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A Characterization of Unbounded Fredholm Operators

Alexander G. Ramm
LMA/CNRS, 31 Chemin Joseph Aiguier
Marseille 13402, cedex 20, France
and Mathematics Department, Kansas State University,
Manhattan, KS 66506-2602, USA
e-mail address: ramm@math.ksu.edu

1 Introduction and the Result

This paper is a continuation of [4], where bounded Fredholm operators are studied. The theory of bounded linear Fredholm-type operators is presented in many texts, see e.g. [1], [2]. This paper is written for a broad audience and the author tries to give simple and short arguments.

We call a linear closed densely defined operator $A : X \to Y$ acting from a Banach space X into a Banach space Y a Fredholm operator, and write $A \in Fred(X, Y)$ if and only if

$$R(A) = \overline{R(A)} \tag{1.1}$$

and

$$n(A) = n(A^*) := n < \infty, \quad n(A) := \dim N(A), \tag{1.2}$$

where $N(A) := \{u : Au = 0, u \in D(A)\}$.

In the literature the Noether operators are sometimes called Fredholm operators. The Noether operators are operators for which (1.1) holds, $n(A) < \infty$, $n(A^*) < \infty$, but $n(A)$ may be not equal to $n(A^*)$. Thus $Fred(X, Y)$ is a proper subset of the Noether operators.

The Noether operators are called in honor of F. Noether, who was the first to study a class of singular integral equations with operators of this class in 1921 [3].

In [4] a simple and short proof of the Fredholm alternative and a characterization of Fredholm operators are given for bounded linear operators. Recall that a linear bounded operator F is called a finite-rank operator if $\dim R(F) < \infty$, where $R(F)$ is the range of F.

In the present paper the results of [4] are generalized to the case of closed unbounded linear operators. Namely, the following result is proved:

Theorem 1.1 *If A is a Fredholm operator, then*

$$A = B - F, \tag{1.3}$$

where B is a linear closed operator, $D(B) = D(A)$, $R(B) = Y$, $N(B) = \{0\}$, and F is a finite-rank operator. Conversely, if (1.3) holds, where $B : X \to Y$ is a linear closed densely defined operator, $R(B) = Y$, $N(B) = \{0\}$, and F is a finite-rank operator, then A is closed, $D(A) = D(B)$, and (1.1) and (1.2) hold, so A is a Fredholm operator.

In section 2 a proof of Theorem 1 is given. In the literature the case of unbounded Fredholm operators is usually not discussed directly. In [5] and in [6], pp.57-61, singularities of the parameter-dependent Fredholm operators are studied, and in [7] applications of the Fredholm operators in branching theory are presented. Theorem 1.1 is useful, for example, in the theory of elliptic boundary value problems, but we do not go into further detail (see, e.g., [1], [2], [7]).

2 Proof

1. Assume that $A : X \to Y$ is linear, closed, densely defined operator, and (1.1) and (1.2) hold. Let us prove that then (1.3) holds, $D(B) = D(A)$, $R(B) = Y$, $N(B) = \{0\}$, B is closed, and F is finite-rank operator.

Let $\{\varphi_j\}_{1\leq j\leq n}$ be a basis of $N(A)$ and $\{\psi_j\}_{1\leq j\leq n}$ be a basis of $N(A^*)$. It is known that

$$R(A)^{\perp} = N(A^*), \tag{2.1}$$

where $R(A)^{\perp}$ is the set of linear functionals $\{\psi_j\}$ in Y^* such that $(\psi_j, Au) = 0 \ \forall u \in D(A)$, where (ψ_j, f) is the value of a linear functional $\psi_j \in Y^*$ on the element $f \in Y$. Clearly, $\psi_j \in N(A^*)$, $1 \leq j \leq n$.

Define

$$Bu := Au + \sum_{j=1}^{n}(h_j, u)\nu_j := (A + F)u, \quad \nu_j \in Y. \tag{2.2}$$

where F is a finite-rank operator, $\{\nu_j\}_{1\le j\le n}$ is a set of elements of Y, biorthogonal to the set $\{\psi_j\}_{1\le j\le n}$, $(\psi_j,\nu_m)=\delta_{jm}:=\begin{cases}0, j\ne m\\ 1, j=m\end{cases}$, and $\{h_j\}_{1\le j\le n}$ is the set of elements of X^*, biorthogonal to the set $\{\varphi_j\}_{1\le j\le n}$, $(h_j,\varphi_m)=\delta_{jm}$. Existence of sets biorthogonal to finitely many linearly independent elements of a Banach space follows from the Hahn-Banach theorem. An arbitrary element $u\in X$ can be uniquely represented as $u=u_1+\sum_{j=1}^n c_j\varphi_j$, $c_j=$ const, and $(h_j,u_1)=0$, $1\le j\le n$.

Let us check that $N(B)=\{0\}$ and $R(B)=Y$. Assume $Bu=0$, that is $Au+\sum_{j=1}^n (h_j,u)\nu_j=0$. Apply ψ_m to this equation, use $(\psi_m,Au)=0$, and get

$$0=\sum_{j=1}^n(\psi_m,\nu_j)(h_j,u)=\sum_{j=1}^n\delta_{mj}(h_j,u)=(h_m,u),\quad 1\le m\le n.$$

Therefore $Au=0$. So $u\in N(A)$, and $u=\sum_{j=1}^n c_j\varphi_j$, $c_j=$ const. Apply h_m to this equation and use $(h_m,\varphi_j)=\delta_{mj}$ to get $c_m=0$, $1\le m\le n$. Thus $u=0$. We have proved that $N(B)=\{0\}$.

To prove $R(B)=Y$, take an arbitrary element $f\in Y$ and write $f=f_1+f_2$, where $f_1=Au_1$ belongs to $R(A)$, and $f_2=\sum_{j=1}^n a_j\nu_j$, $a_j\equiv$ const. Note that

$$Y=R(A)\dot{+}L_n, \tag{2.3}$$

where the sum is direct, L_n is spanned by the elements $\{\nu_j\}_{1\le j\le n}$, and $a_j=(\psi_j,f)$. Indeed,

$$(\psi_m,f)=(\psi_m,Au_1)+\sum_{j=1}^n a_j(\psi_m,\nu_j)=a_m,\quad 1\le m\le n.$$

Given an arbitrary $f\in Y$, $f=Au_1+\sum_{j=1}^n(\psi_j,f)\nu_j$, define $u=u_1+\sum_{j=1}^n(\psi_j,f)\varphi_j$, where $(h_j,u_1)=0$, $1\le j\le n$. Then $Bu=f$. Indeed, using (2.2) one has:

$$B\left[u_1+\sum_{j=1}^n(\psi_j,f)\varphi_j\right]=Au_1+\sum_{j=1}^n(h_j,u_1)\nu_j+\sum_{j=1}^n(h_j,\sum_{m=1}^n(\psi_m,f)\varphi_m)\nu_j=f. \tag{2.4}$$

Here the relations $(h_j,\varphi_m)=\delta_{jm}$ and $(h_j,u_1)=0$ are used. We have proved the relation $R(B)=Y$.

2. Let us now assume that $A=B-F$, where $B:X\to Y$ is a linear closed densely defined operator, $D(A)=D(B)$, $N(B)=\{0\}$, $R(B)=Y$, and F is a finite-rank operator. We wish to prove that (1.1) and (1.2) hold and A is closed.

Let us prove (1.1). Assume that $Au_n:=f_n\to f$ and prove that $f\in R(A)$.

One has $Bu_n - Fu_n \to f$. Since $N(B) = \{0\}$, $R(B) = Y$, and B is closed, B^{-1} is bounded by Banach's theorem. Thus

$$u_n - B^{-1}Fu_n \to B^{-1}f. \tag{2.5}$$

Since F is a finite-rank operator, $B^{-1}F$ is compact. Therefore, if $\sup_n \| u_n \| \leq c$, where c is a constant, then a subsequence, denoted u_n again, can be found, such that $B^{-1}Fu_n$ converges in the norm of X. Consequently, (2.5) implies $u_n \to u$, $u - B^{-1}Fu = B^{-1}f$, so $u \in D(B)$ and $Bu - Fu = f$.

To finish the proof, let us establish the estimate $\sup_n \|u_n\| \leq c$. Assuming $\|u_{n_k}\| \to \infty$ and denoting n_k by n and $B^{-1}F$ by T, define $v_n := \frac{u_n}{\|u_n\|}$, $\|v_n\| = 1$. Then $v_n - Tv_n \to 0$ as $n \to \infty$. One may assume that v_n is chosen in a direct complement of $N(I-T)$ in X. Arguing as above, one selects a convergent in X subsequence, denoted again by v_n, $v_n \to v$, and gets $v - Tv = 0$. Since v belongs to the direct complement of $N(I-T)$, it follows that $v = 0$. On the other hand, since $\|v\| = \lim_{n\to\infty} \|v_n\| = 1$, one gets a contradiction, which proves the desired estimate $\sup_n \|u_n\| \leq c$. Property (1.1) is proved.

Let us prove that A is closed. If $Au_n \to f$ and $u_n \to u$, then $Bu_n - Fu_n \to f$, and the above argument shows that $Bu - Fu = f$ so $Au = f$. Thus A is closed.

Finally, let us prove (1.2).

Let $Au = 0$, i.e. $Bu - Fu = 0$. Applying the bounded linear injective operator B^{-1}, one gets an equivalent equation

$$u - Tu = 0, \quad T := B^{-1}F, \quad T : X \to X, \tag{2.6}$$

with a finite-rank operator T. It is an elementary fact (see [4]) that $\dim N(I-T) := n < \infty$ if T is a finite-rank operator. Since $N(A) = N(I-T)$, one has $\dim N(A) = n < \infty$.

Now let $A^*v = 0$. Then

$$B^*v - F^*v = 0. \tag{2.7}$$

Since $(B^*)^{-1} = (B^{-1})^*$ is a bounded and injective linear operator, the elements v are in one-to-one correspondence with the elements $w := B^*v$, and (2.7) is equivalent to

$$w - T^*w = 0, \quad T^* = F^*(B^*)^{-1}, \tag{2.8}$$

so that T^* is the adjoint to operator $T := B^{-1}F$.

Since T is a finite-rank operator, it is an elementary fact (see [4]) that $\dim N(I - T^*) = \dim(I-T) = n < \infty$. Since $N(A^*) = N(I-T^*)$, property (1.2) is proved.

Theorem 1.1 is proved. □

An immediate consequence of Theorem 1.1 is the Fredholm alternative (see Theorem 1.1 in [4]) for unbounded operators $A \in Fred(X, Y)$.

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