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# Some applications of fuzzy ordered relations

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## 1 Introduction

The notion of fuzzy subset appeared in 1965 in a paper by L. A. Zadeh [38]. This notion tries to show that an object corresponds more or less to the particular category we want to assimilate it to; that was how the idea of defining the membership of an element to a set not on the Aristotelician pair {0,1} any more but on the continuous interval [0,1] was born. Infact the idea of describing all shades of reality was for long the obsession of some logicians.The main purpose of this article is to present recent results regarding the fuzzy ordered relation proved by the author [3-8].

## 2 Fixed point

Fixed point theorems are fundamental tool for solving functional equations. Recently Heilpern [24], Kaleva [26], Bose and Sahani [13], Park and Jeong [34], Lee and Cho [30], Beg [6] and many other authors have proved fixed point theorems in fuzzy setting, specially for fuzzy metric spaces. Zadeh [39] and Negoita and Ralescu [32] have introduced the notion of fuzzy order and similarity in their papers, which was subsequently further developed and refined by Venugopalan [37], Orlovsky [33], Billot [11], Kundu [29], Beg and Islam [9] and Beg [4]. In this section, we present results on fixed points of expansive mappings on fuzzy preordered sets.

Let $X$ be a space of objects, with generic elements of $X$ denoted by $x$. A fuzzy subset $A$ of $X$ is characterize by a membership function $\mu$ which associates with each element in $X$ a real number in the interval $[0, 1]$

**Definition 2.1** *A fuzzy preorder $\mu$ on $X$ is a fuzzy subset of $X \times X$ such that the following conditions are satisfied:*
*(i) for all $x \in X, \mu(x, x) = 1$,*
*(ii) for all $(x, y, z) \in X^3$,*

$$\mu(x, z) \geq \underset{y}{Max} \ [Min\{\mu(x, y), \mu(y, z)\}].$$

*A nonempty set $X$ with fuzzy preorder $\mu$ defined on it, is called fuzzy preordered set and we denote it by $(X, \mu)$ or just by $X$ for simplicity sake when there is no confusion. A fuzzy preordered set is called fuzzy ordered set if:*
*(iii) $\mu(x, y) + \mu(y, x) > 1$ implies $x = y$.*

**Remark 2.2** *Let $X$ be a fuzzy preordered set. The fuzzy preorder $\mu$ is said to be linear if for all $x \neq y$, we have $\mu(x, y) \neq \mu(y, x)$ . A fuzzy subset on which fuzzy preorder is linear is called a fuzzy chain. For a subset $A \subset X$ , an upper bound (strict upper bound) is an element $x \in X$ satisfying $\mu(y, x) \geq \mu(x, y)$ $(\mu(y, x) > \mu(x, y))$ for all $y$ in $A$. An element $x$ is a maximal element of $X$ if $\mu(x, y) \geq \mu(y, x)$ for some $y \in X$, then $\mu(x, y) + \mu(y, x) > 1$. The set of all maximal elements of $X$ will be denoted by $sup(X)$. Minimal elements are defined analogously. A greatest element of $A$ is an $x \in A$ satisfying $\mu(y, x) \geq \mu(x, y)$ for all $y \in A$. Least elements are defined in the obvious fashion.*

*For more details see Zimmerman [41], Billot [11], Beg and Islam [9], Li and Yen [31] and Dubios and Prade [19].*

**Definition 2.3** *A set $X$ is well fuzzy preordered by the linear fuzzy preordered $\mu$ if every nonempty subset of $X$ has a least element.*

**Definition 2.4** *A fuzzy preordered set $X$ is called $A$-inductive if and only if every nonempty well fuzzy preordered subset of it has a least upper bound (abbreviated as lub).*

**Definition 2.5** *Let $X$ be a nonempty fuzzy preordered set. A mapping $f : X \to X$ is called expansive if $\mu(x, f(x)) \geq \mu(f(x), x)$ for every $x \in X$.*

**Definition 2.6** *Let $f : X \longrightarrow X$ , then a point $x \in X$ is called a fixed point of $f$ if $f(x) = x$.*

**Theorem 2.7** *Let $X$ be a nonempty $A$-inductive fuzzy ordered set and $f : X \to X$ be an expansive mapping then $f$ has a fixed point.*

*Proof.* Assume that $f$ has no fixed point. Since fuzzy order $\mu$ is linear therefore for every $x \in X$. $\mu(x, f(x)) > \mu(f(x), x)$. Let $z$ be an arbitrary element of $X$. For every ordinal $p$ we define:

$$f^p(z) = \begin{cases} z \text{ if p=0}, \\ f(f^{p-1}(z)) \text{ if p is a nonzero nonlimit ordinal}, \\ \underset{i<p}{lub} \ f^i(z) \text{ if p is a limit ordinal.} \end{cases}$$

Since $X$ is A-inductive , it follows that for every ordinal $p$, $f^p(z) \in X$. Moreover, for every ordinal $p$ and $q$ if $p \neq q$ then $f^p(z) \neq f^q(z)$. Now , to every $x \in X$, assign an ordinal $F(x)$ as follows:

$$F(x) = \begin{cases} p \text{ if } x = f^p(z), \\ o \text{ otherwise.} \end{cases}$$

Obviously, $F$ is an ordinal valued function, whose domain is $X$. Since $X$ is a set, from the ZF Axiom of Substitution (Kelley [27], page 261) it follows that the range of $F$ is also a set. But range of $F$ is collection of all ordinals, which is not a set. Hence a contradiction. Thus our assumption is wrong and $f$ has a fixed point. □

**Remark 2.8** *Let $X$ be a fuzzy preordered set. Define the fuzzy semilitude subrelation $\sim$ in $X$ by : $x \sim y$ if $\mu(x, y) + \mu(y, x) > 1$. The relation $\sim$ between the different semilitude classes is necessarily antisymmetric (Negoita and Ralescu [32]) . Semilitude classes are not necessarily disjoint. It is obvious that there exists a fuzzy order between semilitude classes.*

**Definition 2.9** *A nonempty subset $E$ of a fuzzy preordered set $X$ is called fuzzy order extremal provided $x \sim y$ for all $x, y \in E$ ; and $x \in E$ whenever $x \in X$ and $\mu(y, x) > \mu(x, y)$ for some $y \in E$.*

**Remark 2.10** *A fuzzy order extremal subset $E$ is always a chain, has a least upper bound and it consists of maximal elements of $X$.*

**Remark 2.11** *A singleton set $E = \{x\}$ is fuzzy order extremal if and only if $x$ is a unique least upper bound of $\{y : y \in X$ and $\mu(x, y) \geq \mu(y, x)\}$.*

**Lemma 2.12** *Let $X$ be a nonempty A-inductive fuzzy preordered set and let $f : X \to X$ be an expansive mapping. Then there exists a fuzzy order extremal f-invariant set $W$ in $sup(X)$.*

*Proof.* Let $Y = \{D_x : D_x = \{y \in X : y \sim x\}\}_{x \in X}$. Define the fuzzy order $\lambda$ in $Y$ by: $\lambda(D_x, D_y) \geq \lambda(D_y, D_x)$ if $\mu(x, y) \geq \mu(y, x)$ (see Remark 2.8) . Since $X$ is A-inductive fuzzy preordered set, it follows that $Y$ is A-inductive fuzzy ordered set. The fuzzy ordered set $Y$ has a maximal element $W$. Otherwise, for each $D \in Y$, there exists a semilitude class $h(D) \in Y$ such that $\lambda(D, h(D)) \geq \lambda(h(D), D)$ and $h(D) \neq D$. Hence Theorem 2.7 is contradicted by the mapping $h : Y \to Y$ . Clearly, the maximal element $W \in Y$ , is order extremal and $W \subset sup(X)$. Moreover, if $x \in W$, then $\mu(x, f(x)) \geq \mu(f(x), x)$. It further implies that $f(x) \in W$. Thus $W$ is f-invariant. □

**Definition 2.13** *An A-inductive fuzzy preordered set $X$ is said to have normal fuzzy order structure if each fuzzy order extremal subset of $X$ is a singleton.*

**Theorem 2.14** *Let $X$ be a fuzzy preordered set with normal fuzzy order structure and $f : X \to X$ be an expansive mapping. Then $f$ has a maximal fixed point.*

*Proof.* Let $X$ be a fuzzy preordered set with normal fuzzy ordered structure. If $\mu(x, f(x)) \geq \mu(f(x), x)$ for every $x \in X$ then the fuzzy ordered extremal set $W \subset sup(X)$, constructed in Lemma 2.12 is a singleton, say $W = \{w\}$. Since $W$ is f-invariant, therefore $f(w) = w$. Maximality of $W$ in $Y$ implies that $w$ is a maximal fixed point. □

## 3 Fuzzy Zorn's lemma

Zorn's lemma is one of the most famous and useful result in mathematics. Chapin [15] studied the basic logical axioms of fuzzy set theory and also introduced the fuzzy axiom of choice.

If a *fuzzy set* $B$ of $X$ is characterized by a membership function $'b'$ which associates with each element in $X$ a real number in the interval $[0, 1]$, with the value of $b(x)$ at $x$ representing the grade of membership of $x$ in $B$. Let $A$ and $B$ be two fuzzy subsets of $X$, then $(a - b)(x) = \min\{a(x),\ 1 - b(x)\}$, $(A \cup B)(x) = \max\{a(x), b(x)\}$ and $(A \cap B)(x) = \min\{a(x), b(x)\}$. A fuzzy set is empty if it is the constant zero function. Using Fuzzy Axiom of Choice (Chapin [15], Ax. 14 ), we choose a function $f$ that assigns to every bounded fuzzy chain $C$ a strict upper bound $f(C)$. For further details and basic logical axioms of fuzzy set theory we refer to Zadeh [38,39], Brown [14], Chapin [15], and Zimmermann [41].

**Definition 3.1** *If $C$ is a fuzzy chain in $X$ and $x \in C$, then we define the fuzzy subset $P(C, x)$ of $C$ by $P(C, x)(y) = r(y, x) - r(x, y)$. A fuzzy subset of a fuzzy chain $C$ that has the form $P(C, x)$ is called an initial segment in $C$.*

**Definition 3.2** *A fuzzy subset $A$ of a fuzzy chain $C$ is called conforming if the following two conditions hold:*
*(iv) Every non-empty fuzzy subset of $A$ has least element.*
*(v) For every $x$ in $A$, $x = f(P(A, x))$.*

**Proposition 3.3** *Let $A$ and $B$ be conforming subsets of a fuzzy chain $C$ and $A \neq B$, then one of these two sets is an initial segment of the other.*

*Proof.* We may assume $A - B \neq \phi$, i.e., $\min\{a(x), 1 - b(x)\} \neq 0$ for some $x \in A$.Define $x_0$ to be the least element of $A - B$. Thus $r(x_0, y) \geq r(y, x_0)$ if $\min\{a(y), 1 - b(y)\} \neq 0$. Therefore, if $r(x_0, y) < r(y, x_0)$, then either $a(y) = 0$or $b(y) = 1$. We claim that $P(A, x_0) = B$. For this we show that:( i) $P(A, x_0) \subseteq B$ and (ii) $B - P(A, x_0) = \phi$.

(i) If $y \in P(A, x_0)$, then $P(A, x_0) > 0$. Thus $r(y, x_0) - r(x_0, y) > 0$. It implies that $r(x_0, y) < r(y, x_0)$. Moreover, $y \in A$ by definition 4. Thus $b(y) = 1$. It further implies that $P(A, x_0) \leq b(y)$. Therefore, $P(A, x_0) \subseteq B$.

(ii) Assume that $B - P(A, x_0) \neq \phi$ and let $y_0$ be the least element of $B - P(A, x_0)$, i.e., $\min\{b(y_0), 1 - P(A, x_0)(y_0)\} \neq 0$ and $r(y, y_0) \leq r(y_0, y)$ for all $y \in B - P(A, x_0) = B - \{y \in A : r(y, x_0) > r(x_0, y)\}$. It implies that $P(B, y_0) \subseteq P(A, x_0)$. Given any element $u \in P(B, y_0)$ and any element $v \in A$ such that $r(v, u) > r(u, v)$. Obviously $v \in P(A, x_0) \subseteq B$. Since $r(u, y_0) > r(y_0, u)$ then $r(v, y_0) > r(y_0, v)$. It further implies that $v \in P(B, y_0)$. Therefore, if $z_0$ is the least element of $A - P(B, y_0)$, we have $P(A, z_0) = P(B, y_0)$. Note that $r(z_0, x_0) \geq r(x_0, z_0)$. But since $z = f(P(A, z_0)) = f(P(B, y_0)) = y_0$ and since $y_0 \in B$, we can not have $z_0 = x_0$.Therefore $r(z_0, x_0) > r(x_0, z_0)$ and we conclude $y_0 = z_0 \in P(A, x_0)$, contradicting the choice of $y_0$. □

**Remark 3.4** *If $A$ is a conforming subset of $X$ and $x \in A$, then whenever $r(y, x) > r(x, y)$, either $y \in A$ or $y$ does not belong to any conforming set. Therefore, it follows that the union $\Omega$ of all conforming subset of $X$ is conforming.*

**Theorem 3.5** *Fuzzy Zorn's lemma: Let $X$ be a fuzzy ordered set with fuzzy order $R$. If every fuzzy chain in $X$ has an upper bound then $X$ has a maximal element.*

*Proof.* Suppose that $X$ has no maximal element. If $C$ is a chain in $X$, then by choosing an upper bound $u$ of $C$ and then choosing an element $x$, $r(u, x) > r(x, u)$. We can obtain an element $x$ in $X$ such that $r(y, x) > r(x, y)$ for every $y$ in $C$. Then $x$ will be a strict upper bound of $C$. If $x = f(\Omega)$ then the set $\Omega \cup [x\}$ is conforming by Remark 3.4. Therefore $x \in \Omega$ , contradicting the fact that $x$ is a strict upper bound of $\Omega$ . □

# 4 Selection

One of the most interesting and important problems in ordered set theory is the extension problem. Two ordered sets $X$ and $Y$ are given, together with a subset $A \subset X$, we would like to know whether every order preserving function $g : A \to Y$ can be extended to an order preserving function $f : X \to Y$. Sometimes there are additional requirements on $f$ e.g., for every $x \in X$, $f(x)$ must be an element of a pre-assigned subset of $Y$. This new problem is clearly more general than the extension problem and is called a selection problem. Even though there is a lot of work in the classical set theory on selection problems ( see; Knaster [28], Tarski [36], Davis [18], Birkhoff [12] and Smithson [35] ). The aim of this section is to prove the existence of a fuzzy order preserving selectors for fuzzy multifunctions under suitable conditions. A fixed point theorem for fuzzy order preserving fuzzy multifunctions is also proved.

Let $X$ be a fuzzy ordered set with a fuzzy order $R$ and $F : X \to [0,1]^X \backslash \{\phi\}$ be a *fuzzy multifunction*, that is, for $x \in X$, $F(x)$ is a nonempty fuzzy subset of $X$. If $F$ maps the points of its domain to singletons, then $F$ is said to be a *single valued fuzzy function*. No distinction will be made between a single valued fuzzy function and a fuzzy multifunction. The fuzzy multifunction $F$ is said to be *fuzzy order preserving* if and only if $x_1, x_2 \in X$ and $y_1 \in F(x_1)$, $r(x_1, x_2) \geq r(x_2, x_1)$ implies that there is a $y_2 \in F(x_2)$ such that $r(y_1, y_2) \geq r(y_2, y_1)$. A *selector* for $F$ is a fuzzy function $f : X \longrightarrow X$ such that $\{f(x)\} \subseteq F(x)$ for each $x \in X$. A point $x \in X$ is a *fixed point* of $F$, if $\{x\} \subseteq F(x)$.

**Theorem 4.1** *Let $X$ be a fuzzy ordered set and let $F$ be a fuzzy order preserving fuzzy multifunction on $X$. If $\sup F(x) \subseteq F(x)$ for all $x \in X$, then there is a fuzzy order preserving selector $f$ for $F$.*

*Proof.* Let $f(x) = \sup F(x)$ for each $x \in X$. Then $f$ is a fuzzy order preserving selector for $F$. Indeed, let $r(x_1, x_2) \geq r(x_2, x_1)$. Since $\{f(x_1)\} \subseteq F(x_1)$ there is a $z \in F(x_2)$ such that $r(f(x_1), z) \geq r(z, f(x_1))$. But $r(z, f(x_2)) \geq r(f(x_2), z)$. Hence $r(f(x_1), f(x_2)) \geq r(f(x_2), f(x_1))$. □

**Theorem 4.2** *Let $X$ be a fuzzy ordered set in which each nonempty fuzzy chain $C$ has a supremum and $X$ contains a least element $u$. If $F : X \to [0,1]^X \backslash \{\phi\}$ is a fuzzy multifunction which satisfies (I)-(III) as follows:*

*(I). Let there be a fuzzy order preserving fuzzy function $g : C \longrightarrow X$ such that $\{g(x)\} \subseteq F(x)$ for all $x \in C$. Then there exists $y_0 \in F(\sup C)$ such that $r(g(x), y_0) \geq r(y_0, g(x))$ for all $x \in C$.*

*(II). Let $r(x_1, x_2) \geq r(x_2, x_1)$ and let $y_1 \in F(x_1), y_2 \in F(x_2)$ with $r(y_1, y_2) \geq r(y_2, y_1)$. If $r(x_1, x) \geq r(x, x_1)$ and $r(x, x_2) \geq r(x_2, x)$ then*

$$F(x) \cap \{z : r(y_1, z) \geq r(z, y_1) \text{ and } r(z, y_2) \geq r(y_2, z)\} \neq \phi.$$

*(III). Let $D = \{z : r(x_1, z) \geq r(z, x_1) \text{ and } r(z, x_2) \geq r(x_2, z)\}$ for $r(x_1, x_2) \geq r(x_2, x_1)$. If $F(x) \cap D \neq \phi$ then $\sup(D \cap F(x)) \in D \cap F(x)$.*

*Then there exists a fuzzy order preserving selector $f$ for $F$ on $X$.*

*Proof.* Let $P$ be the collection of fuzzy subsets $Y$ of $X$ with properties:

(1). $u \in Y$.

(2). If $x \in Y$ and $r(z, x) \geq r(x, z)$ then $z \in Y$, and

(3). There is an order preserving fuzzy function $g : Y \longrightarrow X$ such that $g(x) \in F(x)$ for all $x \in Y$.

Let $(P, g) = \{(Y, g) :\ Y \in P$ and $g$ is a fixed fuzzy function from (3) $\}$. Define a fuzzy order on $(P, g)$ as follows: $(Y_1, g_1) < (Y_2, g_2)$ if and only if $Y_1 \subseteq Y_2$ and $g_1 = g_2|_{Y_1}$. Then by fuzzy Zorn's lemma there is a maximal element $(X_0, f_0)$ of $(P, g)$. If $X = X_0$, we are done. Otherwise, suppose $x \in X \backslash X_0$, and let $C$ be a maximal fuzzy chain containing $u$ and $x$. Then $C \cap X_0 = C_1$ is a fuzzy chain in $X_0$. Let $x_0 = \sup C_1$. Condition (I) and maximality of $X_0$ imply that $x_0 \in X_0$. Now pick a $y' \in F(x)$ such that $r(f(x_0), y') \geq r(y', f(x_0))$. Let $Y = X_0 \cup \{z : r(x_0, z) \geq r(z, x_0)$ and $r(z, x) \geq r(x, z)\}$. Define $f : Y \longrightarrow Y$ as follows:

If $z \in X_0$, then $f(z) = f_0(z)$ and if $z \in \{z : r(x_0, z) \geq r(z, x_0)$ and $r(z, x) \geq r(x, z)\}$ then

$$f(z) = \sup(F(z) \cap \{y : r(f(x_0), y) \geq r(y, f(x_0)) \text{ and } r(y, y') \geq r(y', y)\}).$$

Set $f(x) = y'$. Conditions (II) and (III) show that $f$ is well defined order preserving fuzzy function. This contradicts to the maximality of $X_0$. Hence $X = X_0$. □

**Theorem 4.3** *Let $X$ be a fuzzy ordered set in which each nonempty fuzzy chain $C$ has a supremum. Let $F : X \longrightarrow [0,1]^X \backslash \{\phi\}$ be a fuzzy order preserving fuzzy multifunction such that, given a chain $C$ in $X$ and a single valued order preserving fuzzy function $g : C \longrightarrow X$ satisfying $\{g(x)\} \subseteq F(x)$ for all $x \in C$, there exists a $y_0 \in F(\sup C)$ such that $r(g(x), y_0) \geq r(y_0, g(x))$ for all $x \in C$. If there is a point $p \in X$ and $y \in F(p)$ such that $r(p,y) \geq r(y,p)$, then $F$ has a fixed point.*

*Proof.* Let $p \in X$ and let $y \in F(p)$ with $r(p,y) \geq r(y,p)$. Define a collection $P$ of fuzzy subsets $Y$ of $X$ by:

(i). $p \in Y$.

(ii). If $r(p,z) \geq r(z,p), r(z,x) \geq r(x,z)$ and $x \in Y$ then $z \in Y$; and

(iii). If $x \in Y$, then there is a $z \in F(x)$ such that $r(x,z) \geq r(z,x)$.

Fuzzy order $P$ by inclusion. Since $\{p\} \in P$. Therefore by fuzzy Zorn's lemma there is a maximal element $X_0 \in P$. Let $C$ be a maximal chain in $X_0$ (existence of $C$ is implied by fuzzy Zorn's lemma and let $x_0 = \sup C$.

Element $x_0 \in X_0$. Indeed; let there be a fuzzy subset $C_0 \subseteq C$ such that :

(iv). $x_0 = \sup C_0$, and

(v). There is an order preserving fuzzy function $g : C_0 \longrightarrow X$ such that $\{g(x)\} \subseteq F(x)$ and $r(x, g(x)) \geq r(g(x), x)$ for each $x \in C_0$.

Let $Q$ be the collection of fuzzy subsets of $C$ which satisfies (v). If $C_1, C_2 \in Q$, then fuzzy order the pairs $(C_1, g_1), (C_2, g_2)$, where $g_1$ and $g_2$ are fixed fuzzy functions from condition (v), by $(C_1, g_1) < (C_2, g_2)$, if and only if $C_1 \subseteq C_2$ and $g_1 = g_2|_{C_1}$. By fuzzy Zorn's lemma there is a maximal set $C_0$ with function $g_0$ in $Q$. Let $x' = \sup C_0$. If $x' \neq x_0$ then there is an $x \in C$ such that $r(x', x) \geq r(x, x')$. By hypothesis, we can extend $g_0$ to the set $C_0 \cup \{x\}$ which contradicts the maximality of $C_0$. Thus $\sup C_0 = x_0$. By hypothesis there is a $y \in F(x_0)$ such that $r(g(x), y) \geq r(y, g(x))$ for all $x \in C_0$. But $r(x, g(x)) \geq r(g(x), x)$ for $x \in C_0$ and so $y$ is an upper bound for $C_0$. Thus $r(x_0, y) \geq r(y, x_0)$. Hence, $x_0 \in X_0$.

Since $x_0 \in X_0$, there exists a $y_0 \in F(x_0)$ such that $r(x_0, y_0) \geq r(y_0, x_0)$. If $x_0 = y_0$ , we are done. Otherwise suppose $r(x_0, y_0) > r(y_0, x_0)$. Then put $X_1 = X_0 \cup Z$, where $Z = \{z : r(x_0, z) \geq r(z, x_0)$ and $r(z, y_0) \geq r(y_0, z)\}$.

Since $F$ is fuzzy order preserving, therefore for each $z \in Z$, there is a $w \in F(z)$ such that $r(y_0, w) \geq r(w, y_0)$. But then, $r(z, w) \geq r(w, z)$. Thus (i), (ii) and (iii) are satisfied by $X_1$, which contradicts the maximality of $X_0$. Hence, $x_0 = y_0$ and thus $x_0 \in F(x_0)$. □

**Remark 4.4** *Theorem 4.3. generalizes/extends several known results including among them are Knaster [28], Tarski [36], Abian and Brown [1, Theorem 2], Beg [3, Theorem 2.4] and Beg [6]. Also Theorem 4.1 is a fuzzy analogue of Smithson [35, Theorem 1.1].*

## 5 Extension

In economics, decision analysis, optimization and game theory, it is important to know under what conditions a relation has a maximal element on a nonempty set. Many results are given in the literature to prove the existence of maximal elements for a relation. Among these results, there are basically two streams: One stream assumes a convex cone preferences and focuses on a weakening the topological conditions (see Fan [20], Bergstrom [10] and Zhou and Tian [40]). The other stream assumes a certain nontransitive preference on a compact set with some other topological and/or convexity condition (see Corley [16,17]). The Zorn's lemma [42] is a very powerful mathematical tool to avoid compactness assumption (see also [2]). The main aim of this section is to further weaken the fuzzy transitivity condition without invoking any topological assumptions. A necessary and sufficient condition has been established to completely characterize the existence of maximal elements for general irreflexive nontransitive fuzzy relations.

**Definition 5.1** *Let $X$ be a set with a fuzzy relation $R$. A fuzzy subset $B$ of $X$ is said to be pointwisely dominated in $X$ if for each $x$ in $B$ there is some $y \in X$ such that $y \neq x$ and $r(x,y) \geq r(y,x)$. The fuzzy subset $B$ is called strictly dominated in $X$ if there is some $y \in X \backslash B$ such that $r(x,y) > r(y,x) = 0$ for all $x \in B$. A pointwisely dominant R-fuzzy chain $C$ in $X$ is said to have the dominant property on $X$ , if it is strictly dominated in $X$. When every pointwisely dominated R-fuzzy chain $C \subset X$ is strictly dominated in $X$, we say that the fuzzy relation $R$ has fuzzy chain dominant property on $X$.*

It is clear from the definition that, for a fuzzy relation $R$ on $X$ if there exists an element $x$ in $X$ such that $r(x,x_*) > r(x_*,x) = 0$ for all $x$ in $X$ then $R$ has fuzzy chain dominant property on $X$. In a R-fuzzy chain $C$, the least and greatest elements are unique. A R-fuzzy chain $C_1$ in $(X,R)$ is said to be *maximal chain* if any R-fuzzy chain $C_2$ in $X$ with $C_1 \subset C_2$ implies $C_1 = C_2$.

**Theorem 5.2** *Let $R$ be a fuzzy relation on a nonempty set $X$ and let $x$ be an element in $X$. Then in $X$, there exists a maximal R-fuzzy chain above $x$.*

*Proof.* Let $\mathcal{Z}$ be the set of all fuzzy chains in $X$ above $x$. Since $X$ is nonempty, $\mathcal{Z} \neq \phi$. For any two $C_1, C_2 \in \mathcal{Z}$, if $C_2 \subset C_1$ then we define a (partial) fuzzy order relation $Q$ on $\mathcal{Z}$ by $q(C_2,C_1) = \sup\{C_1(x) - C_2(x) : x \in X\}$. Then any chain in $\mathcal{Z}$ has an upper bound. Indeed, let $N$ be a chain in $\mathcal{Z}$; let $\mathcal{B}$ denote the set of all finite fuzzy subsets of $N$ ordered by $Q$. For each $B \in \mathcal{B}$, define: $C_B = \cup\{c : c \in B\}$ and $\overline{C} = \cup\{C_B : B \in \mathcal{B}\}$. Now $\overline{C}$ is an element of $\mathcal{Z}$ and $q(C,\overline{C}) > q(\overline{C},C)$, for all $C \in N$, i.e., $\overline{C}$ is an upper bound of the chain $N$ in $\mathcal{Z}$. Applying theorem 3.5, we get a maximal element, say $C^*$ in $\mathcal{Z}$. Hence $C^*$ is the maximal R-fuzzy chain in $X$ containing $x$. □

**Theorem 5.3** *Let $R$ be a fuzzy relation on a nonempty set $X$ having the chain dominant property then there exists a maximal element $x_*$ in $X$.*

*Proof.* Since $X$ is nonempty, theorem 5.2 implies that there exists a maximal R-fuzzy chain $C^*$ (say) in $X$. If for each $z$ in $C^*$, there exists $x \in X$ such that $r(z,x) \geq r(x,z)$ Then $C^*$ is pointwisely dominated. By the fuzzy chain dominant property, there exists

$\bar{x} \in X \backslash C^*$ such that $r(z, \bar{x}) > r(\bar{x}, z) = 0$ for all $z$ in $C^*$. In this case, $C^* \cup \{\bar{x}\}$ is again a R-fuzzy chain. Thus we may enlarge the R-fuzzy chain $C^*$ by adding $\bar{x}$ to $C$, and this will violate the maximality of the R-fuzzy chain $C^*$. Hence there exists an element $x_* \in C^*$ such that $r(x_*, x) = 0$ for all $x$ in $X$ with $x \neq x_*$. □

Next we prove that theorem 5.3 is equivalent to the fuzzy Zorn's lemma.

**Theorem 5.4** *Theorem 5.3 is equivalent to fuzzy Zorn's lemma.*

*Proof.* To show the equivalence, we only need to prove that theorem 5.3 implies fuzzy Zorn's lemma.

Let $R^*$ be a fuzzy order relation on a nonempty set $X$. We define $p(y,x) > p(x,y)$ if $r^*(y,x) \geq r^*(x,y)$ and $x \neq y$. Then fuzzy relation $P$ is f-antisymmetric fuzzy relation. Indeed the fact that $p(y,x) > p(x,y)$ and $p(x,y) > p(y,x)$ imply $x \neq y$, $r^*(y,x) \geq r^*(x,y)$ and $r^*(y,x) \geq r^*(x,y)$. Therefore $x = y$ leads to a contradiction. The fuzzy relation $P$ is also f-transitive. For $p(y,x) > p(x,y)$ and $p(z,y) > p(y,z)$ imply $r^*(y,x) \geq r^*(x,y)$ , $r^*(z,y) \geq r^*(y,z)$. Thus $r^*(z,x) \geq r^*(x,z)$. Since $x = z$ is impossible, we have $p(z,x) > p(x,z)$.

Define the completion of $P$ by $P^*$ , i.e., $p^*(y,x) \geq p^*(x,y)$ if $p(x,y) \not> p(y,x)$ .

Next, we claim that, if each $R^*$-fuzzy chain in $X$ has an upper bound, then $P$ has the fuzzy chain dominant property on $X$. Let $C$ be a pointwise dominated P-fuzzy chain in $X$. Since $P$ implies $R^*$, it is clear that this P-fuzzy chain is also a R-fuzzy chain. By our assumption this chain has an upper bound $x_0$ (say) in $X$. Let $x_1, x_2$ be two elements from the chain with $p(x_1, x_2) > p(x_2, x_1)$; then $x_0 = x_1$ will lead to $p(x_0, x_2) > p(x_2, x_0)$, a contradiction to the fact that $x_0$ is an upper bound of $C$. Therefore $p(x, x_0) > p(x_0, x)$ for all $x$ in $C$, except possibly $x_0 = x_2$ for some element $x_2$, in the chain with $p(x, x_2) > p(x_2, x)$ for all $x$ in $C \backslash \{x_2\}$. If such $x_2$ does not exist in the chain, then $x_0$ is a dominator of the chain. If such $x_2$ does exist in the chain, then there are two possibilities: either $x_0 = x_2$ and $p^*(x, x_2) \geq p^*(x_2, x)$ for all $x$ in $X$, or there exists $y_0$ in $X$ such that $p^*(x_0, y_0) \geq p^*(y_0, x_0)$ and $x_0 = x_2$. The first case is impossible by our assumption, so we exclude it. The second case implies that $y_0$ is a dominator of the chain under the transitivity of $P$. So $P$ has the chain dominant property. Now theorem 5.3 guarantee the existence of a maximal element $x_*$ of $P$ on $X$ such that $p^*(x, x_*) \geq p^*(x_*, x)$ for all $x$ in $X$. Let $x \in X$ with $p^*(x_*, x) \geq p^*(x_*, x)$. Since $p(x_*, x) > p(x_*, x)$ will violate $p^*(x, x_*) \geq p^*(x_*, x)$ for all $x$ in $X$. Thus we have $x = x_*$. Hence $x_*$ is a maximal element of fuzzy order relation $R^*$ on $X$. □

**Remark 5.5** *Theorem 5.3 is an extension of the fuzzy Zorn's lemma to general non-transitive fuzzy relations without invoking any topological assumptions or linear structure. We expect that theorem 5.3 can be used for optimization, when the fuzzy relation is nontransitive for which the original fuzzy Zorn's lemma fails to fit.*

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