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Dynamic Negotiations

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SUMMARY. Time varying conflicts are examined with changing Pareto frontier, disagreement payoffs, and break-down probabilities. The monotonicity and limiting properties of the solution obtained by alternating offering are first discussed, and then a differential equation model is derived to model the time dependence of the solution. Several particular models illustrate the general results.

1 Introduction

This paper is concerned with conflicts and negotiations in a dynamic framework, when decisions are made repeatedly in time or the best time period to make decisions is to be determined. Based on the pioneering work of Nash (1950), many researchers have introduced solution concepts and methods for conflict resolution. The axiomatic approach requires the solution to satisfy certain conditions which are called axioms and in most cases the existence and uniqueness of such a solution is proven. The original collection of axioms of Nash (1950) was modified and generalized by several authors. For example, the non-symmetric Nash solutions have been examined by Harsanyi and Selten (1972). Solutions satisfying individual nonotonicity have been introduced by Kalai and Smorodinsky (1975). This solution was further generalized by Anbarci (1995) which is called the reference function solution. The egalitarian solution of Kalai (1977), the super-additive solution of Perles and Maschler (1981), the equal sacrifice solution of Chun (1988) and the equal area solution of Anbarci (1993) are also well known. Nash (1953) has shown that the equilibrium set coincides with the set of Pareto solutions if the problem is considered as a two-person non-cooperative game. We might consider bargaining as a single player decision problem, when the strategy selection of the other player is considered random. If uniform distribution is assumed and each player maximizes his/her own expected payoff, then the common optimal selection is equivalent to the Nash solution. Similar equivalence holds for applying the principle of Zeuthen (1930) in determining the order in which concessions are made. One of the most popular bargaining method is the alternating offer process (Rubinstein, 1982), which has been extended by Howard (1992). In the model of Anbarci (1995), the payoffs of the players depend not only on their offers to themselves but also on how generous their offers are.

In this paper, we will concentrate on the alternating offer process. After the mathematical model is presented, the existence and uniqueness of the solution will be proved. This result is known from the literature, however some details of the proof will be used in the latter parts of the paper. Some monotonicity and limiting properties of the solution will be next discussed, and then the control of the solution by model parameters will be examined. In the case of time-varying Pareto frontiers, a differential equation model will be derived for the solution, and this model will be illustrated by particular examples.

2 The Mathematical Model

A two-person bargaining problem is usually identified by a pair $(S,\mathbf{d})$, where $S \subset R^2$ is the feasible payoff set, and $\mathbf{d} \in S$ is the disagreement payoff vector. It is assumed that the Pareto frontier can be characterized by a strictly decreasing, concave, continuous function:

$$P = \{(x_1, x_2)\,|x_2 = f(x_1), A \le x_1 \le B\} \tag{1}$$

where $A \le d_1 \le B$ and $f(B) \le d_2 < f(A)$. To rule out trivial bargaining problems, it is assumed that $d_2 < f(d_1)$. In each odd bargaining round, player 1 offers a payoff x_2 to player 2, who either accepts or rejects the proposal. In the case of acceptance, the bargaining process terminates. In the case of rejection, the process brakes down with probability δ_1,d is the terminal payoff vector; and with the probability $1 - \delta_1$, player 2 gives a new offer. If his offer x_1 is accepted, then the process terminates. Otherwise, with probability δ_2, bargaining breaks down with payoff vector **d**, and with probability 1- δ_2, player 1 makes a new offer. We assume the disagreement payoff and the break-down probability of the counterpart can be estimated. The process continues until termination. For more details of this process, see Rubinstein (1982).

In the case of subgame perfect equilibria (SPE), the responding player is indifferent between accepting or rejecting the current offer. The stationary SPE proposal **x** of player 1 and that **y** of player 2 are therefore completely characterized by the following system of equations:

$$\begin{aligned} x_1 &= g(x_2), & x_2 &= \delta_1 d_2 + (1-\delta_1) y_2 \\ y_1 &= \delta_2 d_1 + (1-\delta_2) x_1, & y_2 &= f(y_1) \end{aligned} \tag{2}$$

where g is the inverse of f, $g = f^{-1}$.

Exemple 1 An owner of a company and his employees wish to agree on the salary adjustment through a negotiation to terminate a strike. This strike occurred because the original proposal for salary adjustment by the owner was turned down by the employees. The employees started the strike; in the mean time, they also started the negotiation with the owner. In order to illustrate the process of alternating offer bargaining, assume that the Pareto frontier is defined to be:

$$P = \left\{ (x_1, x_2) | x_2 = \sqrt{100 - x_1^2},\ 0 \le x_1 \le 10 \right\},$$

or

$$x_2 = f(x_1) = \sqrt{100 - x_1^2}, \quad \text{and} \quad x_1 = g(x_2) = \sqrt{100 - x_2^2},$$

where x_1 is the expected payoff of the owner, and x_2 is the expected payoff of the employees in terms of percentage of salary adjustment. For example, if the employees choose their expected payoff $x_2 = 5$ units, then the expected payoff to the owner $x_1 = \sqrt{100 - 5^2} = 8.66$ units. Then, for the employees, the percentage of salary adjustment, Q, can be calculated based on the expected payoff x_2, such as a linear relationship that $Q = x_2$.

In the beginning of the negotiation, the owner proposes a six-percent salary adjustment to the employees. In such condition, the expected payoff to the employees is $x_2 = 6$ units, and the expected payoff to the owner is $x_1 = \sqrt{100 - 6^2} = 8$ units. In the case of acceptance, the bargaining process terminates. In the case of rejection (six-percent adjustment was considered too low by the employees), with probability $\delta_1 = 0.3$, the process breaks down (the strike continues and no definite date set for the next negotiation).

If the strike continues, the owner needs to hire temporary workers with higher wages to keep the plant partially operate, the productivity drops, and the employees receive no paychecks from the owner. We assume under such condition, the expected payoff to the owner drops down to three units. Also, the employees who are on strike receive two units per week from the union. That is, the disagreement payoff vector is $\boldsymbol{d}= (3,2)$.

However, with probability $1-\delta_1 = 0.7$, the employees made a new proposal of 8-percent adjustment to the owner. Then the new expected payoff to the employees x_2 was 8 units and the expected total payoff to the owner $x_1 = \sqrt{100-8^2} = 6$ units. If this new proposal was accepted, the negotiation terminated. Otherwise, with probability $\delta_2 = 0.5$, bargaining broke down, and again, $\mathbf{d}= (3,2)$ became the terminal payoff vector. Or with the probability $1-\delta_2 = 0.5$, the owner gave a new offer; for example, a seven-percent adjustment. It is easy to see that if the bargaining continued this way, it was very likely that an agreement could be reached eventually.

Then our example can be modelled as:

$$\begin{aligned} x_1 &= g(x_2) = \sqrt{100-x_2^2}, \\ x_2 &= 0.3\cdot 2 + (1-0.3)y_2 = 0.6 + 0.7\cdot y_2, \\ y_1 &= 0.5\cdot 3 + (1-0.5)x_1 = 1.5 + 0.5\cdot x_1, \quad \text{and} \\ y_2 &= f(y_1) = \sqrt{100-y_1^2}. \end{aligned}$$

In the next section, we will discuss the unique solution to equations (2) that satisfies the conditions $\boldsymbol{x} \geq \mathbf{d}$ and $\boldsymbol{y} \geq \mathbf{d}$. That is, with fixed values of disagreement payoffs, d_1 and d_2, and break-down probabilities, δ_1 and δ_2 there is always a unique stationary SPE.

3 Existence and Uniqueness of the Solution

As in Houba (1993), notice that equations (2) are equivalent to a single equation

$$f(x_1) - \delta_1 d_2 - (1-\delta_1)f(\delta_2 d_1 + (1-\delta_2)x_1) = 0. \tag{3}$$

For the sake of simplicity, let $h(x_1)$ denote the left-hand side. Since f is continuous, h is also continuous; furthermore

$$h(d_1) = f(d_1) - \delta_1 d_2 - (1-\delta_1)f(d_1) = \delta_1(f(d_1) - d_2) > 0,$$

and with the notation $D_1 = g(d_2)$,

$$h(D_1) = d_2 - \delta_1 d_2 - (1-\delta_1)f(\delta_2 d_1 + (1-\delta_2)D_1) = (1-\delta_1)(d_2 - f(\delta_2 d_1 + (1-\delta_2)D_1)) < 0.$$

Hence, there is at least one solution to equation (3). The solution is unique, since function h is strictly decreasing:

$$h(x_1) = \delta_1 \cdot [f(\delta_2 d_1 + (1-\delta_2)x_1) - d_2] + [f(x_1) - f(x_1 - \delta_2(x_1 - d_1))],$$

where both terms are decreasing in x_1. The first term strictly decreases, since f is strictly decreasing. The second term also decreases, since f is concave.

The next simple result gives a sufficient and necessary condition that guarantees that h is strictly decreasing, so we might relax the assumption on the concavity of function f.

Theorem 1 *Assume that f strictly decreases in $[d_1, D_1]$. Function h strictly decreases in x_1 with arbitrary values $\delta_1, \delta_2 \in (0,1)$ if and only if*

$$\psi_\alpha(x_1) = f(x_1) - f((1-\alpha)d_1 + \alpha x_1)$$

is decreasing in x_1 for all $\alpha \in (0,1)$.

Proof. Notice first that h strictly decreases if and only if for all $x < X$,

$$f(x) - \delta_1 d_2 - (1-\delta_1) f(\delta_2 d_1 + (1-\delta_2)x) >$$
$$f(X) - \delta_1 d_2 - (1-\delta_1) f(\delta_2 d_1 + (1-\delta_2)X)$$

which holds if and only if

$$f(X) - f(x) < (1-\delta_1)[f(\delta_2 d_1 + (1-\delta_2)X) - f(\delta_2 d_1 + (1-\delta_2)x)]. \qquad (4)$$

Introduce the notation $\alpha = 1 - \delta_2$. Assume first that relation (4) holds for all $\delta_1, \delta_2 \in (0,1)$. By letting $\delta_1 \to 0$, inequality (4) implies that

$$f(X) - f(x) \leq f(\delta_2 d_1 + (1-\delta_2)X) - f(\delta_2 d_1 + (1-\delta_2)x) \qquad (5)$$

showing that ψ_α decreases. Assume next that ψ_α decreases, then (5) holds with negative right-hand side. Therefore (4) is true with arbitrary $\delta_1 \in (0,1)$. ■

Function h depends on two parameters, however ψ_α has only one. Therefore in practical cases it is easier to check if ψ_α is monotonic rather than to check the same for h. In the case when f is differentiable, an even more simple monotonicity check is provided by the following result.

Theorem 2 *Assume that f is differentiable and strictly decreasing. Then ψ_α decreases for all $\alpha \in (0,1)$ if and only if function $zf'(d_1 + z)$ decreases in $z \in [0, D_1 - d_1]$.*

Proof. Function ψ_α decreases if and only if $\psi'_\alpha(x) \leq 0$ for all x, that is,

$$f'(x) - \alpha f'((1-a)d_1 + \alpha x) \leq 0.$$

This is equivalent to inequality

$$(x - d_1) f'(d_1 + (x - d_1)) \leq \alpha(x - d_1) f'(d_1 + \alpha(x - d_1)).$$

Since $\alpha \in (0,1)$ is arbitrary, this relation is equivalent to the assumption that $zf'(d_1+z)$ decreases. ■

The condition that $zf'(d_1+z)$ is decreasing is a much weaker assumption than the requirement that $f'(d_1+z)$ is decreasing, which is equivalent to the condition that f is concave.

Exemple 2 In the previous example on salary negotiation,

$$\begin{aligned} h(x_1) &= 0.3\cdot[f(0.5\cdot 3+(1-0.5)x_1)-2]+[f(x_1)-f(x_1-0.5(x_1-3))] \\ &= 0.3\cdot[f(1.5+0.5x_1)-2]+f(x_1)-f(0.5x_1+1.5) \\ &= f(x_1)-0.7f(0.5x_1+1.5)-0.6. \end{aligned}$$

Simple differentiation shows that this function is strictly decreasing. Therefore, there is a unique SPE for this example.

The results of this section can be summarized as follows.

Theorem 3 *Under the conditions of Theorem 1 or Theorem 2, the alternating offer procedure has a unique solution.*

4 Monotonicity of the Solution

Let $(x_1^*, x_2^*, y_1^*, y_2^*)$ denote the stationary SPE with fixed values of δ_1 and δ_2. From equations (2) we have

$$\begin{aligned} (x_1^*-d_1)(x_2^*-d_2) &= \left[\frac{1}{1-\delta_2}(y_1^*-\delta_2 d_1)-d_1\right][\delta_1 d_2+(1-\delta_1)y_2^*-d_2] \\ &= \frac{1-\delta_1}{1-\delta_2}(y_1^*-d_1)(y_2^*-d_2). \end{aligned} \tag{6}$$

The offers (x_1^*, x_2^*) and (y_1^*, y_2^*) of the two players therefore have the same Nash product if and only if $\delta_1=\delta_2$. It is also easy to see that

$$x_1^*-y_1^* = x_1^*-\delta_2 d_1-(1-\delta_2)x_1^* = \delta_2(x_1^*-d_1)\geq 0,$$

and

$$y_2^*-x_2^* = y_2^*-\delta_1 d_2-(1-\delta_1)y_2^* = \delta_1(y_2^*-d_2)\geq 0.$$

These inequalities imply that

$$x_1^*\geq y_1^* \text{ and } y_2^*\geq x_2^*.$$

That is, at the stationary SPE, each player offers at least the same payoff value to himself (or herself) as he (she) offers to the other player. If $\delta_1 \longrightarrow 0$, then $y_2^* - x_2^* \longrightarrow 0$; and if $\delta_2 \longrightarrow 0$, then $x_1^* - y_1^* \longrightarrow 0$. This property can be interpreted as in the case when the break-down probabilities converge to zero, the discrepancies between the offers of the two players also tend to zero. This is not surprising, since in the absence of possible break-downs when no threat is present on disagreement penalty, the two players must reach a common solution.

The monotonicity of the stationary SPE on δ_1 and δ_2 will be examined next. First, we note that with fixed vales of d_1, d_2, δ_1 and δ_2, h strictly decreases in x_1, as was demonstrated in the previous section. Since

$$h(x_1) = f(x_1) - f(\delta_2 d_1 + (1-\delta_2)x_1) + \delta_1 \cdot [f(\delta_2 d_1 + (1-\delta_2)x_1) - d_2]$$

with all other variables kept fixed, $h(x_1)$ increases in δ_1. Let $x_1(\delta_1)$ denote now the solution of equation (3), and let $h(x_1, \delta_1)$ denote the left-hand side of equation (3). We will next prove that $x_1^* = x_1(\delta_1)$ increases in δ_1. In contrary, assume that $\delta_1 < \Delta_1$ and $x_1(\delta_1) > x_1(\Delta_1)$. Then

$$0 = h(x_1(\delta_1), \delta_1) < h(x_1(\Delta_1), \delta_1) \leq h(x_1(\Delta_1), \Delta_1) = 0,$$

which is an obvious contradiction. Equations (2) imply that y_1^* also increases and both x_2^* and y_2^* decrease in δ_1. The symmetry of the two players implies that with fixed values of d_1, d_2 and δ_1, x_1^*, and y_1^* decrease in δ_2, and x_2^* and y_2^* increase in δ_2. This property can be interpreted as any increase in break-down probabilities of the other player has an increasing effect on equilibrium payoff.

Assume next that the values of δ_1, δ_2 and d_2 are kept fixed. From equation (3), we see that $h(x_1)$ increases in d_1. Similarly to the previous case, it is easy to prove that x_1^* and y_1^* are increasing in d_1, and x_2^* and y_2^* are decreasing in d_1. By interchanging the two players, we also conclude that with fixed values of δ_1, δ_2 and d_1, x_1^* and y_1^* are decreasing and x_2^* and y_2^* are increasing in d_2. That is, an increase of the disagreement payoff of any player has an increasing effect of his (her) equilibrium payoff.

Assume finally that function f defining the Pareto frontier is changed. Let $\overline{f}$ denote the new function, and assume that for all $x_1, \overline{h}(x_1) > h(x_1)$ with fixed values of all other variables, where $\overline{h}$ is also defined as the left-hand side of equation (3), with $\overline{f}$ instead of f. This condition is necessarily satisfied if for all $x_1, \overline{f}(x_1) > f(x_1)$, and for all $h > 0, \overline{f}(x_1) - \overline{f}(x_1 - h) \geq f(x_1) - f(x_1 - h)$. Let $\overline{x}_1^*, \overline{x}_2^*, \overline{y}_1^*$ and $\overline{y_2^*}$ denote the coordinates of the stationary SPE. Based on the previous idea, one can easily prove that $\overline{x}_1^* > x_1^*$, and therefore $\overline{y}_1^* > y_1^*$. If the inverses of f and $\overline{f}$ satisfy the above conditions, then by interchanging the two players, we conclude that $\overline{x}_2^* > x_2^*$ and $\overline{y}_2^* > y_2^*$.

5 Limit Properties of the Solution

Assume first that the disagreement vector converges to a point on the Pareto frontier, and let (d_1^*, d_2^*) denote the limit point. Since the stationary SPEs are in a compact set, they have at least one limit point $(\bar{x}_1^*, \bar{x}_2^*, \bar{y}_1^*, \bar{y}_2^*)$. Since both should dominate the disagreement vector, $\bar{x}_1^* = \bar{y}_1^* = d_1^*$ and $\bar{x}_2^* = \bar{y}_2^* = d_2^*$ regardless of the selection of δ_1 and δ_2. In the remaining part of this section, we assume that (d_1^*, d_2^*) is not on the Pareto frontier, that is $(d_2^* < f(d_1^*))$, as it was assumed with regard to the disagreement payoff vector in the beginning of this paper. We assume now that, in addition to the disagreement vector, the break-down probabilities δ_1 and δ_2 also converge. Let δ_1^* and δ_2^* denote the limits. The continuity of f implies that equations (2) and (3) hold with d_1^*, d_2^*, δ_1^* and δ_2^*.

Assume first that δ_1^* and δ_2^* are both positive and less than one. Then (x_1^*, x_2^*) and (y_1^*, y_2^*) is the unique stationary SPE with parameters d_1^*, d_2^*, δ_1^* and δ_2^*. So, the stationary SPEs have a unique limit point; hence they are convergent. Assume next that $\delta_1^* = 0$ and $0 < \delta_2^* < 1$. From equation (3) we see that

$$f(x_1^*) - f(\delta_2^* d_1^* + (1 - \delta_2^*) x_1^*) = 0.$$

That is, $x_1^* = y_1^* = d_1^*$ and $x_2^* = y_2^* = f(d_1^*)$. If $0 < \delta_1^* < 1$ and $d_2^* = 0$, then by interchanging the players,

$$x_2^* = y_2^* = d_2^* \quad \text{and} \quad x_1^* = y_1^* = g(d_2^*).$$

Assume next that $\delta_1^* = 1$ and $0 < \delta_2^* < 1$. Then from equation (3) we see that $f(x_1^*) = d_2^*$, that is, $x_1^* = D_1^*$. From equations (2) we conclude that $x_2^* = d_2^*, y_1^* = \delta_2^* d_1^* + (1 - \delta_2^*) D_1^*$ and $y_2^* = f(y_1^*)$. If $\delta_2^* = 1$ and $0 < \delta_1^* < 1$, then by interchanging the players, $x_2^* = f(d_1^*), x_1^* = d_1^*, y_2^* = \delta_1^* d_2^* + (1 - \delta_1^*) f(d_1^*)$, and $y_1^* = g(y_2^*)$.

Assume next that $\delta_1^* = \delta_2^* = 1$. Then, obviously, $x_1^* = D_1^*, x_2^* = d_2^*, y_1^* = d_1^*$, and $y_2^* = f(d_1^*)$.

Consider last the case of $\delta_1^* = \delta_2^* = 0$. Assume first that $\frac{\delta_2^*}{\delta_1^*}$ converges to a positive constant K. A slight modification of the proof presented in Binmore et al. 1986 shows that $x_1^* = y_1^*$, $x_2^* = y_2^*$, and (x_1^*, x_2^*) coincides with the non-symmetric Nash bargaining solution

$$\arg\max \left\{ (x_1 - d_1^*)^{\alpha} (x_2 - d_2^*)^{1-\alpha} | (x_1, x_2) \in S,\ x_1 \geq d_1^*, x_2 \geq d_2^* \right\}$$

with $\alpha = \frac{1}{K+1}$. If $\frac{\delta_2^*}{\delta_1^*}$ does not converge to a positive limit, then we have the following possibilities. For every subsequence where $\frac{\delta_2^*}{\delta_1^*}$ converges to a positive constant, the stationary SPEs converge to the corresponding non-symmetric Nash bargaining solution. Consider now a sub-sequence such that $\frac{\delta_2}{\delta_1}$ converges to zero. By rewriting equation (3) as

$$f(x_1) + (1 - \delta_1) \frac{f(x_1) - f(x_1 - \delta_2(x_1 - d_1))}{\delta_2(x_1 - d_1)} \cdot \frac{\delta_2}{\delta_1}(x_1 - d_1) - d_2 = 0,$$

and using the fact that the left hand side derivative of f is bounded, we conclude that at the limit, $f(x_1^*) = d_2^*$, that is $x_1^* = D_1^*$. Therefore, $x_2^* = y_2^* = d_2^*$ and $y_1^* = D_1^*$. If $\frac{\delta_2}{\delta_1}$ converges to infinity, then by interchanging the two players, we have the limit $x_2^* = f(d_1^*) = y_2^*$ and $x_1^* = y_1^* = d_1^*$.

6 Control by Break-Down Probabilities

Assume first that parameters δ_2, d_1 and d_2 are given and fixed, and only δ_1 is controlled. If x_1^* is the target value for x_1, then equation (3) implies that δ_1 has to be selected as

$$\delta_1 = \frac{-f(x_1^*) + f(\delta_2 d_1 + (1-\delta_2)x_1^*)}{f(\delta_2 d_1 + (1-\delta_2)x_1^*) - d_2}. \tag{7}$$

Notice that if $x_1^* \in (d_1, f^{-1}(d_2))$, then the numerator and denominator are both positive, and $\delta_1 < 1$. The above assumption on x_1^* means that at the equilibrium offers, both players enjoy higher payoff than in the case of disagreement. Hence, x_1^* can be completely controlled by the selection of δ_1. Since $x_2^* = f(x_1^*), x_2^*$ is automatically controlled by x_1^*. In addition, to control the value of y_1^*, the special selection of only δ_1 is not sufficient, since $y_1^* = \delta_2 d_1 + (1-\delta_2)x_1^*$ does not necessarily hold. If $y_1^* \in (d_1, x_1^*)$, then δ_2 has to be selected as

$$\delta_2 = \frac{x_1^* - y_1^*}{x_1^* - d_1},$$

which is always positive and less than one. The above derivation and the symmetry of the players imply the following result.

Theorem 4 *The value of x_1^* (as well as x_2^*) is completely controllable by δ_1, and the value of y_1^* (as well as y_2^*) is completely controllable by δ_2. The pair (x_1^*, y_1^*) (as well as (x_2^*, y_2^*)) is completely controllable by the pair (δ_1, δ_2).*

7 Control by Disagreement Payoffs

Assume next that the values of δ_1, δ_2 and d_2 are given and fixed, and only the value of d_1 can be selected as control. If x_1^* is the desired value, then equation (3) implies that d_1 has to be selected as the solution of equation

$$f(\delta_2 d_1 + (1-\delta_2)x_1^*) = \frac{f(x_1^*) - \delta_1 d_2}{1-\delta_1}. \tag{8}$$

Since the left hand side strictly decreases in d_1, there is a solution for d_1 in the interval

$[A, x_1^*)$ if and only if

$$f(\delta_2 A + (1-\delta_2)x_1^*) \geq \frac{f(x_1^*) - \delta_1 d_2}{1-\delta_1} \tag{9}$$

and

$$f(x_1^*) < \frac{f(x_1^*) - \delta_1 d_2}{1-\delta_1}. \tag{10}$$

The second inequality is necessarily satisfied if $x_1^* < f^{-1}(d_2)$, as assumed before. If inequality (9) is satisfied, then d_1 has to be the unique solution of equation (8). The value of x_2^* is automatically controlled, since $x_2^* = f(x_1^*)$. Similarly to the control with breakdown probabilities, we see that the value of y_1^* in addition cannot be controlled only by the selection of d_1, since relation $y_1^* = \delta_2 d_1 + (1-\delta_2)x_1^*$ does not necessarily hold. The value of y_1 can be controlled by either the selection of only d_2 (by interchanging the two players), or in the case of $d_1 < y_1^* < x_1^*$, there is a unique value of $\delta_2 \in (0,1)$. In summary, we have the following result.

Theorem 5 *The value of x_1^* (as well as x_2^*) can be controlled by selecting d_1 if and only if relation (9) holds. All components of the stationary SPE can be controlled by the simultaneous selection of d_1 and δ_2, if relation (9) and $d_1 < y_1^* < x_1^*$ hold.*

We mention here that similar conditions can be obtained by interchanging the two players.

8 Time-Varying Pareto Frontiers

Assume now that the Pareto frontier, f, is time-dependent: $f = \hat{f}(t, x_1)$. Then, for all t, equation (3) can be rewritten as

$$\hat{f}(t, x_1(t)) - \delta_1 d_2 - (1-\delta_1)\hat{f}(t, \delta_2 d_1 + (1-\delta_2)x_1(t)) = 0. \tag{11}$$

Assume that $\hat{f}$ is differentiable, then implicite differentiation shows that

$$\hat{f}_t(t,x_1) + \hat{f}_x(t,x_1)\dot{x}_1 - (1-\delta_1)[\hat{f}_t(t,\delta_2 d_1 + (1-\delta_2)x_1) + (1-\delta_2)\hat{f}_x(t,\delta_2 d_1 + (1-\delta_2)x_1)\dot{x}_1] = 0$$

that is,

$$\dot{x}_1 = -\frac{\hat{f}_t(t,x_1) - (1-\delta_1)\hat{f}_t(t,\delta_2 d_1 + (1-\delta_2)x_1)}{\hat{f}_x(t,x_1) - (1-\delta_1)(1-\delta_2)\hat{f}_x(t,\delta_2 d_1 + (1-\delta_2)x_1)}. \tag{12}$$

This equation shows how the value of x_1 changes with time. If x_1^* is the solution of (11) at $t=0$, then $x_1(t)$ is the solution of the ordinary differential equation (12) with initial condition $x_1(0) = x_1^*$. Assume furthermore that for all $t \geq 0$, $\hat{f}(t, x_1)$ satisfies the conditions of the static model being outlined in Section 2. The monotonicity and concavity of $\hat{f}$ implies that $\hat{f}_x$ is decreasing, and is negative for $x_1 \geq A$. Therefore, $|\hat{f}_x(t, x_1)| \geq |\hat{f}_x(t, \delta_2 d_1 + (1-\delta_2)x_1)|$, and the denominator of (12) is always negative. Hence the sign of $\dot{x}_1$ is determined by the sign of the numerator. The shape and major properties of the trajectory $\{x_1(t)\}$ depends on the way $\hat{f}$ depends on t. Some special cases will be shown next.

9 Special Cases

In this section, we will discuss some special cases of how Pareto frontier might move in time. Other cases can be examined in a similar manner.

Exemple 3 Assume first that the Pareto frontier is shifted with a constant speed. Then

$$\hat{f}(t, x) = \alpha t + f(x_1 - \beta t), \tag{13}$$

where f satisfies all previous conditions and α and β are given constants. Then equation (12) has the form

$$\begin{aligned} \dot{x}_1 &= -\frac{(\alpha - \beta f'(x_1 - \beta t)) - (1-\delta_1)(\alpha - \beta f'(\delta_2 d_1 + (1-\delta_2)(x_1 - \beta t)))}{f'(x_1 - \beta t) - (1-\delta_1)(1-\delta_2) f'(\delta_2 d_1 + (1-\delta_2)(x_1 - \beta t))} \\ &= \frac{\beta f'(x_1 - \beta t) - \alpha\delta_1 - (1-\delta_1)\beta f'(\delta_2 d_1 + (1-\delta_2)(x_1 - \beta t))}{f'(x_1 - \beta t) - (1-\delta_1)(1-\delta_2) f'(\delta_2 d_1 + (1-\delta_2)(x_1 - \beta t))}. \end{aligned} \tag{14}$$

Notice that the denominator is always negative, as shown before.

If α and β are both positive, then $\dot{x}_1 > 0$, that is, x_1 increases in time. If both α and β are negative, then $\dot{x}_1 < 0$, so x_1 decreases. If α and β have different signs, then the sign of $\dot{x}_1$ is not determined. Assume that $f(x_1) = b - ax_1$ with some $a, b > 0$. Then

$$\dot{x}_1 = \frac{\alpha\delta_1 + \delta_1 a\beta}{a(\delta_1 + \delta_2 - \delta_1\delta_2)}, \tag{15}$$

therefore, if $\alpha + a\beta$ is positive, then x_1 increases with a fixed positive velocity; if $\alpha + a\beta$ is negative, then x_1 decreases with a constant velocity, and if $\alpha + a\beta = 0$, then x_1 remains constant.

If we have a target solution, x_1^0, then it is easy to determine the time when we reach this solution. We have to solve the differential equation (14) with the initial condition $x_1(0) = x_1^*$, and then solve the nonlinear equation $x_1(t) = x_1^0$. In the special case of equation (15), it becomes

$$\frac{\alpha\delta_1 + \delta_1 a\beta}{a(\delta_1 + \delta_2 - \delta_1\delta_2)} t + x_1^* = x_1^0.$$

Exemple 4 Assume next that the Pareto frontier is shifted at nonconstant velocity. We assume that

$$\hat{f}(t, x_1) = \alpha t^u + f(x_1 - \beta t^v), \tag{16}$$

where $u, v > 0$ are given constants. In this case, equation (12) can be rewritten as

$$\dot{x}_1 = -\frac{(\alpha u t^{u-1} + f'(x_1 - \beta t^v)(-\beta) v t^{v-1}) - (1-\delta_1)(\alpha u t^{u-1} + f'(g(x_1, t)))(-\beta) v t^{v-1})}{f'(x_1 - \beta t^v) - (1-\delta_1)(1-\delta_2) f'(g(x_1, t))} \tag{17}$$

where $g(x_1, t) = \delta_2 d_1 + (1-\delta_2)(x_1 - \beta t^v)$. The denominator is always negative, and the numerator can be simplified as

$$f'(x_1 - \beta t^v)\beta v t^{v-1} - \delta_1 \alpha u t^{u-1} - (1-\delta_1) f'(\delta_2 d_1 + (1-\delta_2)(x_1 - \beta t^v))\beta v t^{v-1},$$

which is negative if $\alpha, \beta > 0$ and is positive if $\alpha, \beta < 0$.

Hence $\dot{x}_1$ is positive for $\alpha, \beta > 0$, so, x_1 increases. Similarly, $\dot{x}_1$ is negative for $\alpha, \beta < 0$, therefore, x_1 decreases. As a special case, assume that $f(x_1) = b - a x_1$. Then

$$\dot{x}_1 = \frac{\delta_1 \alpha u t^{u-1} + \delta_1 a \beta v t^{v-1}}{a(\delta_1 + \delta_2 - \delta_1 \delta_2)}, \tag{18}$$

so the trajectory $x_1(t)$ can be easily computed and also the timing for a given outcome can also be determined. In this case, we have to solve the nonlinear equation

$$\frac{\delta_1 \alpha t^u + \delta_1 a \beta t^v}{a(\delta_1 + \delta_2 - \delta_1 \delta_2)} + x_1^* = x_1^0.$$

Exemple 5 Assume next that

$$\hat{f}(t, x_1) = \alpha^t f\left(\frac{x_1}{\beta^t}\right) \tag{19}$$

with some positive α and β. Equation (12) now reduces as

$$\begin{aligned}
\dot{x}_1 \;=\; & -\Big\{\alpha^t \ln \alpha f\left(\frac{x_1}{\beta^t}\right) + \alpha^t f'\left(\frac{x_1}{\beta^t}\right)(-x_1 \beta^{-t} \ln \beta) - (1-\delta_1) \\
& \times \left[\alpha^t \ln \alpha f\left(\frac{\delta_2 d_1 + (1-\delta_2)x_1}{\beta^t}\right) + \alpha^t f'\left(\frac{\delta_2 d_1 + (1-\delta_2)x_1}{\beta^t}\right)\right. \\
& \left. \times (-(\delta_2 d_1 + (1-\delta_2)x_1)\beta^{-t} \ln \beta\right]\Big\} \\
& \Big/ \left(\alpha^t f'\left(\frac{x_1}{\beta^t}\right)\frac{1}{\beta^t} - (1-\delta_1)(1-\delta_2)\alpha^t f'\left(\frac{\delta_2 d_1 + (1-\delta_2)x_1}{\beta^t}\right)\frac{1}{\beta^t}\right)
\end{aligned}$$

The denominator is always negative and the numerator can be rewritten in the following way:

$$
\begin{aligned}
&-\ln \alpha\left[\alpha^t f\left(\frac{x_1}{\beta^t}\right) - (1-\delta_1)\,\alpha^t f\left(\frac{\delta_2 d_1 + (1-\delta_2)\,x_1}{\beta^t}\right)\right] \\
&+x_1\beta^{-t}\ln\beta\alpha^t\left[f'\left(\frac{x_1}{\beta^t}\right) - (1-\delta_1)\,f'\left(\frac{\delta_2 d_1 + (1-\delta_2)\,x_1}{\beta^t}\right)\right] \\
&+(1-\delta_1)\,\delta_2\,(x_1-d_1)\,\frac{\alpha^t}{\beta^t}\ln\beta f'\left(\frac{\delta_2 d_1 + (1-\delta_2)\,x_1}{\beta^t}\right) = \\
&-\ln\alpha\,(\delta_1 d_2) + x_1\beta^{-t}\ln\beta\alpha^t\left[f'\left(\frac{x_1}{\beta^t}\right) - (1-\delta_1)\,f'\left(\frac{\delta_2 d_1 + (1-\delta_2)\,x_1}{\beta^t}\right)\right] \\
&+(1-\delta_1)\delta_2(x_1-d_1)\frac{x^t}{\beta^t}\ln\beta f'\left(\frac{\delta_2 d_1 + (1-\delta_2)x_1}{\beta^t}\right).
\end{aligned}
\tag{20}
$$

If α and β are greater than 1, then $\dot{x}_1 > 0$, which implies that x_1 is strictly increasing. If both α and β are less than one, then $\dot{x}_1 < 0$, so x_1 strictly decreases.

Consider next the relative payoff of player 1 compared to the payoff of player 2: $r(t) = x_1(t)/x_2(t)$. Notice that $x_2(t) = \hat{f}(t, x_1(t))$, so

$$
\begin{aligned}
\dot{r}(t) &= \frac{\dot{x}_1(t)x_2(t) - x_1(t)\dot{x}_2(t)}{x_2(t)^2} \\
&= \frac{\dot{x}_1(t)\hat{f}(t,x_1(t)) - x_1(t)\left[\hat{f}_t(t,x_1(t)) + \hat{f}_{x_1}(t,x_1(t))\dot{x}_1(t)\right]}{\hat{f}(t,x_1(t))^2} \\
&= \frac{\dot{x}_1\left[\hat{f} - x_1\hat{f}_{x_1}\right] - x_1\hat{f}_t}{\hat{f}^2}.
\end{aligned}
\tag{21}
$$

Exemple 6 Assume that $d_1 = d_2 = 0$, and $f(x_1) = a - bx_1$. Then equation (3) has the form

$$a - bx_1 - (1-\delta_1)[a - b(1-\delta_2)x_1] = 0,$$

which implies

$$x_1 = \frac{a\delta_1}{b[1-(1-\delta_1)(1-\delta_2)]}.$$

Therefore,

$$x_2 = a - bx_1 = a - \frac{a\delta_1}{1-(1-\delta_1)(1-\delta_2)} = \frac{a[1-(1-\delta_1)(1-\delta_2)-\delta_1]}{1-(1-\delta_1)(1-\delta_2)}$$

$$= \frac{a(1-\delta_1)\delta_2}{1-(1-\delta_1)(1-\delta_2)},$$

so

$$\frac{x_1}{x_2} = \frac{a\delta_1}{ba(1-\delta_1)\delta_2} = \frac{\delta_1}{b(1-\delta_1)\delta_2}. \tag{22}$$

If the time-variant Pareto frontier is given as

$$\hat{f}(t, x_1) = \alpha t^u + \left(a - b\frac{x_1}{\beta^t}\right) = (a + \alpha t^u) - \frac{b}{\beta^t}x_1 \tag{23}$$

where α, β, u are positive constants, then from equation (22) we see that

$$\frac{x_1}{x_2} = \frac{\delta_1 \beta^t}{b(1-\delta_1)\delta_2},$$

which increases for $\beta > 1$ and decreases for $\beta < 1$.

Exemple 7 Assume next that $d_1 = d_2 = 0$ and $f(x_1) = a - bx_1^p (p \geq 1)$. Then, from equation (3), we have

$$a - bx_1^p - (1-\delta_1)[a - b(1-\delta_2)x_1^p] = 0$$

which implies that

$$x_1 = \left\{\frac{a\delta_1}{b[1-(1-\delta_1)(1-\delta_2)]}\right\}^{1/p}, \quad x_2 = b - ax_1^p = \frac{a(1-\delta_1)\delta_2}{1-(1-\delta_1)(1-\delta_2)}.$$

Therefore,

$$\frac{x_1}{x_2} = \frac{(a\delta_1)^{1/p}[1-(1-\delta_1)(1-\delta_2)]}{a(1-\delta_1)\delta_2 b^{1/p}[1-(1-\delta_1)(1-\delta_2)]^{1/p}}. \tag{24}$$

In the previous time-variant model, a and b have to be replaced by $a + \alpha t^u$ and b/β^t, respectively. Therefore,

$$\frac{x_1}{x_2} = \frac{\delta_1^{1/p}[1-(1-\delta_1)(1-\delta_2)]^{1-(1/p)}}{(a+\alpha t^u)^{1-(1/p)}\left(\frac{b}{\beta^t}\right)^{1/p}(1-\delta_1)\delta_2}.$$

Notice that u, δ_1, δ_2, a, and b are positive constants; therefore the right-hand side is a positive constant multiple of

$$Q(t) = (a + \alpha t^u)^{-1+(1/p)}(\beta^t)^{1/p}. \tag{25}$$

Hence, the major properties of $x_1/\ x_2$ are the same as those of $Q(t)$. As an illustrative numerical example, select $p = \frac{1}{2}$, and u=1. Then

$$Q(t) = (a + \alpha t)\beta^{2t}. \tag{26}$$

Notice first that $Q(0) = a > 0$ and if $\beta < 1$, then $\lim_{t\to\infty} Q(t) = 0$. Differentiation yields

$$Q'(t) = \alpha\beta^{2t} + (a + \alpha t)\beta^{2t}2\ln\beta = \beta^{2t}(\alpha + 2(a + \alpha t)\ln\beta). \tag{27}$$

Assume that $\alpha + 2a\ln\beta > 0$, then $Q'(0) > 0$. Then

$$Q'(t)\begin{cases} > 0 \text{ if } t < \dfrac{-(\alpha + 2a\ln\beta)}{2\alpha\ln\beta}, \\ < 0 \text{ if } t > \dfrac{-(\alpha + 2a\ln\beta)}{2\alpha\ln\beta} \end{cases}$$

and therefore $Q(t)$ has its maximum at

$$t^* = -\frac{\alpha + 2a\ln\beta}{2\alpha\ln\beta}. \tag{28}$$

This maximum property indicates that if player 1 wants to get the largest payoff compared to that of player 2, then the timing of agreement is very important. This ratio is maximal at t^*, so this player has to make the agreement at t^* to achieve this maximum value.

Consider next the general case. If $\beta < 1$, then

$$Q(0) = a^{-1+(1/p)} > 0$$

with $\lim_{t\to\infty} Q(t) = 0$. In addition,

$$\begin{aligned} Q'(t) &= \left(-1 + \frac{1}{p}\right)(a + \alpha t^u)^{-2+(1/p)}\alpha u t^{u-1}\beta^{1/p} + (a + \alpha t^u)^{-1+(1/p)}\beta^{t/p}\frac{1}{p}\ln\beta \\ &= \beta^{t/p}(a + \alpha t^u)^{-2+(1/p)}\left\{\left(-1 + \frac{1}{p}\right)\alpha u t^{u-1} + (a + \alpha t^u)\frac{1}{p}\ln\beta\right\}. \end{aligned} \tag{29}$$

Assume first that $p < 1$ and $u > 1$, then

$$Q'(0) = a^{-2+(1/p)}\left(\frac{a}{p}\ln\beta\right) < 0.$$

If $p < 1$ and $u = 1$, then

$$Q'(0) = a^{-2+(1/p)}\left\{\left(-1 + \frac{1}{p}\right)\alpha + a\frac{1}{p}\ln\beta\right\},$$

which is positive if and only if

$$\left(-1+\frac{1}{p}\right)\alpha+\frac{a}{p}\ln\beta>0. \tag{30}$$

In this case, Q is increasing at $t=0$. It starts from a positive value and converges to zero as $t\to\infty$. Therefore it has a maximal value at a certain time period $t^*>0$.

If $p<1$ and $u<1$, then $Q'(0)=\infty$, hence Q very rapidly increases at 0. Therefore Q has a maximum at some positive t^*. Assume next that $p>1$. Then always $Q'(0)<0$.

In summary, we see that if $p<1$, and $u<1$ or $u=1$ with condition (30), then $Q(t)$, as well as x_1/x_2, has a maximum at a certain positive value t^*.

In practical cases, differential equation initial-value problems and non-linear algebraic equations are solved. For the most popular computer methods, see, for example Szidarovszky and Yakowitz (1986).

10 Conclusions

In this paper, the monotonicity, the limiting properties, and the control of the solution of the alternating offer bargaining process with time-varying Pareto frontier were first examined. A general differential equation model was next introduced to describe the solution. We have also provided several examples to show how to use the general model to design negotiation strategy in order to reach a given outcome or to maximize an outcome. The methodology of this paper can be successfully applied in the designing of negotiations strategies.

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