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International Fishing as Dynamic Oligopoly with Time Delay

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ABSTRACT. International fishing as a special dynamic game will be analyzed, which is a combination of classical population dynamics and oligopoly theory.

The interaction of the countries or firms is through market rules assuming that all markets are open to all participants. In addition, all fishing parties base their activity on the existing common fish stock. The available fish stock and the beliefs of the participants on the fish stock are the state variables. Depending on the possible symmetry of the fishing parties and on their behavior several alternative models can be formed.

The classical competitive model will be first formulated and examined, and the special case of symmetric firms will be introduced. Next, I will assume that a grand coalition is formed, and the total profit of the industry is maximized. Finally, the partially cooperative case will be examined, in which each participant's objective function contains a certain proportion of the profits of the others in addition to its own profits.

Stability analysis will be performed, and the birth of limit cycles will be examined. Numerical examples will illustrate the theoretical results.

1 Introduction

In the classical oligopoly model, the firms only compete through the market. This was expanded in the oligopsony models, where the firms not only compete in the market for their product, but also in the markets for labor and capital (Okuguchi and Szidarovszky, 1999). When oligopoly models are extended to examine renewable natural resources, it must be kept in mind that there is an underlying rule that governs the availability of the natural resource with time. In an oligopoly model of renewable natural resources, the firms must also compete for this natural resource, much like they compete in the factor markets in an oligopsony.

A good example of a renewable natural resource is the fish stock in the ocean that can be harvested through international fishing. The firms (in this case, countries) compete through the price of fish in the various markets, and in their cost for fishing that varies with the amount of fish available. Further, the level of the fish stock is governed by a dynamical system and fishing will modify this system. The economics of fisheries, and the effect of the economic activity on the underlying natural resource has been studied considerably, for example in Conrad and Clarke (1987), and Conrad (1995, 1999). This survey paper is mainly based on the works of the author of this paper (Szidarovszky et al., 2001, Engel et al. 2001, 2002a, 2002b, Engel, 2002).

2 The Mathematical Model

In his earlier paper, Okuguchi (1998) has analyzed international fishing of two countries under imperfect competition. His model and methodology has been extended by Szidarovszky and Okuguchi (1998) to the n-country case, where a general dynamic systems model has been developed to describe the trajectory of the fish stock as a function of time. This model has been extended to the fully cooperative case in Szidarovszky and Okuguchi (2000). In addition they performed a complete equilibrium and stability analysis for both cases. That model is the basis of my analysis.

Assume that n countries harvest fish in an open sea region, and sell it in their markets as well as in the markets of the other harvesting countries. Let x_{ki} denote the amount of fish harvested by country k and sold in country i ($i, k = 1, 2, 3, \ldots, n$). The total amount of fish harvested by country k is given as $X_k = x_{k1} + x_{k2} + \cdots + x_{kn}$, and the total amount of fish sold in country i is $Y_i = x_{1i} + x_{2i} + \cdots + x_{ni}$. The inverse demand function in country i is assumed to be linear:

$$P_i = a_i - b_i Y_i \tag{1}$$

with $a_k, b_k > 0$. The fishing cost of country k is assumed to be quadratic:

$$C_k = c_k + \gamma_k \frac{X_k^2}{X}, \tag{2}$$

where $c_k, \gamma_k > 0$, and X is the total level of fish stock as in Szidarovszky and Okuguchi (1998).

2.1 The fully competitive case

We assume that the n countries behave as Cournot oligopolists, that is, in each time period $t \geq 0$ they form an n-person noncooperative oligopoly and the harvested amounts are determined by the Cournot-Nash equilibrium of the resulting n-person non-cooperative game. In Szidarovszky and Okuguchi (1998) it has been shown that under the above assumptions the total amount of fish harvested is

$$S = \frac{Af(X)}{1+f(X)}, \tag{3}$$

and the amount of fish harvested by country k can be given as

$$X_k = \frac{A-S}{1+2B\gamma_k/X}, \tag{4}$$

where

$$f(X) = \sum_{k=1}^{n} \frac{1}{1+2B\gamma_k/X}$$

with

$$A = \sum_{i=1}^{n} \frac{a_i}{b_i} \text{ and } B = \sum_{i=1}^{n} \frac{1}{b_i}.$$

It is assumed that without commercial fishing the growth rate of the fish stock is a linearly declining function of the fish stock according to the dynamic rule

$$\frac{\dot{X}}{X} = \alpha - \beta X$$

(where $\alpha, \beta > 0$), which has been known to fit well with experimental data for many biological populations (Clark, 1976). The effect of harvesting is to accelerate the decline in the growth rate of the fish stock, therefore in the presence of international commercial fishing, the fish stock changes according to

$$\dot{X} = X\left(\alpha - \beta X - \frac{Af(X)}{X(1+f(X))}\right). \tag{5}$$

Let

$$G(X) = \frac{Af(X)}{X(1+f(X))},$$

then it can be proved that G is strictly decreasing and convex in X, therefore the number of positive equilibria of the differential equation (5) is 0, 1, or 2. In Szidarovszky and Okuguchi (1998) a complete stability analysis of the equilibria was performed.

2.2 The fully cooperative game

The fully competitive model of international fishing presented in section 2.1 was extended by Szidarovszky and Okuguchi (2000) to consider the case where at each time period, the harvesting countries form a grand coalition and their total profit is maximized.

With the price and cost functions given previously, the profit of country k is given by:

$$\Pi_k = \sum_{i=1}^{n} p_i x_{ki} - \left(c_k + \gamma_k \frac{X_k^2}{X}\right). \tag{6}$$

We now assume that the fish harvesting countries form a grand coalition. Therefore the profit of the coalition is the sum of the individual profits of the n countries:

$$\Pi = \sum_{k=1}^{n} \Pi_k = \sum_{i=1}^{n} (a_i - b_i Y_i) Y_i - \sum_{k=1}^{n} \left(c_k + \gamma_k \frac{X_k^2}{X}\right) \tag{7}$$

which is concave in the variables x_{ki}. Assuming interior optimum, the first order conditions for the coalition's profit maximization is given by:

$$\frac{\partial \Pi}{\partial x_{ki}} = a_i - 2b_i Y_i - 2\gamma_k \frac{X_k}{X} = 0$$

for all i and k. Adding these equations over i and simplifying gives

$$\frac{1}{2}A - S - B\gamma_k \frac{X_k}{X} = 0.$$

where S, A and B are defined as before. Therefore the total fish harvest of country k is

$$X_k = \frac{X}{\gamma_k B}\left[\frac{1}{2}A - S\right]. \tag{8}$$

By adding these equations for all values of k and simplifying gives the total fish harvest:

$$S = \frac{ACX}{2(CX + B)} \tag{9}$$

with $C = \sum_{k=1}^{n} \frac{1}{\gamma_k}$. Note that in the purely competitive case, the total fish harvest, S_c is given by

$$S_c = \frac{Af(X)}{1 + f(X)} \tag{10}$$

with

$$f(X) = \sum_{k=1}^{n} \frac{1}{1 + 2B\frac{\gamma_k}{X}}. \tag{11}$$

We will next prove that for all $X > 0$, $S_c > S$. Let $u_k = \frac{X}{2B\gamma_k}$, and note $\sum_{k=1}^{n} u_k = \frac{XC}{2B}$, so equation (11) can be rewritten as

$$f(X) = \sum_{k=1}^{n} \frac{u_k}{u_k + 1}. \tag{12}$$

Similarly,

$$\frac{CX}{CX + 2B} = \frac{1}{1 + \frac{2B}{CX}} = \frac{1}{1 + \frac{1}{\sum_{k=1}^{n} u_k}} = \frac{\sum_{k=1}^{n} u_k}{\sum_{k=1}^{n} u_k + 1} = \sum_{k=1}^{n} \frac{u_k}{\sum_{l=1}^{n} u_l + 1}. \tag{13}$$

Since each term of the right hand side of (12) is larger than the corresponding term of that of equation (13) we have

$$f(X) > \frac{CX}{CX + 2B}.$$

This inequality can be simplified to

$$2Af(X)(CX + B) > ACX\,[1 + f(X)]$$

or

$$\frac{Af(X)}{1 + f(X)} > \frac{ACX}{2(CX + B)},$$

which proves the assertion.

Now, as before, it is assumed that in the absence of fishing, the fish stock changes according to the logistic law:

$$\dot{X} = (\alpha - \beta X) X,$$

where $\alpha, \beta > 0$. Therefore, in the presence of international commercial fishing, the fish stock changes according to

$$\dot{X} = X \left(\alpha - \beta X - \frac{AC}{2(CX + B)} \right). \tag{14}$$

Let

$$G(X) = \frac{AC}{2(CX + B)},$$

then $G(X)$ is strictly decreasing and convex similarly to the competitive case. Therefore the number of positive equilibria of the dynamic system (14) is 0, 1, or 2, and a similar asymptotic study can be performed as in the competitive case.

2.3 The partially competitive case

In this section, the models of international fishing from sections 2.1 and 2.2 will be extended to consider the case of partial cooperations between the the harvesting countries.

Consider the case where a single species of fish is harvested from a single sea region by n countries, and each country sells its harvest in the markets of all n countries. Let x_{ki} denote again the amount of fish harvested by country k and sold in country i ($i, k = 1, 2, \ldots n$). Assume the inverse demand function in country i is linear:

$$p_i = a_i - b_i Y_i \tag{15}$$

with $a_i, b_i > 0$ and $Y_i = \sum_{k=1}^{n} x_{ki}$ is the total amount of fish sold in country i. In addition, as was used previously, assume the fishing cost of country k is quadratic:

$$C_k = c_k + \gamma_k \frac{X_k^2}{X}, \tag{16}$$

where $c_k, \gamma_k > 0$, X is the total level of fish stock and $X_k = \sum_{i=1}^{n} x_{ki}$ is the total amount of fish harvested by country k. As earlier, the profit of country k follows from these inverse demand and cost functions:

$$\Pi_k = \sum_{i=1}^{n} (a_i - b_i Y_i) x_{ki} - \gamma_k \frac{X_k^2}{X}. \tag{17}$$

Now we assume that all n countries partially cooperate, i.e. the payoff of country k is the sum of her profit and certain proportions of the profits of the competitors. Thus the payoff of country k is now

$$\Pi_k = \sum_{i=1}^{n} (a_i + b_i Y_i) x_{ki} - \gamma_k \frac{X_k^2}{X} + \sum_{l \neq k} \alpha_{kl} \left(\sum_{i=1}^{n} (a_i + b_i Y_i) x_{li} - \gamma_l \frac{X_l^2}{X} \right). \tag{18}$$

We assume interior optimum, therefore the first order conditions for the maximization of country k's profit are given by

$$\frac{\partial \Pi_k}{\partial x_{ki}} = a_i + b_i Y_i - b_i x_{ki} - \frac{2\gamma_k X_k}{X} + \sum_{l \neq k} \alpha_{kl} (-b_i x_{li}) = 0 \tag{19}$$

or simplifying

$$x_{ki} + \sum_{l \neq k} \alpha_{kl} x_{li} = \frac{a_i}{b_i} - Y_i - \frac{2\gamma_k X_k}{b_i X}. \tag{20}$$

This equation for all k ($k = 1, 2, \dots n$) can be combined in the matrix equation

$$\mathbf{H}\,\mathbf{z}_i = \left(\frac{a_i}{b_i} - Y_i \right) \mathbf{1} - \frac{2}{b_i X} \mathbf{C}\,\mathbf{X} \tag{21}$$

where

$$\mathbf{H} = \begin{pmatrix} 1 & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & 1 & \cdots & \alpha_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & 1 \end{pmatrix}, \mathbf{z}_i = \begin{pmatrix} x_{1i} \\ x_{2i} \\ \vdots \\ x_{ni} \end{pmatrix}$$

$$\mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} \gamma_1 & 0 & 0 & \cdots & 0 \\ 0 & \gamma_2 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \gamma_n \end{pmatrix}, \mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Equation (21) can be solved for $\mathbf{z}_i$ to get

$$\mathbf{z}_i = \left(\frac{a_i}{b_i} - Y_i \right) \mathbf{H}^{-1}\,\mathbf{1} - \frac{2}{b_i X} \mathbf{H}^{-1}\,\mathbf{C}\,\mathbf{X}.$$

Adding these equations over i ($i = 1, \dots n$) and simplifying gives

$$\mathbf{X} = (A - S) \left(\mathbf{H} + \frac{2B}{X} \mathbf{C} \right)^{-1} \mathbf{1}, \tag{22}$$

where A and B are as before. Since $\mathbf{1}^T\mathbf{X} = S = \sum_{k=1}^{n} X_k$, we have

$$S = (A - S)\,\mathbf{1}^T\left(\mathbf{H} + \frac{2B}{X}\mathbf{C}\right)^{-1}\mathbf{1}.$$

Solving for S results in

$$S = \frac{Af(X)}{1 + f(X)} \tag{23}$$

where $f(X) = \mathbf{1}^T\left(\mathbf{H} + \frac{2B}{X}\mathbf{C}\right)^{-1}\mathbf{1}$. Combine equations (22) and (23) to get

$$\mathbf{X} = \left(\frac{A}{1 + f(X)}\right)\left(\mathbf{H} + \frac{2B}{X}\mathbf{C}\right)^{-1}\mathbf{1}. \tag{24}$$

By using equation (23), it is clear that in the case of partial cooperation, dynamic systems (5) and (14) are modified as

$$\dot{X} = (\alpha - \beta X - G(X))\,X \tag{25}$$

where in this case

$$G(X) = \frac{Af(X)}{X(1 + f(X))}.$$

3 Time Lag

In classical oligopoly theory, it is assumed that all information was immediately available to the firms for use in making harvesting decisions. This is not a realistic assumption; there are always time lags between obtaining and implementing information about the competitors' production levels. These time lags can be modeled as either fixed time lags or continuously distributed time lags. Fixed time lags have been studied in Russel et al. (1986), where they lead to a differential-difference equation. The characteristic equation of such systems is a combination of polynomial and exponential functions with an infinite eigenvalue spectrum, as shown in the next simple example.

Example 1 Consider the system

$$\dot{x}(t) = x(t) + x(t-1).$$

We seek a solution of the form $x(t) = ce^{\lambda t}$. The system then simplifies to a mixed exponential-polynomial equation

$$c\lambda = c + ce^{-\lambda}.$$

This infinite eigenvalue spectrum makes the computation of eigenvalues and the application of classical bifurcation theory extremely difficult. In addition, in economic models the actual time lag is not typically known. For this reason, models using continuously distributed time lags are more appropriate which compute the expected profits with respect to the randomized delays. Continuously distributed time lags were used in mathematical biology (see, for example, Cushing, 1977). Invernizzi and Medio (1991) made the first use of continuously distributed time lags in the economic literature, and Chiarella and Szidarovszky (2002) have applied them for dynamic oligopolies.

The investigation of instabilities deriving from the addition of continuously distributed time delays is based on recent research in the qualitative theory of nonlinear differential equations. In particular, we are looking for special bifurcation types, namely for the birth of limit cycles using the Hopf Bifurcation Theorem (see for example, Guckenheimer and Holmes, 1983). In this document, we will show several examples of the birth of limit cycles.

3.1 The fully competitive case

One of our key assumptions in this paper is that when the harvesting countries make their decisions on the harvesting amounts, they have only delayed information on the fish stock. The delay itself is uncertain, so a continuously distributed time delay (to be described below) models this situation appropriately. It is also assumed that the information delays of different countries are not necessarily the same.

For each country we form X^{Ek}, the expectation of country k on the fish stock at time period t, according to

$$X^{Ek}(t) = \int_0^t w(t-s, T_k, m_k)\, X(s)\, ds. \tag{26}$$

Note that X^{Ek} is the expected value of earlier data on the fish stock with the density function $w(t-s, T_k, m_k)$.

As in the case of dynamic oligopolies Szidarovszky and Chiarella (2001) we select the weighting function

$$w(t-s, T, m) = \begin{cases} \frac{1}{T} e^{-\frac{t-s}{T}} & \text{if } m = 0 \\ \frac{1}{m!}\left(\frac{m}{T}\right)^{m+1} (t-s)^m e^{-\frac{m(t-s)}{T}} & \text{if } m \geq 1 \end{cases} \tag{27}$$

where m is an integer and T is a positive parameter. If $m = 0$, then weights are exponentially declining with most weight given to the most current data, and if $m \geq 1$, then zero weight is assigned to the most current data, rising to a maximum at $t - s = T$ and declining exponentially thereafter. As m increases, the weighting function becomes more peaked around $t - s = T$, and if $m \to \infty$ or $T \to 0$, the function tends to the Dirac delta function centered at T or zero, respectively.

Each country harvests according to the harvesting function given in equation (4), but using as argument $X^{Ek}(t)$, its estimate of the fish stock at time t. We note that the harvesting function(equation (4)) of country k can be expressed as

$$g_k(X) = \frac{A}{(1 + f(X))(1 + 2B\gamma_k/X)} \quad (k = 1, 2, \ldots, n).$$

Then the dynamics of the fish stock, equation (5), can be expressed as

$$\dot{X}(t) = X(t)(\alpha - \beta X(t)) - \sum_{k=1}^{n} g_k\left(X^{Ek}(t)\right) \tag{28}$$

The dynamic equation (28), with $X^{Ek}(t)$ defined according to equation (26), is a Volterra integro-differential equation. It is in fact equivalent to a system of nonlinear ordinary differential equations as shown in Szidarovszky et al. (2001). Therefore standard methods known from the theory of differential equations can be used to solve the equations and investigate the asymptotical properties of the solution.

We will use linearization around an equilibrium $\overline{X}$. The linearized version of the integro-differential equation (28) can be written as

$$\dot{X}_\delta(t) = (\alpha - 2\beta\overline{X})X_\delta(t) - \sum_{k=1}^{n} g_k'\left(\overline{X}\right)\int_0^t w(t-s, T_k, m_k)X_\delta(s)\,ds, \tag{29}$$

where X_δ is the deviation of X from its equilibrium level. Using the usual techniques (expounded for example by Miller, 1972) of seeking the solution in the form $X_\delta(t) = e^{\lambda t}v$. By substituting this solution into equation (29) and letting $t \to \infty$, we have

$$\lambda - (\alpha - 2\beta\overline{X}) + \sum_{k=1}^{n} g_k'\left(\overline{X}\right)\left(1 + \frac{\lambda T_k}{q_k}\right)^{-(m_k+1)} = 0. \tag{30}$$

At any equilibrium $\overline{X}$, $\alpha - \beta\overline{X} = G\left(\overline{X}\right)$, so $\alpha - 2\beta\overline{X} = 2G\left(\overline{X}\right) - \alpha$ which can be positive, negative, or even zero depending on the shape of function G and the location of the equilibrium. In general, the solution of equation (30) is impossible to give in a closed form, it can only be found by using computer methods. In order to obtain analytic and not only experimental results, special cases will be examined.

First, we shall focus on the case of symmetric firms, where $T_1 = \cdots = T_n = T$, $a_1 = \cdots = a_n = a$, $b_1 = \cdots = b_n = b$, and $\gamma_1 = \cdots = \gamma_n = \gamma$ implying that $A = na/b$, $B = n/b$,

$$f(X) = \frac{n}{1 + 2B\frac{\gamma}{X}} = \frac{nX}{X + 2B\gamma},$$

$$G(X) = \frac{A\frac{nX}{X+2B\gamma}}{X\left(1 + \frac{nX}{X+2B\gamma}\right)} = \frac{An}{(n+1)X + 2B\gamma},$$

and

$$g(X) = \frac{A}{\left(1 + \frac{nX}{X+2B\gamma}\right)\left(1 + \frac{2B\gamma}{X}\right)} = \frac{AX}{(n+1)X + 2B\gamma},$$

where $g_1(X) = \cdots = g_n(X) = g(X)$.

The equilibrium is a solution of the equation

$$\alpha - \beta X - \frac{An}{(n+1)X + 2B\gamma} = 0$$

which is quadratic:

$$X^2\beta(n+1) + X(2B\beta\gamma - \alpha(n+1)) + (An - 2\alpha B\gamma) = 0.$$

In this case

$$\alpha - 2\beta\overline{X} = 2G(\overline{X}) - \alpha = \frac{2An}{(n+1)\overline{X} + 2B\gamma} - \alpha$$

without a definite sign and

$$g'(X) = \frac{A[(n+1)X + 2B\gamma] - AX(n+1)}{[(n+1)X + 2B\gamma]^2} > 0$$

for all X. We will use these relations later.

Equation (30) can be rewritten as follows:

$$[\lambda - (\alpha - 2\beta\overline{X})]\left(1 + \frac{\lambda T}{q}\right)^{m+1} + ng'(\overline{X}) = 0, \tag{31}$$

which is a polynomial equation with $m + 2$ real or complex roots. Analytic solution is possible for only small values of m.

3.2 The case of $m = 0$

In the case of $m = 0$, equation (31) is quadratic:

$$\left[\lambda - \left(\alpha - 2\beta\overline{X}\right)\right](1 + \lambda T) + ng'\left(\overline{X}\right) = 0,$$

that is

$$T\lambda^2 + \lambda\left(1 - T\left(\alpha - 2\beta\overline{X}\right)\right) + \left(ng'\left(\overline{X}\right) - \alpha + 2\beta\overline{X}\right) = 0. \quad (32)$$

We distinguish three cases depending on the magnitude of $\alpha - 2\beta\overline{X}$.

(i) ***Assume first that*** $\alpha - 2\beta\overline{X} \leq 0$.

Then all coefficients are positive implying that the real parts of the eigenvalues are negative. So, the equilibrium is asymptotically stable.

(ii) ***Assume next that***

$$\alpha - 2\beta\overline{X} > 0. \quad (33)$$

If

$$\alpha - 2\beta\overline{X} > ng'\left(\overline{X}\right), \quad (34)$$

then the constant term is negative, showing the existence of two real roots, one is negative and the other is positive. In this case, the equilibrium is unstable.

If

$$\alpha - 2\beta\overline{X} = ng'\left(\overline{X}\right), \quad (35)$$

then the constant term is zero implying the existence of two real eigenvalues, at least one of them is zero. If

$$ng'\left(\overline{X}\right) = \frac{1}{T},$$

then both roots are zero, and if

$$ng'\left(\overline{X}\right) > \frac{1}{T},$$

then the nonzero eigenvalue is positive implying the instability of the equilibrium. If

$$ng'\left(\overline{X}\right) < \frac{1}{T},$$

then the nonzero eigenvalue is negative.

In the first and third cases no conclusion can be reached on the stability of the equilibrium, however, in the second case the equilibrium is unstable.

(iii) ***Assume finally that in addition to (33),***

$$\alpha - 2\beta\overline{X} < ng'\left(\overline{X}\right). \tag{36}$$

In this case, the constant term is positive. If

$$\alpha - 2\beta\overline{X} < \frac{1}{T},$$

then the coefficients are positive implying that the equilibrium is asymptotically stable. If

$$\alpha - 2\beta\overline{X} > \frac{1}{T},$$

then the linear coefficient becomes negative implying the existence of roots with positive value or positive real parts, so the equilibrium is unstable. Assume finally that

$$\alpha - 2\beta\overline{X} = \frac{1}{T}, \tag{37}$$

then there are two pure complex roots.

The existence of a pair of pure complex roots shows that there is the possibility that the real parts change sign implying the change in the asymptotic behavior of the equilibrium. In order to guarantee that such change really occurs we need to show that the real part is a strictly monotonic function of some model parameter. Select T as this bifurcation parameter.

Differentiating equation (32) with respect to T, we have

$$\lambda^2 + 2T\lambda\dot{\lambda} + \dot{\lambda}\left(1 - T\left(\alpha - 2\beta\overline{X}\right)\right) - \lambda\left(\alpha - 2\beta\overline{X}\right) = 0. \tag{38}$$

From (37), the critical value of T is

$$T^* = \frac{1}{\alpha - 2\beta\overline{X}},$$

and from (38),

$$\dot{\lambda} = \frac{\lambda\left(\alpha - 2\beta\overline{X}\right) - \lambda^2}{2T\lambda + \left[1 - T\left(\alpha - 2\beta\overline{X}\right)\right]}.$$

Notice that at the critical value T^*,

$$\dot{\lambda}\Big|_{T=T^*} = \frac{1 - T^*\lambda}{2\,T^{*\,2}}, \tag{39}$$

so

$$Re\,\dot{\lambda}\Big|_{T=T^*} = \frac{1}{2\,T^{*2}} \neq 0$$

implying the change in stability and by the Hopf bifurcation theorem, the existence of a limit cycle. Hence we have the following existence theorem.

Theorem 1 *If* $m = 0$ *and* $\alpha - 2\beta\overline{X} = \frac{1}{T}$, *then there is a limit cycle around the equilibrium.*

3.3 The case of $m = 1$

When m=1, the characteristic equation (31) becomes

$$\left[\lambda - \left(\alpha - 2\beta\overline{X}\right)\right]\left(1 + 2\lambda T + \lambda^2 T^2\right) + ng'\left(\overline{X}\right) = 0,$$

which is the cubic:

$$\begin{aligned}\lambda^3 T^2 + \lambda^2\left(2T - T^2\left(\alpha - 2\beta\overline{X}\right)\right) + \lambda\left(1 - 2T\left(\alpha - 2\beta\overline{X}\right)\right)& \\ + \left(ng'\left(\overline{X}\right) - \alpha + 2\beta\overline{X}\right) &= 0.\end{aligned} \tag{40}$$

The Routh-Hurwitz stability criterion implies that all roots have negative real parts if and only if all coefficients are positive and

$$\left(2T - T^2\left(\alpha - 2\beta\overline{X}\right)\right)\left(1 - 2T\left(\alpha - 2\beta\overline{X}\right)\right) - T^2\left(ng'\left(\overline{X}\right) - \alpha + 2\beta\overline{X}\right) > 0$$

which is equivalent to the quadratic inequality of the form

$$2\left(\alpha - 2\beta\overline{X}\right)^2 T^2 - T\left(ng'\left(\overline{X}\right) + 4\left(\alpha - 2\beta\overline{X}\right)\right) + 2 > 0. \tag{41}$$

Eigenvalue Analysis

(i) ***Assume first that*** $\alpha - 2\beta\overline{X} < 0$.

Then all coefficients of (40) are necessarily positive. The discriminant of the left hand side of (41) has the form

$$ng'\left(\overline{X}\right)\left[ng'\left(\overline{X}\right) + 8\left(\alpha - 2\beta\overline{X}\right)\right],$$

so we have to consider three possibilities:

(a) If

$$\alpha - 2\beta\overline{X} = -\frac{1}{8}ng'\left(\overline{X}\right), \tag{42}$$

then (41) holds for all

$$T \neq \frac{ng'\left(\overline{X}\right) + 4\left(\alpha - 2\beta\overline{X}\right)}{4\left(\alpha - 2\beta\overline{X}\right)^2} = -\frac{1}{\alpha - 2\beta\overline{X}},$$

and in this case the equilibrium is asymptotically stable.

(b) If

$$\alpha - 2\beta\overline{X} < -\frac{1}{8}ng'\left(\overline{X}\right), \tag{43}$$

then the right hand side of (41) has no real root, so (41) holds for all $T > 0$. Therefore, in this case the equilibrium is asymptotically stable.

(c) If

$$\alpha - 2\beta\overline{X} > -\frac{1}{8}ng'\left(\overline{X}\right), \tag{44}$$

then (41) has two real roots:

$$T_{1,2} = \frac{ng'\left(\overline{X}\right) + 4\left(\alpha - 2\beta\overline{X}\right) \pm \sqrt{ng'\left(\overline{X}\right)\left[ng'\left(\overline{X}\right) + 8\left(\alpha - 2\beta\overline{X}\right)\right]}}{4\left(\alpha - 2\beta\overline{X}\right)^2}. \tag{45}$$

In order to examine the signs of these roots, notice first that

$$ng'\left(\overline{X}\right) + 4\left(\alpha - 2\beta\overline{X}\right) > ng'\left(\overline{X}\right) - \frac{1}{2}ng'\left(\overline{X}\right) > 0,$$

that is, both roots are positive. therefore (41) holds if and only if $T < T_1$ or $T > T_2$, where the roots are indexed so that $T_1 < T_2$. Therefore, in this case, the equilibrium is asymptotically stable for $T < T_1$ or $T > T_2$. If $T_1 < T < T_2$, then the equilibrium is unstable.

(ii) ***Consider next the case when*** $\alpha - 2\beta\overline{X} = 0$.

Then all coefficients of (40) are positive, and (41) reduces to the simple inequality

$$-Tng'\left(\overline{X}\right) + 2 > 0$$

which holds if and only if

$$T < \frac{2}{ng'\left(\overline{X}\right)}.$$

So, if this relation holds, then the equilibrium is asymptotically stable.

(iii) ***Consider next the case when*** $\alpha - 2\beta\overline{X} > 0$.

All coefficients of (40) are positive if and only if

$$\alpha - 2\beta\overline{X} < \min\left\{\frac{2}{T}, \frac{1}{2T}, ng'\left(\overline{X}\right)\right\}. \tag{46}$$

The quadratic polynomial (41) has two roots, given by (45), both are positive. By using (46), the larger root satisfies the following relation:

$$T_2 > \frac{5\left(\alpha - 2\beta\overline{X}\right) + \sqrt{9\left(\alpha - 2\beta\overline{X}\right)^2}}{4\left(\alpha - 2\beta\overline{X}\right)^2} = \frac{2}{\alpha - 2\beta\overline{X}} > 4T,$$

so T is never larger than T_2, thus (41) holds if and only if $T < T_1$. Combining this condition with (46), we see that in this case the equilibrium is asymptotically stable, if $T < \min\left\{T_1, \frac{1}{2T}, ng'\left(\overline{X}\right)\right\}$.

Hopf Bifurcation Analysis

Let's now turn our attention to the case of pure complex eigenvalues. A pure complex number $\lambda = ir$ solves equation (40) if and only if

$$\begin{aligned} -ir^3T^2 - r^2\left(2T - T^2\left(\alpha - 2\beta\overline{X}\right)\right) + ir\left(1 - 2T\left(\alpha - 2\beta\overline{X}\right)\right) & \\ + \left(ng'\left(\overline{X}\right) - \alpha + 2\beta\overline{X}\right) &= 0. \end{aligned}$$

Equating the real and imaginary parts to zero, we have

$$r^2 = \frac{1 - 2T\left(\alpha - 2\beta\overline{X}\right)}{T^2} = \frac{ng'\left(\overline{X}\right) - \alpha + 2\beta\overline{X}}{2T - T^2\left(\alpha - 2\beta\overline{X}\right)}. \tag{47}$$

Since $T^2 > 0$, real r exists only if

$$1 - 2T\left(\alpha - 2\beta\overline{X}\right) > 0$$

and (41) is satisfied with equality:

$$2\left(\alpha - 2\beta\overline{X}\right)^2 T^2 - T\left(ng'\left(\overline{X}\right) + 4\left(\alpha - 2\beta\overline{X}\right)\right) + 2 = 0. \tag{48}$$

A positive solution for T exists only if the discriminant is nonnegative and the linear coefficient is negative:

$$\alpha - 2\beta\overline{X} \geq -\frac{1}{8}ng'\left(\overline{X}\right). \tag{49}$$

Let T_1^* and T_2^* ($T_2^* \geq T_1^*$) denote the roots. The bifurcation parameter is selected again as T, then T_1^* and T_2^* are the critical values. Differentiate equation (40) with respect to T implicitly to have

$$\begin{aligned}3\lambda^2\dot{\lambda}T^2 + \lambda^3 2T + 2\lambda\dot{\lambda}\left(2T - T^2\left(\alpha - 2\beta\overline{X}\right)\right) + \lambda^2\left(2 - 2T\left(\alpha - 2\beta\overline{X}\right)\right) & \\ +\dot{\lambda}\left(1 - 2T\left(\alpha - 2\beta\overline{X}\right)\right) + \lambda\left(-2\left(\alpha - 2\beta\overline{X}\right)\right) &= 0\end{aligned}$$

implying that

$$\dot{\lambda} = -\frac{2T\lambda^3 + \lambda^2\left(2 - 2T\left(\alpha - 2\beta\overline{X}\right)\right) - 2\lambda\left(\alpha - 2\beta\overline{X}\right)}{3\lambda^2T^2 + 2\lambda\left(2T - T^2\left(\alpha - 2\beta\overline{X}\right)\right) + \left(1 - 2T\left(\alpha - 2\beta\overline{X}\right)\right)}. \tag{50}$$

At $\lambda = ir$, the real part of this derivative is the following:

$$Re\ \dot{\lambda}\Big|_{\lambda=ir} = \frac{A}{B} \tag{51}$$

where

$$B = \left[-3r^2T^2 + 1 - 2T\left(\alpha - 2\beta\overline{X}\right)\right]^2 + \left[2r\left(2T - T^2\left(\alpha - 2\beta\overline{X}\right)\right)\right]^2 > 0$$

and

$$\begin{aligned}A &= r^2\left[2 - 2T\left(\alpha - 2\beta\overline{X}\right)\right]\left[-3r^2T^2 + 1 - 2T\left(\alpha - 2\beta\overline{X}\right)\right] \\ &\quad + \left[2Tr^3 + 2r\left(\alpha - 2\beta\overline{X}\right)\right]\left[2r\left(2T - T^2\left(\alpha - 2\beta\overline{X}\right)\right)\right] \\ &= r^2\left[1 + r^2T^2\right]\left[-3r^2T^2 + r^2T^2\right] \\ &\quad + \left[2Tr^3 + 2r\frac{1 - r^2T^2}{2T}\right] 2r\left[2T - T^2\frac{1 - r^2T^2}{2T}\right] \\ &= r^2\left(-r^4T^4 + 2r^2T^2 + 3\right),\end{aligned}$$

which is nonzero if $r^2T^2 \neq 3$. In this case, there is a limit cycle as a consequence of the Hopf bifurcation theorem. Notice that if $r^2T^2 = 3$, then from (47),

$$T\left(\alpha - 2\beta\overline{X}\right) = -1,$$

and relation (48) implies that

$$-Tng'\left(\overline{X}\right) + 8 = 0.$$

From these equations, we see that

$$\alpha - 2\beta\overline{X} = -\frac{1}{8}ng'\left(\overline{X}\right), \tag{52}$$

that is, (49) holds with equality. Assume next that (52) holds, then (48) can be rewritten as

$$2\left(\alpha - 2\beta\overline{X}\right)^2 T^2 + 4T\left(\alpha - 2\beta\overline{X}\right) + 2 = 2\left(\left(\alpha - 2\beta\overline{X}\right)T + 1\right)^2 = 0$$

implying that

$$T\left(\alpha - 2\beta\overline{X}\right) = -1,$$

and from (47), we see that $r^2T^2 = 3$. We can summarize the above derivation as follows.

Theorem 2 *If $m = 1$ and $\alpha - 2\beta\overline{X} > -\frac{1}{8}ng'\left(\overline{X}\right)$, then there is a limit cycle around the critical value $T \neq \frac{\sqrt{3}}{r}$, otherwise the existence of a limit cycle is not guaranteed.*

The fully cooperative case, and the partially cooperative case with symmetric countries, the form of the equations governing the level of the fish stock is the same, thus the behavior is similar.

4 Numerical Examples

First, we select $m = 0$, $\alpha = \beta = 1$, $a = 1$, $b = 3$, $\gamma = 0.8$ and $n = 2$. In this case, $A = B = \frac{2}{3}$,

$$\begin{aligned} f(X) &= \frac{2X}{X + \frac{16}{15}}, \\ G(X) &= \frac{\frac{4}{3}}{3X + \frac{16}{15}}, \\ g(X) &= \frac{\frac{2}{3}X}{3X + \frac{16}{15}}. \end{aligned}$$

The critical value of T is 9. Figure 1 shows the birth of a limit cycle with this value of T. In figures 2 and 3, we changed the critical value to 8.8 and 9.2. In the first case, a shrinking cycle is shown, and in the second case, an expanding cycle is obtained, showing the asymptotically stable system became unstable by increasing the value of T.

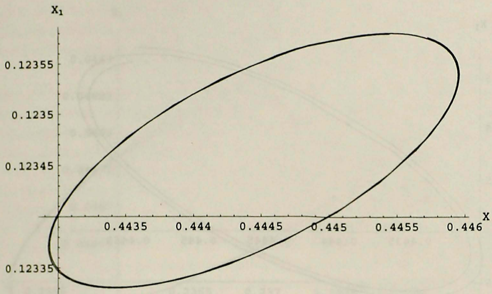

Figure 1: Birth of a limit cycle with $m = 0$

Next, the case of $m = 1$ is illustrated. The selected parameter values are $\alpha = 0.75$, $\beta = 1.5$, $a = 1$, $b = 3$, $\gamma = 2$ and $n = 2$. In this case, $A = B = \frac{2}{3}$,

$$f(X) = \frac{2X}{X + \frac{8}{3}},$$
$$G(X) = \frac{\frac{4}{3}}{3X + \frac{8}{3}},$$
$$g(X) = \frac{\frac{2}{3}X}{3X + \frac{8}{3}}.$$

The critical values of T are 4.38014 and 145.212 as the solutions of equation (48). Figure 4 shows the occurrence of a limit cycle with the smaller value of T. Similar phenomenon is obtained with the large root, as well. Shrinking and expanding limit cycles can also be generated by lowering and raising the value of T from the critical values.

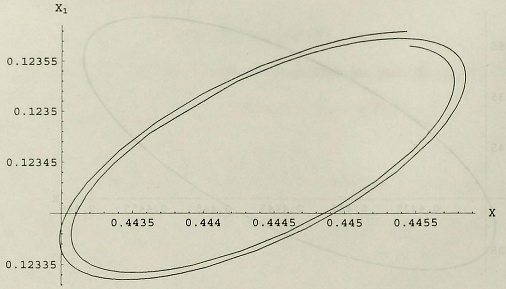

Figure 2: A shrinking cycle for $T = 8.8$

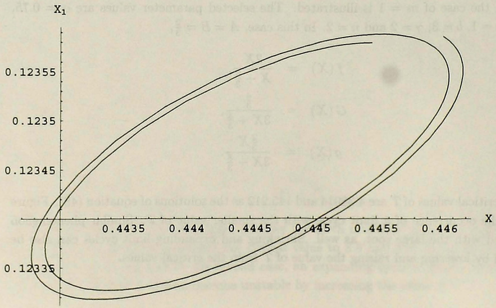

Figure 3: An expanding cycle with $T = 9.2$

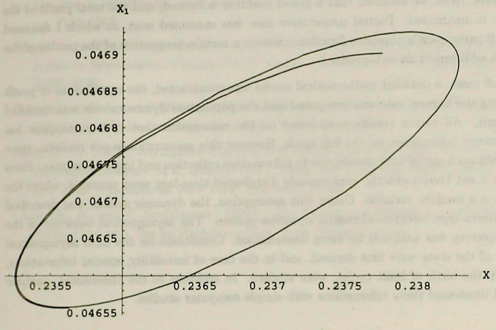

Figure 4: Birth of a limit cycle with $m = 1$

5 Conclusions

In this paper, a special class of dynamic games was analyzed. This type of dynamic games arises from the combination of the classical population dynamics theory and the well known oligopoly theory of mathematical economics.

International fishing is a typical example of this type of dynamic games in which several countries, or firms, fish on a sea region. Their interaction is through market rules when we assume that all markets are open to all participants. In addition, all fishing parties base their activity on the existing common fish stock. The combination of the market economic rules with population dynamics results in a special dynamic system, in which the available fish stock and the beliefs of the participants on the fish stock are the state variables. Depending on the behavior of the participants several alternative models can be formed.

The classical competitive model was first formulated and examined. After the general model was constructed, a special case was introduced. We assumed the countries, or firms, are identical. In this special case, I derived simple analytic results about the asymptotic behavior of the state trajectory, which is tractable in the general case only by computer simulations. Next, we assumed that a grand coalition is formed, and the total profit of the industry is maximized. Partial cooperative case was examined next, in which I assumed that each participant's objective function contains a certain proportion of the profits of the others in addition to its own profits.

In all cases, a detailed mathematical model was constructed, the equilibrium or profit optimizing the harvest rate was computed and the population dynamics rule was modified accordingly. All earlier results were based on the assumption that each participant has instantaneous information on the fish stock. However this assumption is not realistic, since there is always a gap of information due to information collection and implementation. Since the time is not known exactly, continuously distributed time lags were assumed, where the time lag is a random variable. Under this assumption, the dynamic system was described by a Volterra-type integro-differential equation system. The asymptotical behavior of the state trajectory was analyzed by using linearization. Conditions for the local asymptotical stability of the state were first derived, and in the case of instability, special bifurcations, especially the birth of limit cycles, were studied. In addition to the theoretical, analytic results, I illustrated these bifurcations with simple computer studies.

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