2, t)dt$$

We get the same integral formula for the function $\partial^2 h / \partial v_1 \partial y_2$. □

The equation (24) is called the John equation. It can be written in the form

$$\frac{\partial^2 h}{\partial s^2} + \frac{\partial^2 h}{\partial v^2} - \frac{\partial^2 h}{\partial t^2} - \frac{\partial^2 h}{\partial u^2} = 0$$

after a linear coordinate change. This equation, called ***ultrahyperbolic***, does belong neither to elliptic nor to hyperbolic type. The Cauchy problem is ill posed. There are, however, well posed characteristic boundary problems that relate to inversion formulae for the ray transform.

5.2 Characteristic Cauchy problem

We can consider the reconstruction problem as a kind of boundary value problem for the John equation (24) with data on the hypersurface (pencil) $\Sigma \subset A_1(E^3)$. If we can solve this equation by means of this boundary data, we obtain Mf on $A_1(E)$ and apply reconstruction of f by means of above Corollary. We know from the theory of partial differential equations that one need to fix two functions on a non characteristic hypersurface S, for example, $g|S$ and $\partial_\nu g|S$ where ∂_ν means the normal to S derivative to ensure local uniqueness of solution of the Cauchy problem. In the reconstruction problem only the function $f|\Sigma$ is known. This implies the necessary condition: the pencil Σ is *characteristic* at each point. This condition turns to be sufficient under additional assumptions. To clarify the idea, let us consider the general second order equation

$$a(y, D)\, u = 0 \tag{25}$$

in an open set U in an Euclidean space E^n.

Proposition 10 *Let K be a compact in U with smooth boundary Γ, $x \in K\backslash\Gamma$ and F be a solution to $a^*F = \delta_x$ defined in a neighborhood of K such that the restriction $F_x|\Gamma$ is well defined as distribution on Γ. An arbitrary solution u can be reconstructed in the point x from data $u|S$ provided $S \subset \Gamma$ is a characteristic hypersurface for a and* supp $F_x|\Gamma \Subset S$.

PROOF. Take a smooth function ϕ in U such that $\phi \geq 0$ in K, $\phi < 0$ in $U\backslash K$ and $|\nabla\phi| = 1$ on Γ. The function $\theta = \Theta(\phi)$ is the indicator function of $\mathbb{R}_+$ where $\Theta(t) = 1$ for $t \geq 0$ and $\Theta(t) = 0$ otherwise. Write

$$u(x) = \delta_x(\theta u) = \int_W a^*(F)\, \theta u dV_y = \int_W F a(\theta u)\, dV_y$$

By the Leibniz formula

$$a(y, D)(\theta u) = \theta a(y, D)\, u + \sum_i \theta_i a^{(i)}(y, D)\, u + \frac{1}{2}\sum_{i,j} \theta_{i,j} a^{(i,j)}(y, D)\, u$$

where $\theta_i = \partial_i\theta$, $a^{(i)}(x, D)$ is the differential operator with symbol $a^{(i)}(x,\xi) = \partial a(x,\xi)/\partial\xi_i$;the operator $a^{i,j}$ is defined similarly. The first term vanishes in virtue of (25), the second and the third terms are supported in Γ. We have $\theta_i = \phi_i\, \delta(\phi)$ where δ is the delta-function, hence, the second term is equal to $\delta(\phi)\, \phi_i a^{(i)}(y, D)\, u$. Write $a = a_2 + a_1 + a_0$ where a_j is a homogeneous differential operator of order j. Then $\phi_i a^{(i)}(y, D) = \tau + a_1(\phi)$, where $\tau = \phi_i(y)\, a_2^{(i)}(y, D)$ is a vector field. We have

$\tau(\phi) = \phi_i a_2^{(i)}(y, \nabla\phi) = 2a_2(y, \nabla\phi) = 0$ in S, since S is characteristic. This means that τ is tangent to S. We have further

$$\theta_{i,j} = \phi_i \phi_j \delta'(\phi) + \phi_{i,j}\,\delta(\phi),$$

$$\frac{1}{2}\sum \theta_{i,j} a^{(i,j)}(y, D) = a_2(y, \nabla\phi)\,\delta'(\phi) + a_2(y, D)(\phi)\,\delta(\phi)$$

The first term vanishes in S since $a_2(y, \nabla\phi) = 0$. This yields $Fa(\theta u) = (\tau + \alpha)\, uF\delta(\phi)$ where $\alpha \doteq a_1(\phi) + a_2(\phi)$. The product $F\delta(\phi)$ is well defined as a distribution on Γ by the condition which yields

$$u(x) = \int_\Gamma (\tau + \alpha)\, uF\delta(\phi)\, dV_y = \int_S (\tau + \alpha)\, uF dS$$

where $dS = dV/d\phi$ is the Euclidean surface area element in Γ. The integral depends only on $u|S$, since S contains a neighborhood of the support of the distribution $F\delta(\phi)$. □

6 Reconstruction from ray integrals

Now we study the case $\dim E = 3$ in more details. There are several cases where there is well-defined reconstruction formulae:

1. Choose a plane $H \subset E$ and consider the pencil Σ_H of straight lines that are parallel to H. Take an arbitrary plane H' that is parallel to H. Any line $L \subset H'$ belongs to Σ_H hence we know the line transform $M_1 f(L) = \int_L f ds$. Apply the inversion of the Radon transform in H' and reconstruct the function $f : H' \to \mathbb{C}$ for each plane H' that is parallel to H.

2. Take a curve $C \subset \mathbb{P}(E)$ such that any plane $H \subset E$ has non-empty intersection with C at infinity. Consider the pencil $\Sigma(C)$ of lines L that meet C at infinity. There exists a reconstruction method for this pencil. Indeed, take a plane H; let $c \in C$ be a point where H meets C. Any line $L \subset H$ that contains the point c at infinity belongs $\Sigma(C)$. Such lines L are parallel one to another and makes a foliation of H. Let $T \subset H$ be a line that is orthogonal to L. By Fubini's theorem

$$\int_H f dS = \int_T dt \int_L f ds$$

where dt is the Euclidean density in T. Thus we know the Radon transform $M_2 f$ for any plane H in E and can reconstruct the function f.

3. Let Γ be a curve in E and $\Sigma(\Gamma)$ be the pencil of rays with vertices in Γ. A function f with compact support can be reconstructed if the completeness condition (Sec.5.6) is satisfied. The pencil $\Sigma(\Gamma)$ is characteristic.

Note that the pencil Σ_H as in case **1** is equal to the pencil $\Sigma(C)$ where $C = H \cap \mathbb{P}(E)$ is an improper line. The class **2** can be reduced to the class **3,** since the curve C can be transported to E by a suitable projective transformation P. The mapping P transforms lines to lines, hence, $P(\Sigma(C)) = \Sigma(P(C))$. By Proposition 3

a reconstruction formula for $\Sigma(C)$ can be translated to a reconstruction formula for $\Sigma(P(C))$ and vice versa.
4. Let S be a surface in E and $\Sigma(S)$ be the pencil of rays with vertices in S that are tangent to S. It is also characteristic. A reconstruction of a function f with compact support is possible under the completeness condition. Class **3** is, in a sense, contained in the closure of class **4,** see Remark below.

6.1 Rays with verticis on a curve

Theorem 11 *[Grangeat,Finch] Let Γ be a curve in an Euclidean space E^3 and the plane $H(\omega,p) = \{x; \omega\cdot x = p\}$ meets Γ in a point y. Then for an arbitrary $f \in C^1(E)$ such the set* supp $f \cap H$ *is compact we have*

$$\frac{\partial}{\partial p}M_2 f(H) = \int \frac{\partial}{\partial q}M_1 f(L(q,v))|_{q=0}d\phi(v) \quad (26)$$

where $L(q,\omega) = \{x = y + t(v + q\omega),\ t\in\mathbb{R}_+\}$, v is a unit vector in $H(\omega,0)$, and $d\phi$ is the area element on a unit circle in $H(\omega,0)$.

Proof. We have

$$M_1 f(L(q,v)) = \left(1+q^2\right)^{1/2}\int f(y + t(v+q\omega))dt$$

since $\left(1+q^2\right)^{1/2} = |v+q\omega|$. Take the derivative

$$\frac{\partial}{\partial q}\left(1+q^2\right)^{-1/2} M_1 f(L(q,v)) = \int g(y+t(v+q\omega))tdt$$

where $g = \omega\cdot\nabla f$. Integrating against the element $d\phi$ and setting $q=0$ we get

$$\int\frac{\partial}{\partial q}[\left(1+q^2\right)^{-1/2}M_1 f(L(q,v))]|_{q=0}d\phi(v) = \int\frac{\partial}{\partial q}M_1 f(L(q,v))|_{q=0}d\phi(v)$$
$$= \int\int \omega\cdot\nabla f(y+t\lambda)tdtdv = \int_H \omega\cdot\nabla f dS = \frac{\partial}{\partial p}M_{n-1}f(H)$$

since $dS = tdtd\phi$ is the Euclidean surface element in H. □

Thus the quantity $\partial_p M_2 f(H)$ is reconstructed from data of ray integrals of f for the family of rays starting from a point $y\in H\cap\Gamma$ that are close to H. If the conditions of this Theorem are satisfied for any hyperplane H in E that meets supp f, we know the derivative $\partial_p M_2 f(H)$ for all H. Remind that the inversion formula for the Radon transform in the case $n=3$ depends only on second derivative of $\partial_p^2 M_2 f$. Therefore the information we have is sufficient to apply this formula and recover the function f.

6.2 Rays tangent to a surface

Let C be a curve in E^3; a function $s: C\to\mathbb{R}$ is called a natural parameter if the differential ds is equal the Euclidean density, i.e. if $|s(x)-s(y)|$ for any points $x,y\in C$ is equal the length the arc of C between these points.

Theorem 12 *[Denisjuk-Palamodov] Let S be a closed surface in E, H be a plane in E transversal to S. Then for an arbitrary $f \in C^1(E)$ such the set $\operatorname{supp} f \cap H$ is compact in $H \backslash S$ we have*

$$\frac{\partial}{\partial p} M_2 f(H) = \frac{1}{z} \int_C \left(\frac{\kappa}{[x', \nu, \omega]} \frac{\partial}{\partial q} - \frac{\partial}{\partial s} \left(\frac{\nu \cdot \omega}{[x', \nu, \omega]} \right) \right) M_1 f(L(s,q))|_{q=0} ds \tag{27}$$

where $x = x(s)$ is the equation of the curve $C \doteq S \cap H$, s is a natural parameter, and $\kappa = [x', x'', \omega]$ is the curvature of C; $L(s,q)$ is the ray given by

$$x = x(s) + t[x'(s) + q(x'(s) \times \nu)], \ t \in \mathbb{R}_+$$

and $\nu = \nu(x(s))$ is a continuous normal vector field to S. We assume that the set $\xi^{-1}(\operatorname{supp} f)$ is compact for the mapping

$$\xi : C \times \mathbb{R} \to H, \quad \xi(s,t) = x(s) + tx'(s)$$

and define

$$z(x) \doteq \sum_{x = \xi(s,t)} \operatorname{sgn} \kappa(s)$$

The number z does not depend on $x \in K$; we assume that $z \neq 0$.

The vector $x'(s)$ is orthogonal to $\nu(x(s))$ hence the line L is tangent to the curve C at the point $x(s)$ and belongs to the pencil $\Sigma(S)$. If the conditions of this theorem are satisfied for any plane H we can calculate the first derivative of $M_2 f$. Then by means of the inversion formula for the Radon transform we recover f.

PROOF. We have

$$\int f'(\xi) \cdot x(s)\, t dt = \int \frac{\partial}{\partial t} f(\xi) dt = 0 \tag{28}$$

since the function $f|L(s,0)$ has compact support. Therefore

$$\frac{\partial}{\partial s} M_1 f(L(s,0)) = \int \frac{\partial}{\partial s} f(\xi) dt = \int f'(\xi) \cdot (x'(s) + tx''(s))\, dt$$

$$= \int f'(\xi) \cdot x''(s)\, t dt = \kappa \int f'(\xi) \cdot \omega \times x'(s)\, t dt \tag{29}$$

since $x'' = \kappa\omega \times x'$. Farther

$$\frac{\partial}{\partial q} M_1 f(L(s,q))|_{q=0} = \int f'(\xi) \cdot x'(s) \times \nu\, t dt$$

The vectors ω, $x' \times \omega$, $x' \times \nu$ are orthogonal to x' hence $x' \times \nu = [x', \nu, \omega]\omega + \nu \cdot \omega\, x' \times \omega$. This yields

$$\frac{\kappa}{[x', \nu, \omega]} \frac{\partial}{\partial q} M_1 f(L(s,q))|_{q=0} = \kappa \int (f'(\xi), \omega)\ t dt + \frac{\nu \cdot \omega}{[x', \nu, \omega]} \frac{\partial}{\partial s} M_1 f(L(s,0))$$

Integrate both sides over $H \cap S$ against the density ds:

$$\int \frac{\kappa}{[x',\nu,\omega]}\frac{\partial}{\partial q}M_1 f(L(s,0))ds + \int \frac{\nu\cdot\omega}{[x',\nu,\omega]}\frac{\partial}{\partial s}M_1 f(L(s,0))ds$$

$$= \int \kappa \int f'(\xi)\cdot\omega\, tdtds = \frac{\partial}{\partial\omega}\int\kappa\int f(\xi)tdtds \qquad (30)$$

Consider the system of coordinates $s,t,p=\omega\cdot x$ in a neighborhood of C. We have $\partial x/\partial(t,s,p) = \det(x', x'+tx'', \omega) = \kappa t$. The Euclidean volume element in H is equal dV/dp, hence $\partial h/\partial(s,t) = \kappa t$. Therefore

$$\int\kappa\int f(\xi)tdtds = z\int_H fdS = zM_2 f(H)$$

hence the right side of (30) is equal to $z\partial_p M_2 f(H)$ and

$$z\frac{\partial}{\partial p}M_2 f(H) = \int \frac{\kappa}{[x',\nu,\omega]}\frac{\partial}{\partial q}M_1 f(L(s,0))ds + \int \frac{\nu\cdot\omega}{[x',\nu,\omega]}\frac{\partial}{\partial s}M_1 f(L(s,0))ds$$

Integrating by parts in the second term, we get (27). □

Remark. Theorem (11) in the case $n=3$ is a limiting case of the above result. Really, take an arbitrary compact smooth curve $\Gamma \subset V$ and consider its ε-neighborhood Γ_ε for some $\varepsilon > 0$. If the number ε is sufficiently small, the boundary S_ε of Γ_ε is a smooth surface. Take the pencil $\Sigma(S_\varepsilon)$ of tangent lines and apply formula (27). It is easy to see that this pencil tends to the pencil $\Sigma(\Gamma)$ as $\varepsilon \to 0$ and formula (27) tends to (26).

One of the reasons this theorem is restricted to the case $n=3$ is that in general the affine transform M_k fulfils a complicated system of differential equations. The arguments of Sec.5 based on a characteristic Cauchy problem can not be applied, at least, no straightforward adaptation is known. In Sec.8 we obtain a reconstruction method for the case $k=n-2$ by means of duality argument. It is plausible that there are more geometrically defined of pencils of affine spaces admitting an explicit reconstruction formula.

6.3 Reconstruction from plane collimated measurements

Let $\varphi = fdV$ be a density of sources of a radiation in the Euclidean space E^3 with compact support. For a plane $P \subset E$ and a point $a \in P$ the radiation from sources in P measured by a detector in position a, is given by the integral

$$I(a,P;\varphi) \doteq \int_P \frac{f(x)dS}{|x-a|}$$

where dS is the area element in P. The problem is to reconstruct f from knowledge of data of integrals $I(a,P)$. Suppose that these integrals are known for all sources a on a smooth curve Γ in E. The completeness condition looks as follows: any plane P that meets $\operatorname{supp}\varphi$ contains a point $a \in \Gamma$. Denote by K the convex hull of $\operatorname{supp}\varphi$.

Theorem 13 *Let Γ be a smooth curve in $E\backslash K$ that fulfils the completeness condition for* $\operatorname{supp}\phi$. *The function f can be reconstructed from the knowledge of $I(a,P;\varphi)$ for such $a\in\Gamma$ and $P\ni a$.*

PROOF. Take a point $a\in\Gamma$ and choose an Euclidean coordinate system x,y,z centered at a such that $x>0$ in $\operatorname{supp}\varphi$. Apply the projective transformation

$$\xi=\frac{1}{x},\ \eta=\frac{y}{x},\ \zeta=\frac{z}{x}$$

By Proposition 3 $dxdy=\xi^{-3}d\xi d\eta$, hence

$$\frac{dxdy}{|p-a|}=\frac{d\xi d\eta}{\xi^2\sqrt{1+\eta^2+\zeta^2}}$$

Choose a number ζ and take the plane $P(\zeta)=\{z=\zeta x\}$. We have $dS=(\cos\theta)^{-1}dxdy$, where θ is the angle between P and the x-axis, and

$$\cos\theta\, I(a,P_\zeta;\varphi)=\int_{P_\zeta}\frac{fdxdy}{|p-a|}=\int\frac{fd\xi}{\xi^2}\frac{d\eta}{\sqrt{1+\eta^2+\zeta^2}} \tag{31}$$

$$=\int_{-\infty}^{\infty}\left(\int_0^\infty f(x,\eta x,\zeta x)dx\right)\frac{d\eta}{\sqrt{1+\eta^2+\zeta^2}}$$

Denote

$$Lf(\eta,\zeta)\doteq\int_{R(\eta,\zeta)}fdx=\left(1+\eta^2+\zeta^2\right)^{-1/2}\int_{R(\eta,\zeta)}fds, \tag{32}$$

$$R(\eta,\zeta)\doteq\{y=\eta x,z=\zeta x,\ x>0\}$$

where ds is the Euclidean density in the ray R. The equation (31) is equivalent to

$$\cos\theta\, I(a,P_\zeta;\varphi)=\int Lf(\eta,\zeta)d\eta,$$

An arbitrary plane through the origin that touch $\operatorname{supp}\varphi$ can be written in the form $P=\{\alpha y+\beta z=x\},\alpha^2+\beta^2\neq 0$. We have

$$\cos\theta\ I(a,P;\varphi)=\int_{P^*}Lf(\eta,\zeta)d\sigma,\ \cos\theta=\left(1+\left(\alpha^2+\beta^2\right)^{-1}\right)^{-1/2}$$

where $P^*\doteq\{(\eta,\zeta):\alpha\eta+\beta\zeta=1\}$ and $d\sigma$ is the Euclidean measure in the line P^*. This equation shows that the ray transform of the function $Lf(\eta,\zeta)$ is known for any straight line P^* in the Euclidean (η,ζ)-plane, hence the function Lf can be reconstructed be means of the inversion of the plane Radon transform. From (32) we find the ray integrals $\int_R fds$ for all rays R with the vertex a. Therefore the ray transformation of f is known for any ray starting from a point $a\in\Gamma$ and the function f can be reconstructed by means of Theorem 11. □

7 Duality in integral geometry

7.1 Fourier transform of homogeneous functions

Consider an Euclidean space $\mathbf{E}$ of dimension $n+1$ with an Euclidean coordinate system $x = (x_0, ..., x_n)$. The form $dV = dx_0...dx_n$ is the Euclidean volume element. We write the Fourier integral for a density $\phi = f dV \in L_1(\mathbf{E})$ in the form (3). The Fourier image $\widehat{f} = F(f dV)$ is a function on the dual Euclidean space $\mathbf{E}^*$. Let $S(\mathbf{E})$ be the Schwartz space of smooth functions in $\mathbf{E}$. It is the class of smooth C^∞-functions f in $\mathbf{E}$ such that $D^i f = O\left((1+|x|)^{-q}\right)$ for any $i = (i_0, ..., i_n)$ and arbitrary natural q. Let $S(\mathbf{E}^*)$ be the Schwartz class of functions in $\mathbf{E}^*$. An element the space $S(\mathbf{E}^*)dV^*$ is a density in $\mathbf{E}^*$ of the form $\rho = \psi dV^*$, where ψ is an element of the Schwartz space $S(\mathbf{E}^*)$ and $dV^* = d\xi_0...d\xi_n$ for the dual coordinate system $\xi_0, ..., \xi_n$. The dual space $S'(\mathbf{E})$ is the space of tempered distributions and $(S(\mathbf{E}^*)dV^*)'$ is the space of tempered generalized functions. Replace the space $\mathbf{E}$ by its dual and consider the conjugated transform (4). This operator maps $S(\mathbf{E}^*)dV^*$ continuously to $S(\mathbf{E})$; the dual operator is $F' : S'(\mathbf{E}) \to (S(\mathbf{E}^*)dV^*)'$. The latter is called the Fourier transform of tempered distributions; the image of such a distribution is a tempered generalized function. The operator F' agrees with (4) for distributions $\phi \in L_1$.

A distribution or a generalized function u in $\mathbf{E}$ is called homogeneous of degree $\lambda \in \mathbb{C}$, if it satisfies the equation $L_e u = \lambda u$ where L_e denotes the Lie derivative along the Euler field $e = \sum x_i \partial/\partial x_i$. This equation means that $u(L_e\phi) = -\lambda u(\phi)$ where $L_e\phi = e(\phi)$ for any test function ϕ and L_e is the Lie derivative along e. This definition agrees with the classical one, since for a smooth function or distribution u and test density, respectively, function ϕ the following equation holds

$$\int L_e u\phi + \int u L_e \phi = \int L_e(u\phi) = 0$$

In particular, the volume density dV is a homogeneous distribution of degree $n+1$ and the delta-function $\delta_0(\phi dV) = \phi(0)$ is a homogeneous generalized function of degree $-n-1$. Any homogeneous distribution or generalized function is tempered.

7.2 Duality for Minkowski-Funk transform

We denote by $\Sigma(\mathbf{E}^*)$ the unit sphere in the Euclidean space $\mathbf{E}^*$ and by $\sigma_{\mathbf{E}}$ the corresponding projective volume form. For a subspace $L \subset E$ the polar L° is the space of covectors $y \in \mathbf{E}^*$ such that $(x, y) = 0$ for all $x \in L$.

Theorem 14 *Let L be an arbitrary proper subspace of $\mathbf{E}$ and L° be its polar. We have*

$$\int_{\Sigma(L^\circ)} \widehat{f}\sigma_{L^\circ} = \int_{\Sigma(L)} f\sigma_L \tag{33}$$

for any homogeneous function f in $\mathbf{E}$ of degree $-k$, $k = \dim L$.

7.3 Duality in Euclidean space

Definition. Let E be an Euclidean space of dimension n with coordiantes $x = (x_1, ..., x_n)$. Take an affine subspace $A \subset E \setminus \{0\}$ of dimension k and consider the system of equations for $x \in E$:

$$\langle x, y \rangle_E + 1 = 0, \; y \in A$$

The set $\tilde{A}$ of solutions is an affine subspace E of dimension $n - k - 1$. We call this space *dual* to A. The double dual space to A coincides with A.

Let M_E be the affine integral transform in E. It turns that the values of M_E on A and $\tilde{A}$ are related as follows. Consider the standard embedding $e : E \to \mathbf{E}$ by $x \mapsto (1, x)$ where $\mathbf{E} = \mathbb{R} \dotplus E$ be an Euclidean space with coordiantes $x_0, ..., x_n$. Fix an integer $k, 0 < k < n$; let f be a function in E such that

$$\left(1 + |x|^{2k+2}\right) f(x) \in L_1(E) \tag{34}$$

Define the function in $\mathbf{E}$

$$g(x_0, x) \doteq x_0^{-k-1} \left(1 + \left|\frac{x}{x_0}\right|^2\right)^{(k+1)/2} f\left(\frac{x}{x_0}\right)$$

It is homogeneous of degree $-k - 1$. By (34) the density $g dx_0 dx_1 ... dx_n$ is locally integrable and the Fourier transform $\hat{g}$ is well defined in $\mathbf{E}^* \cong \mathbf{E}$. It is a homogeneous generalized function of degree $k - n$. We call

$$\tilde{f}(x) \doteq \left(1 + |x|^2\right)^{(k-n)/2} \hat{g}(1, x)$$

k-dual function to f. It is easy to see that the function $n - k - 1$-dual to $\tilde{f}$ is equal to f provided that $\left(1 + |x|^{2n-2k}\right) \tilde{f}(x) \in L_1(E)$. Denote $d(A) \doteq \operatorname{dist}(A, 0)$.

Theorem 15 *Let f be a function in E satisfying (34) for some integer k, $0 < k < n$. Then for arbitrary affine subspace $A \subset E$ of dimension k we have*

$$d^{1/2}\left(\tilde{A}\right) M\tilde{f}\left(\tilde{A}\right) = d^{1/2}(A) M f(A) \tag{35}$$

where $\tilde{f}$ is the k-dual function.

PROOF. The density $g\sigma$ is integrable on the hemisphere $\Sigma_+ \doteq \left\{x_0^2 + |x|^2 = 1,\right.$ $\left. x_0 > 0\right\}$ in $\mathbf{E}$. Let L be the linear span of $e(A)$ in $\mathbf{E}$. By (14) we have

$$\int_{\Sigma_+ \cap L} g\sigma = \left(1 + d^2(A)\right)^{1/2} \int_A \left(\left(1 + |x|^2\right)^{-(d+1)/2} g(1, x)\right) dV_A \tag{36}$$
$$= \left(1 + d^2(A)\right)^{1/2} M_E f(A)$$

On the other hand, by (33) we have

$$\int_{\Sigma_+ \cap L} g\sigma_L = \frac{1}{2}\int_{\Sigma(L)} g\sigma_L = \frac{1}{2}\int_{\Sigma(L^\circ)} \widehat{g}\sigma_{L^\circ} = \int_{\Sigma_+ \cap L^\circ} \widehat{g}\sigma_{L^\circ}$$

The polar space L° is equal to the span of $\tilde{A}$. We apply the equation (36) to the right side:

$$\int_{\Sigma_+ \cap L^\circ} \widehat{g}\sigma_{L^\circ} = \left(1 + d^2\left(\tilde{A}\right)\right)^{1/2} M_E \tilde{f}\left(\tilde{A}\right) \tag{37}$$

Now (35) follows from (36) and (37) since $d(A)\, d\left(\tilde{A}\right) = 1$. □

Example 1. Let S be a surface in E^3 with non-vanishing Gaussian curvature and $\Lambda(S)$ the variety of lines tangent to S. Then we have

$$\tilde{\Lambda}(S) = \Lambda(\tilde{S}),$$

where $\tilde{S}$ is the dual surface, i.e. the envelope of hyperplanes $\tilde{y}$, $y \in S$.

Example 2. Let Γ a smooth curve in $E^n \setminus \{0\}$ and $\Lambda(\Gamma)$ the variety of lines that meet Γ. The dual variety $\tilde{\Lambda}(\Gamma)$ consists of affine $n-2$-planes A that are contained in hyperplanes $\tilde{\gamma}$, $\gamma \in \Gamma$ where $\tilde{\gamma}$ is dual to the point γ. The family $\{\tilde{\gamma}, \gamma \in \Gamma\}$ have an envelope S which is a smooth hypersurface in E if Γ generic, for instance the vectors $x'(s), x''(s), ..., x^{(n)}(s)$ are independent in each point $x = x(s)$ of Γ. The variety $\tilde{\Lambda}(\Gamma)$ is the family of $n-2$-planes that are tangent to S. Note that S is a hypersurface of very special form. For the variety $\Lambda(\Gamma)$ we have the following reconstruction method which is a direct generalization of Theorem 11.

Theorem 16 *Let Γ be a curve in an Euclidean space E^n and the hyperplane $H(\omega, p) = \{x; \omega \cdot x = p\}$ meets Γ in a point y. Then for an arbitrary $f \in C^{n-2}(E)$ such the set* supp $f \cap H$ *is compact we have*

$$\frac{\partial^{n-2}}{\partial p^{n-2}} M_{n-1} f(H) = \sum_{k=0}^{[n/2]-1} c_{n,k} \int \frac{\partial^{n-2-2k}}{\partial q^{n-2-2k}} M_1 f(L(q,v))|_{q=0} dS \tag{38}$$

where $L(q, \omega) = \{x = y + t(v + q\omega),\ t \in \mathbb{R}_+\}, v$ is a unit vector in $H(\omega, 0)$, dS is the area element on a unit sphere $S^{n-2} \subset H(\omega, p)$ and

$$c_{n,k} = (-1)^k ((2k-1)!!)^2 \binom{n-2}{2k}$$

By the duality we get a reconstruction method for the variety $\tilde{\Lambda}(\Gamma)$.

8 Spherical mean transform

Let E be a Euclidean space of dimension n. The spherical mean transform of a function f in X is the integral

$$M_S f(x, r) \doteq \frac{1}{|S^{n-1}| r^{n-1}} \int_{|y-x|=r} f dS,\ |S^{n-1}| = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

The function $M_S f$ is defined in the space $X_S \doteq X \times \mathbb{R}_+$ where $\mathbb{R}_+$ stands for the closed half-line; we have $M_S f(x, 0) = f(x)$.

For an arbitrary continuous function f in an Euclidean space E^n the spherical mean transform $g \doteq M_S f$ satisfies the Euler-Poisson-Darboux equation in $X = E \times \mathbb{R}_+$

$$Dg \doteq \left(r\frac{\partial^2}{\partial r^2} + (n-1)\frac{\partial}{\partial r} - r\Delta_x \right) g(x,r) = 0$$

The principal part of the Darboux operator is equal to the wave operator $r\Box$ where $\Box = \partial_r^2 - \Delta_x$ with the velocity 1. A hypersurface $\Sigma \subset X$ is *characteristic* for the Darboux (and for the wave) operator at a point $(x, r) \in X, r > 0$, if the principal symbol vanishes on the conormal vector ν to Σ at x i.e., if $\nu_r^2 - \nu_x^2 = 0$, where $\nu_x \in E$ and $\nu_r \in \mathbb{R}$ are the components of ν.

Proposition 17 *The variety $\Sigma(Y)$ is characteristic at each its point.*

It follows that the reconstruction problem can be reduced to the characteristic Cauchy problem: *given a solution g of the Darboux equation is known on a characteristic hypersurface, to find this solution on the boundary $E \times \{0\}$*. We have then $f(x) = g(x, 0)$. The method of Sec.5.2 can be applied for reconstruction of f from the data $M_S f|\Sigma(Y)$, provided the variety $\Sigma(Y)$ is complete at each point $x \in \operatorname{supp} f$.

9 Integral transform of differential forms

Let V be a vector space of dimension n, α be a smooth differential form in V with compact support. Suppose we know the integrals

$$M\alpha(\gamma) = \int_\gamma \alpha, \ \gamma \in \Gamma \tag{39}$$

for a family Γ of closed algebraic submanifolds $\gamma \subset V$. The problem is which information on α can be recovered? All integrals (39) vanish if $\alpha = d\beta$ for a form β with compact support, hence, we can not expect to reconstruct the form α uniquely. The problem can be specified in the following way:

Problem: is it possible to reconstruct the form $d\alpha$ from the integral information (39)? To avoid the redundance we ought to assume that Γ is a n-parametric family. This is, in fact, equivalent to reconstruction of the form α modulo the subspace of exact forms $d\beta$ such that supp β is compact. There is a large variety of interesting special cases of this problem. We consider here only two simplest examples.

Radon transform: Γ is the family of hyperplanes in V

Proposition 18 *The form $d\alpha$ can be uniquely reconstructed from data of $M\alpha|H$ for all hyperplanes $H \subset V$ and a form α with compact support.*

Proof. Let H be a hyperplane with an orientation and $V(H)$ be a half-space bounded by H. We have by Stokes'

$$\int_{V(H)} d\alpha = \int_H \alpha$$

Choose an Euclidean structure in V; let dV be the volume element in this structure. Write $d\alpha = \phi dV$ for a function ϕ and set $H = H(\omega, p)$. Taking p-derivative we get

$$\partial_p \int_{H(\omega,p)} \alpha = \int_{H(\omega,p)} \phi dV = R\phi(\omega, p)$$

We recover the function ϕ by means of inversion of the Radon transform. □

Integrals over quadrics. Consider a family of quadratic hypersurfaces $q(x) = 0$ in V. Fix N points $p_1, ..., p_N \in V$ in general position, where $N = n(n+1)/2$ and take the family Γ_N of quadrics that contain these points: $q(p_j) = 0, j = 1, ..., N$. The family of quadratic functions q satisfying these equation is a vector Q space of dimension $n+1$, hence the family Γ_N has n parameters.

We show that the data $M\alpha|\Gamma_N$ is sufficient for reconstruction, at least, locally. Choose a basis $q_0, ..., q_n$ in Q; it defines the algebraic mapping $q : V \to \mathbb{R}^{n+1}$, $\xi_j = q_j(x)$, $j = 0, ..., n$. Consider the projective space $\mathbb{P}^n = \mathbb{P}(\mathbb{R}^{n+1})$ and choose an affine chart, say, the chart $\mathbb{W} \cong \mathbb{R}^n$ with coordinates $y_1 = \xi_1/\xi_0, ..., y_n = \xi_n/\xi_0$. The mapping $q : V \backslash \{q_0 = 0\} \to \mathbb{W}$ is well defined; choose an open subset $V' \subset V \backslash (q_0 = 0)$ such that the mapping $V' \to \mathbb{W}$ is an embedding. This means that the functions $y_j = q_j(x)/q_0(x), j = 1, ..., n$ are coordinates in V'. Any linear equation $a_0 + a_1 y_1 + ... + a_n y_n = 0$ is equivalent to $a_0 q_0 + ... + a_n q_n = 0$, i.e. any hyperplane H belongs to the family Γ_N. For an arbitrary form α of degree $n-1$ such that $\operatorname{supp} \alpha \subset V'$ all the hyperplane integrals are known. According to Proposition 18 the form $d\alpha$ can be reconstructed from this data.

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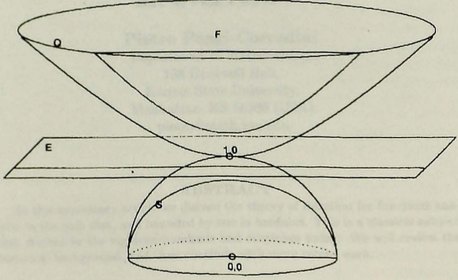

Figure 1: Intersection of S_+, E, Q with a plane F through origin