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The Weyl algebra and its field of fractions *

Paul M. Cohn
Department of Mathematics
University College London
Gower Street
London WC1E 6BT, Great Britain
pmc@math.ucl.ac.uk

At present non-commutative algebra is the flavour of the day, and one of its prime examples is the Weyl algebra. This is a mathematical system that has found wide application, not merely in various parts of mathematics, but also in physics, where it first originated. The object of this brief article is to present some of its history and properties; the aim is not to be comprehensive, there are now good accounts such as the book by Coutinho [3]. Here we merely discuss some of its aspects which show its differences from the commutative situation.

By the *n-th Weyl algebra* $A_n(k)$ over a field k (named after Hermann Weyl, who laid much of the mathematical foundations in his book [12]), one understands an algebra over k on $2n$ generators $x_1, \cdots, x_n, y_1, \cdots, y_n$ such that each variable commutes with all but one of the others, while

$$x_i y_i - y_i x_i = 1 \quad (i = 1, \quad , n).$$

We shall be interested in the simplest case, where $n = 1$, and we shall omit the subscripts; thus we deal with the first Weyl algebra, here just called the *Weyl algebra* $A_1(k)$ generated by x, y subject to the single defining relation

*This article is based on a lecture given on the occasion of the 80th birthday of the late Kurt A. Hirsch on 14 January, 1986 at Queen Mary College, London.

$$xy - yx = 1. \tag{1}$$

We shall frequently abbreviate $xy-yx$ as $[x,y]$, so that (1) reads: $[x,y]=1$.

The simplest way of reaching this form is via differential operators. If we are dealing with functions of a variable t and write $D = \frac{d}{dt}$, then the rule for differentiating a product gives

$$D(tf) = tDf + f,$$

which in operator form reads

$$Dt - tD = 1,$$

and this is essentially (1). This accounts for much of the importance of this algebra.

Nevertheless it did not become established until the advent of quantum theory in the 1920's; thus Wedderburn in his paper [11] on infinite-dimensional algebras in 1924 does not list it among his examples.

One reason why it was thought odd is that it has no finite-dimensional representations, at least in characteristic zero. This is well known and follows by taking traces in (1): if ρ is a representation, then $\operatorname{tr}(\rho(xy-yx)) = \operatorname{tr}(\rho(x)\rho(y) - \rho(y)\rho(x)) = 0 \neq \operatorname{tr}(I)$. Of course an infinite-dimensional representation is easily written down: using matrix units e_{ij}, $i,j = 1,2,\cdots$, we can put $x = \sum ne_{nn+1}$, $y = \sum e_{n+1n}$. Then we obtain

$$\begin{aligned} xy - yx &= \textstyle\sum ne_{nn} - \sum ne_{n+1n+1} \\ &= \textstyle\sum [n-(n-1)]e_{nn} = I. \end{aligned}$$

It is not hard to see that $A_1(k)$ is simple, when k has characteristic zero: Every element has the form $f = \sum a_{ij}x^iy^j$, where $a_{ij} \in k$. In any non-zero ideal I take $f \neq 0$ of least total degree. Then I contains $[f,y] = \frac{\partial}{\partial x}\cdot f$, which, being of lower degree in x, must be 0, hence $f = \sum a_jy^j)$ $(a_j \in k)$, and taking f to be a polynomial in y of least degree in I, we have $[x,f] = \frac{\partial}{\partial y}\cdot f = 0$, hence f is a non-zero element of k, so $I = A_1(k)$. When k has prime characteristic p, $A_1(k)$, or rather, its skew field of fractions, becomes an algebra of finite dimension p^2 over $k(x^p, y^p)$, as is easily verified.

In quantum mechanics one interprets momentum as conjugate of the position variable and the Hamiltonian theory suggests writing q for y and $\frac{\partial}{\partial q}$ for x. This again leads to (1) except for a scalar factor, and if p, q are momentum and position, it means that here we have two observables which do not commute and so (regarding them as matrices) they cannot be simultaneously diagonalized. This derivation of the Heisenberg uncertainty relation has an air of magic about it. But there is a more down-to-earth (though more technical) explanation by Pascual Jordan [5] in terms of the Thomas-Kuhn formula for dispersion of coherent radiation.

In the early 1930's many papers were written which dealt with the Weyl algebra, at least implicitly. Some of these were concerned with differential operators, e.g. [6], others were more algebraic. Among the latter was a paper by Dudley E. Littlewood [7] in which he studies real and complex algebras. Everything was very much couched in 19th century language; a defining relator was called a 'modulus' and Theorem X in his paper states (in the above notation, though using his terminology):

No second modulus is compatible with the modulus $xy - yx - 1$.

In modern terms this just states that $A_1(k)$ is simple; the proof was by a normal form argument, and in essence the same as the above simplicity proof. Much of the rest of the paper was taken up with the second normal form ($\sum b_{rs}y^r x^s$ as against $\sum a_{ij}x^i y^j$) and a comparison of the two, and so does not concern us here. What is of interest is that Theorem X gave rise to a short paper by Kurt A. Hirsch [4], in which he gives a general proof that Weyl algebras are simple. More precisely, he shows: If R is an algebra over a field of characteristic 0, with generators $x_1, \cdots, x_r$ and defining relations $x_j x_j - x_j x_i = a_{ij}$, for a skew-symmetric matrix $A = (a_{ij})$, then R is simple if and only if A is non-singular. The proof is straightforward, using the well known reduction of A to normal form. The language of the paper is thoroughly modern, describing R by generators and relations (having first defined a free associative algebra), and taking quite a general commutative field k, but pointing out that the characteristic needs to be zero for the result to hold.

The Weyl algebra is an integral domain. This was already noted by Ludwig Schlesinger in his book [8] on differential equations (and proved again by Littlewood). It is easily seen to be Noetherian, and so has a division ring or more briefly, a skew field of fractions (though we shall usually omit the qualifying adjective). This last is actually proved by Littlewood, who shows that the Ore condition is satisfied. Of course the Ore condition also ensures that the least field of fractions is determined up to isomorphism. We shall call the field of fractions of $A_1(k)$ the Weyl field and denote it by $D_1(k)$.

In forming fields of fractions, one has two choices: to illustrate this by the polynomial ring $k[x]$, we can either form the field $k(x)$ consisting of all fractions $\frac{f}{g}$, or we can first form the power series ring $k[[x]]$, a local ring with maximal ideal (x), and on localizing, get the field of formal Laurent series in x:

$$k((x)) = k[[x]]_x.$$

All this still goes through for *skew* polynomial rings. If K is a field (possibly skew) with an automorphism α, we define $K[x;\alpha]$ as the ring generated by K and x with defining relations

$$ax = xa^{\alpha} \quad \text{for all } a \in K. \tag{2}$$

Again we have a field of fractions $K(x;\alpha)$ and a Laurent series field $K((x;a))$. Sometimes the commutation formula (2) is replated by

$$ax = xa^{\alpha} + a^{\delta} \quad \text{for all } a \in K. \tag{3}$$

Here α is again an automorphism (or at least an endomorphism), while δ is a linear mapping such that

$$(ab)^{\delta} = a^{\delta}b^{\alpha} + ab^{\delta} \quad \text{for all } a, b \in K, \tag{4}$$

as we see by equating terms in $(ab)x = a(bx)$. A linear mapping satisfying (4) is called an α-*derivation* and the polynomial ring with the commutation rule (3) is called a *skew polynomial ring*, denoted by $K[x;\alpha,\delta]$. For example, the Weyl algebra may be written in the form $K[y;1,']$, where $'$ is a derivation (short for 1-derivation).

If we want to form Laurent series, or even just power series, we have a problem when $a^{\delta} \neq 0$, for each term cx^n gives rise to terms of lower degrees and we do not get convergence. Briefly, the multiplication $x \mapsto ax$ is nor continuous; this problem was noticed by I. Schur [9], who overcame this difficulty by replacing x by x^{-1}. Writing $z = x^{-1}$, we obtain from (3),

$$za = a^{\alpha}z + za^{\delta}z = a^{\alpha}z + a^{\delta\alpha}z^2 + za^{\delta\delta}z^2 = \cdots$$

and in this way we can obtain a power series expression for za. If we bear in mind that x represents differentiation with respect to x, this is just an expression of the fact that integration improves convergence, whereas differentiation makes it worse.

Another way to deal with this problem is to put $z = xy$. Then (1) becomes

$$xz - zx = x^2y - xyx = x,$$

hence we obtain

$$zx = x(z-1). \tag{5}$$

Thus we can form $F = k(z)$ with the shift automorphism $\alpha : f(z) \mapsto f(z-1)$ and then obtain a field containing $A_1(k)$ by taking $F(x;\alpha)$, or also $F((x;\alpha))$. Littlewood actually noted this possibility (however without explaining why it worked).

This power series representation is often useful. Suppose that we want to show that any element commuting with x is a function of x alone. We have $f = \sum x^j c_i$, where $c_i = c_i(z)$. If $xf = fx$, then we have

$$0 = xf - fx = \sum x^{i+1} c_i - \sum x^i c_i x = \sum x^{i+1}(c_i(z) - c_i(z-i)).$$

By the uniqueness of this form we have $c_i(z) = c_i(z-1) = c_i(z-2) = \cdots$, hence c_i is independent of z and so f is a function of x alone.

Strictly speaking, one still needs to show that a rational function of x, z which is a power series in x alone is a rational function of x, but that is not hard to see, using the classical criteria for the rationality of power series (see e.g. [2], p.69). Of course the same holds for y instead of x, because we have an automorphism of $A_1(k)$ (and its field of fractions) given by $x \mapsto y$, $y \mapsto -x$. A similar result holds for $x+y$, using the automorphism $x \mapsto x$, $y \mapsto x+y$, and similarly for other cases.

For any variables u, v one has the well known Baker-Hausdorff formula

$$e^u e^v = e^w,$$

where w is a Lie element, i.e. obtained by forming repeated commutators [1]. Explicitly,

$$w = u + v + \frac{1}{2}[u,v] + \frac{1}{12}[[u,v],v] + \frac{1}{12}[[u,v],u] + \cdots$$

It follows that in $A_1(k)$ we have $e^x e^y = e^{x+y+\frac{1}{2}}$, a relation noted already in [6].

In some respects the Weyl field $D_1(k)$ behaves like a field of rational functions of one variable, in other respects like a field of rational functions in two variables. Let us consider the possible valuations of D over k (see Shtipelman [10]). For comparison we look at the commutative case first; this was treated by Zariski in [13]. Any function field E of two variables is an algebraic extension of the rational function field $k(x,y)$. So in classifying valuations on E, it is enough to consider valuations on the latter, because the value group for $k(x,y)$ is of finite index in the value group of E, and the residue class field undergoes a finite extension.

A valuation is called *d-dimensional* if the residue class field has transcendence degree d over the field of constants. For a non-trivial valuation this must go down from the dimension of the given field, so we have just a 0-dimensional or a 1-dimensional case.

1-dimensional valuation. This is a principal valuation defined by a prime divisor in $k[x, y]$. Geometrically it is a curve defined on $k(x, y)$.

0-dimensional valuation. We denote the value group by Γ and distinguish several cases.

i) Γ is discrete of rank 1. Choose a uniformizer ξ; then we have an embedding $k(x, y) \to k((\xi))$. Geometrically this is a non-algebraic curve on the surface. E.g. for any $f \in k(x, y)$ define $v(f)$ as the order (in t) of $f(t, e^t)$. This gives a 0-dimensional valuation, corresponding to the curve $y = e^x$.

ii.a) Γ is of rank 1, non-discrete but rational. Every subgroup of the additive group of rational numbers $\mathbb{Q}$ is determined by a 'supernatural number' $N = \prod_p p^{\alpha_p}$, where α_p is a natural number or ∞. The divisors of N form the set of denominators. We thus get a fractional power series:

$$y = c_1 x^{\frac{m_1}{n_1}} + c_2 x^{\frac{m_2}{n_2}} + \cdots, \quad \frac{m_1}{n_1} < \frac{m_2}{n_2} < \cdots$$

ii.b) Γ is of rank 1, non-discrete and not rational. Let $v(x) = 1, \quad v(y) = \tau$, where τ is irrational. We obtain a branch of the curve $y = x^\tau$.

iii) Γ is of rank 2. In this case the valuation is composed of a 1-dimensional valuation of $k(x, y)$, followed by a valuation of the residue class field. Geometrically it defines a place on an algebraic curve on the surface.

Consider now the Weyl field $D = k(x, z)$ with $zx = x(z - 1)$. It turns out that there are fewer possibilities here. We note that a priori the value group need not be abelian, although we shall soon find that in fact it is so. As we saw, a 1-dimensional valuation is defined by a prime divisor, but since $A_1(k)$ is simple, this case cannot arise now. Thus our valuation will be 0-dimensional. We write again $F = k(z)$ with shift automorphism $\alpha : f(z) \mapsto f(z - 1)$, so that $D = F(x; \alpha)$, and first consider the valuation restricted to F. The possible valuations of F over k are associated with 1) an irreducible polynomial in z, 2) the trivial case or 3) associated with z^{-1}.

1) Let $p \in k[z]$ be irreducible such that $v(p) > 0$. Then p^α is again irreducible and $p^\alpha \neq p$ so p^α is prime to p. We have $v(p) > 0$, $v(p^\alpha) = 0$ but $px = xp^\alpha$, hence $v(p) + v(x) = v(x) + v(p^\alpha)$ and so $v(p^\alpha) > 0$, a contradiction. It follows that this case cannot occur.

2) v is trivial on $k(z)$. Let V be the valuation ring in D and suppose that $x \in V$. Then V meets $F[x]$ in a prime ideal (p) and the polynomial p must be invariant, i.e. the left or right ideal generated by it is two-sided. This means that $p = x^n$, for some $n \geq 1$. By irreducibility, $n = 1$, so we have the x-adic valuation on D.

If $x \notin V$, then $x^{-1} \in V$, and a similar argument shows that the valuation is associated with x^{-1}. In both cases we have a 1-dimensional residue class field, namely $k(z)$. The first case may be realized by expressing our element as a power series in x with polynomials in z as coefficients: $f = \sum x^i c_i$ and taking $v(f)$ to be the order, i.e. the least degree occurring; similarly for the second case we write our element as a power series in x^{-1}

3) v is associated with z^{-1}. Thus $v(z^{-1}) > 0$, and we may take $v(z^{-1}) = 1$, without loss of generality. The relation (5) may be written $xz^{-1} = z^{-1}x(1 - z^{-1})$; on writing $z^{-1} = -u$ we obtain $xu = ux + uxu$, which yields the commutation formula

$$xu = ux + u^2x + \cdots = \sum u^{n+1}x. \tag{6}$$

Suppose that $v(x) = \lambda$; since $v(u) = 1$, (6) shows that $\lambda + 1 = 1 + \lambda$; it follows that Γ is abelian. We now distinguish various cases depending on the structure of Γ:

i) Γ is of rank 1 and discrete. Taking a uniformizer t, we can expand x, u in power series in t, and so find that $xu = ux$, a contradiction. Hence this case cannot occur.

ii.a) Γ is of rank 1, non-discrete, but rational. Now Γ consists of all rational numbers with denominator dividing some supernatural number N. We can express x as a fractional power series, and again find that $xu = ux$, so this case is again ruled out.

ii.b) Γ is of rank 1 , non-discrete, thus Γ is non-rational. Then λ is irrational and distinct monomials have distinct values. Writing our element as a double power series, we have $f = \sum c_{ij}u^ix^j$, where $c_{ij} \in k$; here $v(f) = min\{i + \lambda j | c_{ij} \neq 0\}$.

iii) Γ is of rank 2 , and λ is not real (e.g. λ is infinitely large or infinitely small, or more generally, infinitely close to some real number). Here v takes the same form as in ii.b).

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