CUBO *a Mathematical Journal*
Vol. 6, Nº 1, (123-138). March 2004.

Symplectic geometry and related structures [1] [2]

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ABSTRACT

We discuss symplectic, contact and locally conformal symplectic structures. We show how they are connected and how they organize themselves inside the category of Jacobi structures. An emphasis is put on the role of their automorphism groups since they encode the corresponding geometry in the spirit of the Erlanger Programme. We also evoque the problems of existence and (local and global) classification of these structures. For the background of the material discussed in this article, see [30].

1 Some basic results in Symplectic Geometry

Symplectic Geometry is the geometry of a smooth manifold M equipped with a 2-form Ω satisfying:

(1) $d\Omega = 0$, i.e Ω is closed,

(2) Ω is non-degenerate.

Such a form is called a **symplectic form**.

Condition (2) means that the bundle map $\tilde{\Omega} : T(M) \to T^*(M)$, assigning to a vector field X the 1-form $\tilde{\Omega}(X)$ denoted also $i(X)\Omega$, such that $\tilde{\Omega}(X)(\xi) = \Omega(X, \xi)$, for all ξ, is an isomorphism.

[1] 1991 *Mathematics Subject Classifications*: 53C12; 53C15

[2] *Key words and phrases*: symplectic form, contact form, locally conformal symplectic form, Jacobi structure, Poisson structure, complex structure, almost complex structure, hamiltonian diffeomorphisms, Arnold conjecture, Erlanger Programme, Lichnerowicz cohomology

This implies that the dimension of M is even, say $2n$. The non-degenerecy condition (2) is also equivalent to requiring that

$$\Omega^n = \Omega \wedge \Omega \wedge \ldots \wedge \Omega$$

(n times) is everywhere non-zero. The volume form Ω^n fixes an orientation of M.

The couple (M, Ω) of a smooth manifold M with a symplectic form Ω is called a symplectic manifold.

A smooth function $f : M \to \mathbb{R}$ on a symplectic manifold defines a vector field X_f, called the hamiltonian vector field, by the equation:

$$X_f = \tilde{\Omega}^{-1}(df)$$

i.e $i(X_f)\Omega = df$. The system of first order differential equations

$$\dot{x} = X_f$$

is called the Hamilton equations.

This naming comes from the fact that symplectic geometry started off as a setting for Classical Mechanics, in which the equations above are the equations of the motion of a particule in the "phase space" M.

For each smooth function f, the hamiltonian vector field X_f satisfies :

$$L_{X_f}\Omega = di(X_f)\Omega + i(X_f)d\Omega = d^2 f = 0.$$

Here L_X is the Lie derivative in the direction of a vector field X and we used the Cartan formula. The equation above says that if ϕ_t is the local 1-parameter group of diffeomorphisms generated by X_f, then $\phi_t^*\Omega = \Omega$.

A diffeomorphism $\phi : M \to M$ of a symplectic manifold (M, Ω) is said to be a symplectic diffeomorphism, or a symplectomorphism, if $\phi^*\Omega = \Omega$. We just saw how to get a symplectomorphism by integrating a hamiltonian vector field X_f, where $f : M \to \mathbb{R}$ is a function with compact support. This suggests that the set $Diff_\Omega(M)$ of all symplectomorphisms of (M, Ω) is very large. How much of $Diff_\Omega(M)$ we get by integrating X_f? We will answer this question in section 3.

It is clear that $Diff_\Omega(M)$ is a group, we call the group of **symplectic diffeomorphisms of** (M, Ω). This is the automorphism group of the symplectic geometry of (M, Ω). It acts transitively on the symplectic manifold (provided that it is connected). Hence the symplectic manifold (M, Ω) can be viewed as a homogeneous space of $Diff_\Omega(M)$. The group $Diff_\Omega(M)$ has an even deaper significence: it encodes the symplectic geometry, according to Klein's Erlanger creed [3].

Theorem 1 (9) *Let (M_1, Ω_1) and (M_2, Ω_2) be two symplectic manifolds. Then there exists a diffeomorphism $h : M_1 \to M_2$ such that $h^*\Omega_2 = \lambda\Omega_1$ for some constant λ if and only if the group $Diff_{\Omega_1}(M_1)$ is isomorphic to the group $Diff_{\Omega_2}(M_2)$.*

The first example of a symplectic manifold, directly related to Classical Mechanics, is the Euclidean spac $\mathbb{R}^{2n}$ with the 2-form

$$\Omega_S = dx_1 \wedge dx_{1+n} + dx_2 \wedge dx_{2+n} + \ldots + dx_n \wedge dx_{2n}$$

where $(x_1, \ldots x_{2n})$ are coordinates on $\mathbb{R}^{2n}$. Clearly $d\Omega_S = 0$ and $\Omega_S^n \neq 0$ everywhere.

The form Ω_S is called the "canonical" or "standard" symplectic form on $\mathbb{R}^{2n}$.

The example $(\mathbb{R}^{2n}, \Omega_S)$ is the local model of any symplectic manifold. Namely, we have the following

Theorem 2 *Darboux theorem*
Each point in a symplectic manifold (M, Ω) of dimension $2n$ has an open neighborhood U, which is the domain of a local chart $\phi : U \to \mathbb{R}^{2n}$ such that $\phi^*\Omega_S = \Omega_{|}U$.

This theorem says that all symplectic manifolds look alike locally. Therefore there are no local invariants in Symplectic Geometry. However they are many global ones, for instance the de Rham cohomology class $[\Omega] \in H^2(M, \mathbb{R})$ of the symplectic form. If M is compact, then $[\Omega^k] \in H^{2k}(M, \mathbb{R}) \neq 0, k = 1, .., n$. This is an immediate consequence of Stokes theorem.

The canonical symplectic form Ω_S combines the usual inner product $<.>$ on $\mathbb{R}^m$ and the complex structure J on $\mathbb{R}^{2n}$. Recall that $< U, V >= \sum_{i=1}^m u_i v_i$ if $u = (u_1, ..u_m), v = (v_1, .., v_m)$ and $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ maps $(v_1, ..., v_n, v_{n+1} .., v_{2n})$ to $(v_{n+1}, ..., v_{2n}, -v_1, ..., -v_n)$. We have $J^2 = -I$.

If $U = \sum U_i \partial/\partial x_i$ and $V = \sum V_i \partial/\partial x_i$, then

$$\Omega_S(U, V) = \sum_{i=1}^{n} (U_i V_{i+n} - U_{i+n} V_i =< U, JV >$$

The global equivalent on a smooth manifold M of the inner product $<, >$ on $\mathbb{R}^m$ is a **Riemannian metric** g , and the equivalent of the complex structure J on $\mathbb{R}^{2n}$ is a bundle map $J : TM \to TM$ such that $J^2 = -I$. This bundle map is called an **almost complex structure**. A very mild condition ensures that all smooth manifolds admit riemannian metrics. However they are serious obstructions for even dimensional manifolfds to admit almost complex structures. (these are of homotopy theoretical nature).

The situation in the Euclidean space generalizes to any symplectic manifold [27]

Theorem 3 *On any symplectic manifold (M, Ω), there exist infinitely many riemannian metrics g, and almost complex structures J such that $g(JX, JY) = g(X, Y)$ and $\Omega(X, Y) = g(X, JY)$ for all vector fields X, Y.*

Almost complex structures like in the theorem above are said to be **compatible** with the symplectic form Ω.

Theorem 4 *The set $\mathcal{J}(\Omega, M)$. of almost complex structures compatible with a symplectic form Ω on M (with the compact open topology) is contractible.*

Therefore, if we pick $J \in \mathcal{J}(\Omega, M)$, the Chern classes $c_i \in H^{2i}(M, \mathbb{Z})$ of the complex tangent bundle (TM, J) are independent of the choice of J. Hence the c_i are invariants of the symplectic manifold (M, Ω).

2. Examples of symplectic manifolds and the problems of existence and classification

2.1 The canonical symplectic form Ω_S on $\mathbb{R}^{2n}$ is invariant under translations: hence it descends to a symplectic form Ω on the torus $T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$.

2.2 Any oriented surface is a symplectic manifold. The volume form is a symplectic form in that case.

2.3 If $(M_i, \Omega_i), i = 1,2$ are symplectic manifolds, then $\lambda_1\pi_1^*\Omega + \lambda_2\pi_2^*\Omega$ where $\pi_i : M \times M \to M$ is the projection on the i th factor, $\lambda_i \in \mathbb{R}$, is a symplectic form on $M_1 \times M_2$.

2.4 Let $M = T^*X$ be the cotangent space of a smooth manifold X and $\pi : T^*X \to X$ the natural projection. An element $A \in T^*X$ is a couple (a, θ) where $a = \pi(A)$ and $\theta \in T_a^*X$. For $\xi \in T_A(T^*X)$, consider the canonical Liouville 1-form:

$$\omega_X(A)(\xi) = \theta((T_A\pi)\xi).$$

It is easy to check that $\Omega_X = d\omega_X$ is a symplectic form on M.

We thus see that to any smooth manifold M we can assign the symplectic manifold (T^*M, Ω_M). Given a smooth map $f : M \to N$, we get a smooth map $f^* : T^*N \to T^*M$ such that $f^*\Omega_M = f^*\Omega_N$. Hence we get a functor from the category of smooth manifolds (with smooth maps) into the category of symplectic manifolds (with symplectic maps).

2.5 Many examples come from complex geometry: all Kaehler manifolds are symplectic manifolds. Let M be a complex manifold with complex structure J.

A **Kaehler form** is a closed 2-form Ω such that $\Omega(X, Y) = g(X, JY)$ for some hermitian metric g.

It was believed for longtime that a symplectic manifold is always Kaehler until Thurston [33] produced the following example: Let $\mathcal{H}$ be the discrete subgroup of symplectomorphisms of $\mathbb{R}^4$ generated by the following diffeomorphisms:

$h_1(x_1, x_2, y_1, y_2) = (x_1, x_2 + 1, y_1, y_2)$

$$h_2(x_1, x_2, y_1, y_2) = (x_1, x_2, y_1, y_2 + 1)$$

$$h_3(x_1, x_2, y_1, y_2) = (x_1 + 1, x_2, y_1, y_2)$$

$$h_4(x_1, x_2, y_1, y_2) = (x_1, x_2 + y_2, y_1 + 1, y_2)$$

The symplectic form Ω_S descends to a symplectic form Ω on the quotient $M = \mathbb{R}^4/\mathcal{H}$. This form can not be Kaehler since one sees easily that the 3rd Betti number of M is 3, and classical results assert that odd Betti numbers of Kaehler manifolds must be even.

In section 4, example 4, we give an example of a natural symplectic (in fact Kaehler) form Ω_C on the complex projective space $\mathbb{C}P^n$. The following embedding theorem is due to Tischler [32], see also Gasqui [15]:

Theorem 5 *Let $M, \Omega)$ be a symplectic manifold, such that the cohomology class $[\Omega] \in H^2(M, \mathbb{R})$ actually belongs to $H^2(M, \mathbb{Z})$ (we say that Ω has integral periods), then there is an embedding $e : M \to \mathbb{C}P^m$ for some m such that $e^*\Omega_C = \Omega$.*

We saw plenty of examples of symplectic manifolds. Yet the problem of existence of symplectic structures on a given manifold is still unsolved. They are two cases: the compact case and non compact one.

The non compact case has been settled by Gromov, using his "h-principle" : an open manifold has a symplectic form if and only if it has an almost complex structure [17]. This is a homotopy theoretical problem. Namely the vanishing of Wu's characteristic classes is the necessary and sufficient condition to guarantee the existence of an almost complex structure.

On the othe hand, nothing much is known in the compact case.

The classification problem remains a mystery as well. Recently, Taubes, using a formidable machinery (Gromov-Witten invariants) [31], showed that all symplectic structures on the complex projective space $\mathbb{C}P^2$ are equivalent to the symplectic form in example 4, in section 4. Besides this result, there is an old elegant result of Moser[28] stating the following :

Theorem 6 *Let Ω_t be a smooth family of symplectic forms on a compact manifold M such that the cohomology classes $[\Omega_t] \in H^2(M, \mathbb{R})$ are independent of t, then there is a smooth family of diffeomorphisms ϕ_t , with $\phi_0 = Id$ and $\phi_t^*\Omega_t = \Omega_0$.*

This beautiful theorem is "weak" since the hypothesis that two symplectic forms are connected by a smooth path of symplectic forms is very hard to check. There are a few simple examples where this is true:

1. When the symplectic forms Ω_1 and Ω_2 are C^1 close.
2. When Ω_1 and Ω_2 have the same compatible almost complex structure, in particular if they are both Kaehler in the same complex manifold.

3. Symplectic diffeomorphisms

The support of a diffeomorphism $\phi : M \to M$ is the closure of the subset $\{x \in M | \phi(x) \neq x\}$. Let $Diff_\Omega(M)_c$ be the subgroup of the symplectomorphisms of a symplectic manifold (M, Ω) with compact support. We endow the subgroup $Diff_\Omega(M)_K$ of $Diff_\Omega(M)_c$ formed by those transformations with support in a fixed compact set K with the C^∞ compact-open topology, and topologize $Diff_\Omega(M)_c$ as the direct limit of $Diff_\Omega(M)_K$. Let $G_\Omega(M)$ denote the identity component in $Diff_\Omega(M)_c$.

The structure of the group $Diff_\Omega(M)_c$ has been studied in [2]. See [3] for a full exposition. For instance the following result was obtained:

Theorem 7 *There is a surjective homomorphism S from $G_\Omega(M)$ to a quotient $H^1_c(M, \mathbb{R})/\Gamma$ of $H^1_c(M, \mathbb{R})$ (the de Rham cohomology with compact supports).*

The Kernel of S is equal to the commutator subgroup $[G_\Omega(M), G_\Omega(M)]$.

If M is compact, then $KerS = [G_\Omega(M), G_\Omega(M)]$ is a simple group.

In particular $KerS$ is equal to the group generated by time-one flows of hamiltonian vector fields.

The group $KerS$ is also called the group of hamiltonian diffeomorphisms, and is often refered to as $Ham_\Omega(M)$. Elements of $Ham_\Omega(M)$ are those diffeomorphisms ϕ such that $\phi = \psi_1$, where ψ_t is an isotopy such that there exists a smooth family of functions $f_t : M \to \mathbb{R}$ on M so that:

$(\partial/\partial t)\psi_t(x) = X_{f_t}(\psi_t(x))$ and $\psi_0(x) = x$.

The group of hamiltonian diffeomorphisms occupied the thought of several generations of mathematicians primarly because of the Arnold conjecture [1]. This conjecture has been the driving force for Symplectic Geometry in last two decades.

Arnold Conjecture.

Let ϕ be a hamiltonian diffeomorphism of a compact symplectic manifold (M, Ω). Suppose that each fixed point of ϕ is non-degenerate: i.e the graph of ϕ meets the diagonal $\Delta = \{x, x\} \subset M \times M$ transversally.

Then the number of fixed points of ϕ is bounded from below by the sum of Betti numbers $b_i = dim H^i(M, Q)$.

The conjecture is motivated by considering ϕ a time one flow of a hamiltonian vector field with hamiltonian a Morse function. The conjecture reduces to one of the famous Morse inequalities. The proof of this conjecture for hamiltonian diffeomorphisms close C^1 close to the identity is an easy consequence of Morse theory [7].

In 1982, Conley and Zehnder [14] found a proof of the Arnold conjecture for the particular symplectic manifold (T^{2n}, Ω) of example 2.

They translated the problem into a variatonnal problem in infinite dimension.

But the decisive move came from Floer in 1985. In his paper " Witten complex and Infinite Morse Theory" [20] he constructed (under several hypothesis) a homology theory, where the chains are free abelian groups with generators the fixed points of the hamiltonian diffeomorphism, and then showed that this homology is isomorphic with the ordinary singular homology. The conjecture follows (under the required hypothesis). This homology is called nowadays the "Floer homology". Several poeple including Floer himself, Ruan-Tian, Liu-Tian, Fukaya-Ono etc.. have been working to remove the additional hypothesis. The goal seems to be "almost " achieved nowadays, although the author thinks they are still unckecked details (herhaps mistakes) here and there. For a comprehensive exposition of Floer homology, see [29].

Before we close this section on symplectomorphisms, let us mention that from the work of Gromov [18], it emerged that **Symplectic Geometry is topological in nature**. This can be rephrased in terms of the group of symplectomorphisms [27]:

Theorem 8 *(Eliashberg-Hofer)*
Let (M,Ω) *be a symplectic manifold. The group* $Diff_\Omega(M)_c$ *is* C^0*- open in the group of all* C^∞ *diffeomorphisms of* M.

4. Contact manifolds

The odd dimensional analogue of Symplectic Geometry is **Contact Geometry**. This is the geometry of an odd dimensional manifold, say of dimension 2n+1, equipped with a 1-form α such that $\alpha \wedge (d\alpha)^n$ is everywhere nonzero. Such a 1-form is called a contact form. For instance, the 1-form $\alpha_S = y_1 dx_1 + ... + y_n dx_n + dz$ on $\mathbb{R}^{2n+1}$, with coordinates $(x_1,..,x_n,y_1,..,y_n,z)$ is a contact form. If $\pi : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n}$ is the projection $(x_1,..,x_n,y_1,..,y_n,z) \to (x_1,..,x_n,y_1,..,y_n)$, then $d\alpha_S = \pi^*\Omega_S$, where Ω_S is the standard symplectic form on $\mathbb{R}^{2n}$.

There is also a Darboux theorem, asserting that, locally, all the contact forms look like α_S.

Let α be a contact form on M. There exists a unique vector field ξ, called the characteristic vector field of α, or the Reeb field, such that $\alpha(\xi) = 1$ and $i(\xi)d\alpha = 0$.

Let α be a contact form on a smooth manifold M. The hyperplane field $E \subset T(M)$ of kernels of α is called the contact structure defined by α. Clearly, if $f : M \to \mathbb{R}$ is a smooth non vanishing function on M , then $f\alpha$ is again a contact form with the same kernel as α, i.e. defining the same contact structure. We may also say that the contact structure E is the same thing as the equivalence class of contact forms where α and α' are equivalent iff $\alpha' = f\alpha$, for some nowhere zero function. The couple (M,E), where $E = Ker\alpha$, is called a contact manifold, defined by the contact form α.

The restriction of the form $d\alpha$ to the bundle $E = ker\alpha$ makes it a symplectic vector bundle (i.e. for each $x \in M$, $d\alpha(x)$, restricted to E_x is a non-degenerate bilinear two form). The sections $\Gamma(E)$ of E are called basic vector fields and the sections $\Gamma(E^*)$ are called semi-basic 1-forms, and $d\alpha$ induces an isomorphism $\tilde{d\alpha} : \Gamma(E) \to \Gamma(E^*)$ like in the symplectic case.

An automorphism of the contact structure E defined by a contact form α is a diffeomorphism ϕ such that $\phi^*\alpha = f\alpha$, for some nowhere zero function. The set of such diffeomorphism is denoted $Diff_E(M)$, or $Diff(M,\alpha)$, and is called the group of contact diffeomorphisms. We denote by $\mathcal{L}(M,\alpha)$ the Lie algebra of vector fields on M whose local 1-parameter groups belong to $Diff_E(M)$, and call it the Lie algebra of contact vector fields. We have $\mathcal{L}(M,\alpha) = \{X | L_X\alpha = \lambda\alpha\}$ for some function λ. Each function $f : M \to \mathbb{R}$ on M defines a vector field, we denote by Y_f by the equation:

$$Y_f = f\xi + (\tilde{d\alpha})^{-1})((df(\xi))\alpha - df)$$

where ξ is the characteristic vector field of α. It is easy to check that $L_{Y_f}\alpha = (df(\xi))\alpha$, i.e. that Y_f is a contact vector field. In fact the map above from the space $C^\infty(M)$ of smooth functions on M to $\mathcal{L}(M,\alpha)$ is an isomorphism.

Like in the symplectic case, we see that $Diff_E(M)$ is very large. It acts transitively on the contact manifold, (if this manifold is connected), and also encodes the contact geometry [10]:

Theorem 9 *Let (M_i, E_i) be two contact connected manifolds, where E_i are defined by two contact forms α_i. There exists a diffeomorphism $h : M_1 \to M_2$ such that $h^*\alpha_2 = f\alpha_1$, for some function f on M_1 if and only if the group $Diff_{E_1}(M_1)$ is isomorphic to $Diff_{E_2}(M_2)$*

Unfortunately, the algebraic structure of the group of contact diffeomorphisms remains a mystery.

Contactization and symplectification: If α is a contact form on a manifold M, then $\Omega = d(e^t\alpha)$ is a symplectic manifold on $M \times \mathbb{R}$, here t is the projection of $M \times \mathbb{R}$ to $\mathbb{R}$. The obtained symplectic manifold is called the symplification of the contact manifold (M,α).

If Ω is an exact symplectic form on M, i.e. $\Omega = d\theta$, then if $N = M \times \mathbb{R}$, and $\pi : N \to M$, $t : N \to \mathbb{R}$ the projections on each factor, then $\alpha = \pi^*\theta + dt$ is a contact form on N: this contact manifold is called the contactization of the symplectic manifold (M,Ω).

The following result, proved by Boobthy and Wang [13] gives another construction of a contact manifold out of a symplectic manifold;

Theorem 10 *Let (M,Ω) be a symplectic manifold such that the cohomology class $[\Omega] \in H^2(M,\mathbb{R})$ is in fact in $H^2(M,\mathbb{Z})$. Then M is the base of a principal circle bundle $\pi : P \to M$ where P has a contact form α such that $\pi^*\Omega = d\alpha$. The characteristic vector field of α generate the action of S^1 on P.*

Examples of contact manifolds

1. The contactization of $(T^*M, d\omega_M)$ gives a contact structure on $T^*(M) \times \mathbb{R}) \approx J^1(M)$, the bundle of 1-jets of functions on M.

2. Martinet proved that any compact oriented 3-dimensional manifold has a contact form [26].

3. Very recently, it was proved that odd dimensional tori carry contact forms [12]. A contact form on T^3 is well known: the 1-form $\theta = cos(2\pi z)dx + sin(2\pi z)dy$ on $\mathbb{R}^3$ defines a contact form on T^3. A contact form on T^5 was known to Lutz for longtime. The recent result generelizes Lutz constuction.

4. All odd dimensional spheres have a contact form. Let $\theta = \sum_{i+1}^{n+1} x_i dy_i - y_i dx_i$, in coordinates $(x_1, .., x_{n+1}, y_1, .., y_{n+1})$ of $\mathbb{R}^{2n+2}$. The restriction α of θ to S^{2n+1} is a contact form. Its characteristic vector field is $\xi = \sum_{i=1}^{n} x_i \partial/\partial y_i - y_i \partial/\partial x_i$. The orbits of ξ generate an action of S^1 on S^{2n+1}. The quotient space S^{2n+1}/S^1 is the projective space $\mathbb{C}P^n$, and the projection $\pi : S^{n2+1} \to \mathbb{C}P^n$ is the Hopf fibration. The kernel of $d\alpha$ is generated by the vector field ξ. Moreover, $d\alpha$ is invariant under ξ, hence descends to a closed non degenerate (i.e a symplectic) form on $\mathbb{C}P^n$.

The pull back of the Hopf fibration by the Tischler-Gasqui embedding (theorem 5) provides an S^1 principal bundle over an integral symplectic manifold. Compare with theorem 10.

Existence and classification of contact structures

The general problem of existence of contact structures is still an open problem. However, they are more results than in the symplectic case. For instance, Martinet's theorem of existence in dimension 3. Thomas and Geiges have proved several existence theorems in higher dimensions.

As far as the classification is concerned, the only general result available is an old result of Gray [17], reproved by Martinet [25]:

Theorem 11 *Let M be a compact manifold endowed with two contact forms α and α' inducing the same orientation on M. There is a diffeomorphism h of M such that $h^*\alpha' = f\alpha$, for some positive function f if and only if there is a smooth family of contact forms α_t such that $\alpha_0 = \alpha$ and $\alpha' = \alpha_1$.*

5. Locally conformal symplectic structures

The existence of a symplectic form on a smooth manifold puts strong restrictions on the topology of the manifold. We already obseved that the problem of existence of symplectic structures on compact manifold is still unsettled. The esistence of a non-degenerate 2-form is the same as the existence of an almost complex structure. This is a homotopy problems and the obstructions are known (Wu characteristic classes). The real difficult problem is to determine whether the non-degenerate 2-form is closed. We may want to generalize this last condition. One comes up with the notion of **locally conformal symplectic structures**

A **locally conformal symplectic (lcs) form** on a smooth manifold M is a non-degenerate 2-form Ω such that there exists an open cover $\mathcal{U} = (U_i)$ and smooth positive functions λ_i on U_i such that

$$\Omega_i = \lambda_i(\Omega_{|U_i})$$

is a symplectic form on U_i . If for all i , $\lambda_i = 1$, the form Ω is a symplectic form. Lee [.] observed that the 1-forms $\{d(ln\lambda_i)\}$ fit together into a closed 1-form ω such that

$$d\Omega = -\omega \wedge \Omega. \tag{1}$$

Such 1-form is uniquely determined by Ω and is called the Lee form of Ω.

Conversely, if a non degenerate 2-form Ω satisfies (1), and $\mathcal{U} = (U)_i$ is an open cover with contractible open sets, then $\omega_{|U_i} = dln\lambda_i$, for some positive function λ_i on U_i and $\lambda_i\Omega_{|U_i}$ is symplectic.

We have the following "Darboux" type theorem :

Theorem 12 *Each point in smooth manifold equipped with a lcs form Ω has an open neigborhood U and local coordinates $(x_1, .., x_n, y_1, .., y_n)$, with $y_1 \neq 0$, such that*

$$\Omega_{|U} = y_1(\sum_{i=1}^{n} dx_i \wedge dy_i)$$

and $\omega_{|U} = \frac{dy_1}{y_1}$, where Ω is the Lee form of Ω.

Two lcs forms Ω, Ω' on a smooth manifold M are said to be (conformally) equivalent if $\Omega' = f\Omega$, for some positive function f on M.

A locally conformal symplectic (lcs) structure $\mathcal{S}$ on a smooth manifold M is an equivalence of lcs forms. If a lcs forms Ω is a representative of $\mathcal{S}$, we write: $\Omega \in \mathcal{S}$.

Let ω be a closed 1-form on a smooth manifold M and let $\Lambda^p(M)$ be the set of p-forms on M. One defines an operator

$$d_\omega : \Lambda^p(M) \to \Lambda^{p+1}(M), \qquad \theta \mapsto d\theta + \omega \wedge \theta$$

It is easy to see that $(d_\omega)^2 = 0$, and hence that (Λ^*, d_ω) is a complex. Its cohomology, which was introduced by Lichnerowizc, is denoted $H^*_\omega(M)$. We will called it the Lichnerowicz cohomology. It is easy to see that if ω is an exact 1-form, then $H^*_\omega(M)$ is isomorphic to the de Rham cohomology.

The condition $\dot{\Omega} = -\omega \wedge \Omega$ means then that Ω is d_ω-closed. Hence a locally conformal symplectic form is a non-degenerate d_ω-closed 2-form. This definition matches the definition of a symplectic form in case ω is exact.

The category of locally conformal symplectic manifolds is larger than the category of symplectic manifolds:

The cartesian product$M = N \times S^1$ of a contact manifold (N, α) with the circle S^1 has a locally conformal symplectic form. Let $\Omega = d_\omega(\pi_1^*\alpha)$, where π_i are the projections on each factor, and $\omega = \pi_2^*(\mathcal{L})$, where $\mathcal{L}$ is the length form on S^1. It is easy to see that Ω is a locally conformal symplectic form. For instance $S^{2n+1} \times S^1$ admits a locally conformal symplectic form, and we know it does not admit a symplectic form since $H^2(S^{2n+1} \times S^1, \mathbb{R}) = 0$.

Observe that on the compact manifold $S^{2n+1} \times S^1$, we have the d_ω-exact locally conformal symplectic form $\Omega = d_\omega(\pi_1^*\alpha)$. We saw that a symplectic form on a compact manifold can not be exact! In theorem 17, we characterise d_ω exact locally conformal symplectic forms.

The example above can be generalized in the following result (see for instance [5], [34]):

Theorem 13 *The total space of a flat S^1-principal bundle over a contact manifold carries a locally conformal symplectic form.*

The following simple remark [6] gives a link between symplectic geometry and locally conformal symplectic geometry:

Theorem 14 *Let (M, S) be a lcs manifold, and let $\Omega \in S$. Let $\pi : \tilde{M} \to M$ be the minimum regular covering of M associated with the 1-form ω. Let $\lambda : \tilde{M} \to \mathbb{R}$ be a positive function on $\tilde{M}$ such that*

$$\pi^*\omega = d(ln\lambda).$$

Then $\tilde{\Omega} = \lambda(\pi^*\Omega)$ is a symplectic form on $\tilde{M}$ and its conformal class $\tilde{S}$ depends only on S, i.e. is independent of the choice of $\Omega \in S$ and of λ.

It is well known that the group $\mathcal{A}$ of automorphisms of the covering $\tilde{M}$, is equal to the group of periods of ω. It is easy to see that for any $\tau \in \mathcal{A}$, $(\lambda \circ \tau)/\lambda = c_\tau$ is a constant number, independent of the choice of λ and $\tau \mapsto c_\tau$ is a group homomorphism c from $\mathcal{A}$ to the multiplicative group $\mathbb{R}^+$ of positive real numbers.

Let $Diff_S(M)$ be the group of automorphisms of a lcs structure S on a smooth manifold M. It is clear that for any lcs $\Omega \in S$, then $Diff_S(M)$ is the set of all diffeomorphisms ϕ of M such that $\phi^*\Omega = f_\phi\Omega$, where f_ϕ is a smooth function on M.

Theorem 15 (21) *The group $Diff_S(M)$ determines the locally conformal symplectic geometry.*

In local conformal symplectic geometry, we also have a version of Moser theorem [4]

Theorem 16 *Let Ω_t be a smooth family of lcs forms on a compact manifold M. Suppose that for all t, the Lee form of Ω_t can be written as $\omega_t = \omega + df_t$, where f_t is a smooth family of functions, and that $\Omega_t - \Omega_0$ is d_ω- exact, then there exist a smooth family of diffeomorphisms ϕ_t with $\phi_0 = id$ and a smooth family of functions f_t such that $\phi_t^*\Omega_t = f_t\Omega_0$.*

The Lie algebra $\mathcal{X}_S(M)$ of infinitesimal automorphisms of $\mathcal{S}$, consists of vector fields X on M such that $L_X\Omega = \delta_X\Omega$, where δ_X is a smooth function on M, and $\Omega \in \mathcal{S}$.

A short calculation shows that for such vector field X:

$$d(\omega(X)) = L_X\omega = -d\delta_X.$$

Hence $\omega(X) + \delta_X$ is a constant $l(X)$, and the correspondance $X \mapsto l(X)$ is a Lie algebra homomorphism called the extended Lee homomorphism. If M is compact and Ω is normalized so that the volume $V = \int_M(\Omega)^n = 1$, then [5]:

$$l_X = \int_M \omega(X)(\Omega)^n.$$

For any $X \in \mathcal{X}_S(M)$, setting $\theta = i(X)\Omega$, we have :

$$d_\omega\theta = l(X)\Omega.$$

This shows that if there exists an infinitesimal automorphism X with $l(X) \neq 0$, then Ω is d_ω-exact. The converse is also true.

Theorem 17 *A locally conformal symplectic form Ω with Lee form ω is d_ω-exact if and only if there exists $X \in \mathcal{X}_S(M)$ with $l(X) \neq 0$.*

Another interesting feacture of the extended Lee homomorphism is the following fact [6]:

Theorem 18 *Let $X \in \mathcal{X}_S(M)$ and $\tilde{X}$ a lift of X to the cover $\tilde{M}$, then:*

$$L_{\tilde{X}}\tilde{\Omega} = l(X)\tilde{\Omega}$$

Hence if the Lee homomorphism is identically zero, the infinitesimal automorphisms of the local conformal symplectic structure on M lift into symplectic vector fields for the symplectic manifold $(\tilde{M}, \tilde{\Omega})$. This implies that non d_ω-exact lcs manifolds (M, Ω) behave like the symplectic manifold $\tilde{M}, \tilde{\Omega})$.

On the other hand, d_ω-exact lcs manifolds are very special. We have the following

Theorem 19 *Let (M,Ω) be a compact lcs manifold with $\Omega = d_\omega\theta$, where ω is the Lee form. Suppose $d\theta$ has constant rank. Then M is fibered over S^1 and the restriction of θ on each fiber is a contact form.*

The Lee homomorphism can be globalized at the group level [6]:

Theorem 20 *Let (M,S) ba a connected lcs manifold, $\Omega \in S$ with Lee form ω, $\pi : \tilde{M} \to M$ the covering associated with ω, a function $\lambda : \tilde{M} \to \mathbb{R}$ such that $\pi^*\omega = d(ln\lambda)$.*

For each $\phi \in Diffs(M)_0$, let $\tilde{\phi} : \tilde{M} \to \tilde{M}$ be a diffeomorphism covering ϕ, i.e such that $\pi \circ \tilde{\phi} = \phi \circ \pi$, then

$$\frac{\lambda \circ \tilde{\phi}}{\lambda}.(f_\phi \circ \pi) \tag{3}$$

is a non zero constant $b_{\tilde{\phi}}$, independent of the choice of λ.

If $\hat{\phi}$ is another lifting of ϕ, then $b_{\hat{\phi}} = \sigma.b_{\tilde{\phi}}$, where $\sigma \in \Delta = c(\mathcal{A})$.

The correspondance : $\phi \mapsto b_{\tilde{\phi}}$ is a well defined group homomorphism:

$$\mathcal{L} : Diffs(M)_0 \to \mathbb{R}^+/\Delta$$

which does not depend on the choice of $\Omega \in S$, i.e. is a conformal invariant.

The number $b_{\tilde{\phi}}$ is the similitude ratio of $\tilde{\phi}$, i.e. $\tilde{\phi}^*\tilde{\Omega} = b_{\tilde{\phi}}\tilde{\Omega}$. Hence the Kernel G of $\mathcal{L}$ is a normal subgroup which can be identified with a quotient of a connected subgroup of the group of symplectic diffeomorphisms of $(\tilde{M},\tilde{\Omega})$.

Let ϕ_t be the local 1-parameter group of diffeomorphisms generated by an infinitesimal automorphism $X \in \mathcal{X}_S(M)$, then:

$$\frac{d}{dt}(ln(b_{\tilde{\phi}_t}))|_{t=0} = l(X) \tag{4}$$

As a consequence if l is surjective, then $\mathcal{L}$ is non trivial. This in turn implies that $\tilde{\Omega}$ is exact.

6. Jacobi structures [24]

If Ω is an locally conformal symplectic form on a smooth manifold M, then $\tilde{\Omega} : T(M) \to T^*(M), X \mapsto i(X)\Omega$ is an isomorphism. The inverse of $\tilde{\Omega}$ defines a section P of $\Lambda^2T(M)$, i.e. a bivector: $P(A,B) = \Omega((\tilde{\Omega})^{-1}(A),(\tilde{\Omega})^{-1}(B)$ for all $A,B \in T^*(M)$.

The condition $d\Omega = -\omega \wedge \Omega$ translates as

(i) $[[P,P]] = 2E \wedge P$ and (ii) $[[E,P]] = 0$,

where $E = P(\omega)$, and $[[,]]$ is the Schouten bracket (this is a natural extension of the Lie derivative to skew symmetric contravariant tensor fields).

A couple (E, P) where E is a vector field and $P \in \Lambda^2(M)$ is a bivector, satisfying the equations:

$$[[P, P]] = 2E \wedge P \text{ and } [[E, P]] = 0$$

is called a **Jacobi structure**.

Therefore a local conformal symplectic structure gives raise to a Jacobi structure (P, E) on M, where P comes from an invertible bundle map from $T^*(M)$ to $T(M)$.

A Jacobi structure (P, E) where $E = 0$ is called a Poisson structure. It is simply given by a bivector P such that $[[P, P]] = 0$. If P is invertible, then P comes from a symplectic form. Hence Poisson manifolds, i.e the couples (M, P) of a smooth manifold M and a bivector P satisfying $[[P, P]] = 0$, are particular cases of Jacobi manifolds, and generalize symplectic manifolds.

It is easy to check that contact manifolds also are Jacobi manifolds. Hence all the structures presented in this expository paper are Jacobi structures. A good conclusion of this paper is the following result of Guerida and Lichnerowiz [19] :

Theorem 21 *Every Jacobi manifold (M, P) admits a generalized foliation (in the sense of Stefan-Sussman), with the property that even dimensional leaves are local conformal symplectic submanifolds and odd dimensional leaves are contact submanifolds.*

I would like to end this paper by a

Question

It is easy to define the group $Diff(M, (P, E))$ of automorphisms of a Jacobi manifold $(M, (P, E))$. Does this group determine the Jacobi structure (P, E)?

Acknowledgements

I would like to thank Claudio Cuevas for soliciting this expository paper about of some areas of Symplectic Geometry that are most connected with my research.

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