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A FIRST ENCOUNTER WITH VARIATIONAL METHODS IN DIFFERENTIAL EQUATIONS

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Introduction

Many times, solutions of a given *differential equation problem*

$$\mathcal{D}[u] = 0$$

can be found by *variational techniques*, in other words, by seeking points of *Minimum*, *Maximum*, or even *Saddle-Points* of an *associated real-valued functional*

$$\Phi : X \longrightarrow \mathbb{R} .$$

Here, X is a suitable (huge) *space* or *set of functions* which includes the possible solutions(s) of our differential equation problem. And the functional $\Phi[u]$ is such that its 'derivative' is equal to $\mathcal{D}[u]$, so that our given problem reads $\Phi'[u] = 0$, or $D_u\Phi[u] = 0$.

The basic idea is borrowed from Calculus, when one looks for 'stationary' or 'critical points' of a given function.

The functional $\Phi[u]$ has often the meaning of an *Energy*, a *Cost Functional*, etc. The classical example is the famous **Dirichlet Principle** [4]:

> *"The* **solution** *of the Laplace equation* $\Delta u(x) = 0$ *on (say) a* nice *plane domain* Ω, *satisfying the condition* $u(x) = h(x)$ *on the boundary of* Ω *(for a given nice* $h(x)$*), is precisely that function* $u_0(x)$ *which* **minimizes** *the* **Energy Functional**
>
> $$\Phi[u] = \int_\Omega |\nabla u(x)|^2 \, dx$$
>
> in the **admissible set** X of all functions $u(x)$ defined on Ω that satisfy the given boundary condition."

Here, one could think of the given boundary function $h(x)$ (or its graph) as giving the *shape* of a fixed curved wire in space, and the graphs of the admissible functions $u(x)$ as the possible *shapes* of elastic membranes which may have the given wire as contour. Then, Dirichlet's Principle states that the shape of the elastic membrane which will adjust to the given wire is given by the function $u_0(x)$ which minimizes the energy functional above [The values of $\Phi[u]$ represent the (elastic) *potential energies* of the candidates to the various membrane configurations].

The goal of these notes is to introduce the reader to some of the modern techniques of the *Calculus of Variations*. By means of simple, motivating examples of ordinary differential equations, we will describe in a somewhat informal manner the main ideas behind the so-called *Variational Methods*. Besides the *Basic Minimization Result* of the Calculus of Variations, the reader will also encounter two other basic results of a *Minimax* nature. For those interested in further studying the variational methods and their applications, we recommend the standard references [6, 8, 10, 11], as well as the monograph [3] (in portuguese) by the present author.

We would like to thank to Prof. Claudio Cuevas and the Editorial Committee of Cubo for the invitation to write these notes. We also thank Prof. Hossein Tehrani for having read the manuscript and for his suggestions.

1 Critical Points in Calculus

All of us who took the calculus series in college are familiar with the problem of finding the *stationary* or *critical points* of a given *nice function* $F : \mathbb{R}^n \longrightarrow \mathbb{R}$, that is, of finding the solutions (if any) of the problem

$$\nabla F(x) = 0 ,$$

and of deciding whether a critical point is a point of *(local) mimimum*, a point of *(local) maximum*, or *neither*.

The following figure, in the one-dimensional case $n = 1$, illustrates a situation with all three possibilities: a point x_1 of local maximum, a point x_2 of local minimum, and a point x_3 which is neither a point of local minimum nor local maximum (We recall that, in this example, the point x_3 is called a point of *inflection* since it gives a local minimum from one side and a local maximum from the other).

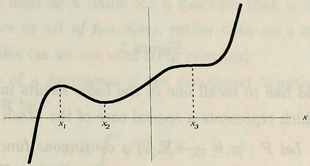


Figure 1

Of course, critical points which are neither local minimum nor local maximum can be more complicated than points of inflection, as the example $F(x) = x^2 sin(\frac{1}{x})$ below shows:

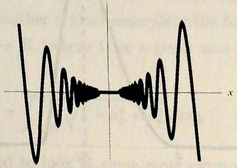


Figure 2

Another typical example, in the two-dimensional case $n = 2$, is given by the function $F(x, y) = x^2 - y^2$, which has $(x, y) = (0, 0)$ as its only critical point. We recall that, in this example, (0.0) is called a *saddle-point* by virtue of the fact that it is a local minimum along a certain direction (the x-axis, in this case) and a local maximum along another direction (the y-axis, in this case):

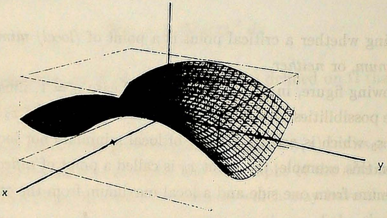


Figure 3

Finally, we would like to recall one of the basic results in the one-dimensional situation ($n = 1$), which represents a special case of the *Mean Value Theorem*:

Rolle's Theorem: *Let $F : [a, b] \longrightarrow \mathbb{R}$ be a continuous function, which is differentiable on the open interval (a, b) and is such that $F(a) = F(b) = 0$. Then, there exists some point x_0 in (a, b) such that $F'(x_0) = 0$.*

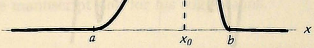


Figure 4

So, Rolle's Theorem guarantees the existence of (at least) one critical point when our nice function F is at same level (zero) at two different points (a and b). The figure depicted above illustrates Rolle's Theorem in a situation where the critical point x_0 is a point of maximum. We now close this section with a natural question, whose answer will be given later on.

Question 0: Is there an analogue of Rolle's Theorem in dimension $n > 1$?

2 The Classic and Modern Periods of the Calculus of Variations

The so called Classic Period of the Calculus of Variations is that period of time that followed the introduction of Calculus by Newton and Leibniz, in which questions of finding *extrema* (maximum or minimum) of *functionals* were considered. Here, the word *functional* is used as a name for a function that is defined on a given (*admissible*) *linear space* or *set of functions*, rather than on a space or set with a finite number of variables (as we are used to in calculus).

A typical example of a *functional* is the functional 'length', defined for *nice* functions $u : [a, b] \longrightarrow \mathbb{R}$ by

$$L = \int_a^b \sqrt{1 + u'^2}\, dx \ .$$

Here, for each function $u : [a, b] \longrightarrow \mathbb{R}$ having a continuous derivative, one considers its *length* $L = \Phi[u]$ given by the above formula. Thus, we have a functional $\Phi : X \longrightarrow \mathbb{R}$ defined on the (*infinite dimensional*) *linear space* of all continuously differentiable functions $u : [a, b] \longrightarrow \mathbb{R}$. Another typical example is the functional 'area', defined for *nice* functions $u : \Omega \longrightarrow \mathbb{R}$ (where Ω is a given *nice* bounded region of the xy-plane) by the formula:

$$A = \int\int_\Omega \sqrt{1 + u_x^2 + u_y^2}\, dxdy \ .$$

The reader is encouraged to look at some more examples in, for example, [4] (Chapter IV) or [9] (Chapter 9). Our main purpose in this section is to explain how one goes about 'finding' maximum or minimum (or more general critical points) of such functionals Φ. For simplicity, we will assume that we are dealing with functionals defined on a linear space X (so that linear combinations of functions in X are again in X). The idea is to reduce the problem to a one-dimensional problem by considering the function

$$\delta \mapsto \Phi[u + \delta h]$$

of one variable $\delta \in \mathbb{R}$, where u and h are fixed functions in the space X. Then, a given u_0 is said to be a *critical point* of the functional Φ if

$$\frac{d}{d\delta}\Phi[u_0 + \delta h]|_{\delta=0} = 0 \tag{2.1}$$

for <u>all</u> h in X.

Remark 2.1. *We note that formula* (2.1) *says that the* directional derivative of Φ at u_0, in the direction of an arbitrary h, is equal to zero* :

$$D\Phi[u_0] \cdot h := \lim_{\delta \to 0} \frac{\Phi[u_0 + \delta h] - \Phi[u_0]}{\delta} = 0 \ . \tag{2.2}$$

Formula (2.2) leads to the so called *Euler-Lagrange equation of* Φ. Let us see how this is done for functionals of the form

$$\Phi[u] = \int_a^b \mathcal{F}(x, u, u')\, dx \ , \tag{2.3}$$

where the integrand $\mathcal{F}(x, y, z)$ is a *nice* function and the admissible space X consists of continuously differentiable functions $u : [a, b] \longrightarrow \mathbb{R}$ satisfying $u(a) = u(b) = 0$. Indeed, if we 'formally calculate' the limit in (2.2) in this case, we will find that

$$D\Phi[u_0] \cdot h = \int_a^b \{\mathcal{F}_y(x, u_0, u_0')\ h + \mathcal{F}_z(x, u_0, u_0')\ h'\}\ dx \ .$$

Therefore, since h satisfies $h(a) = h(b) = 0$, if u_0 is a *nice* function (say twice continuously differentiable) and we integrate by parts the second term of the integral

*Strictly speaking, this is less demanding than the requirement that the "derivative" $\Phi'[u_0]$ vanish.

above and factor out the common term h, we see that (2.2) reads

$$D\Phi[u_0]\cdot h = \int_a^b \{\mathcal{F}_y(x,u_0,u_0') - \frac{d}{dx}\mathcal{F}_z(x,u_0,u_0')\}h\ dx = 0$$

for <u>all</u> such h. From this, it can be shown† that u_0 must be a solution of the equation

$$\mathcal{F}_u(x,u,u') - \frac{d}{dx}\mathcal{F}_{u'}(x,u,u') = 0\ , \tag{2.4}$$

which is the so-called **Euler-Lagrange equation** of the functional (2.3). In other words, **any nice critical point of the functional** (2.3) **is necessarily a solution of the Euler-Lagrange equation** (2.4).

Example. *Let us consider the functional length* $L = \Phi[u]$ *(mentioned earlier) defined on the admissible set of (nice) functions* $u : [a,b] \longrightarrow \mathbb{R}$ *satisfying the boundary conditions* $u(a) = c$ *and* $u(b) = d$*, for some given* $c, d \in \mathbb{R}$*. In order to have such functions forming a linear space* X *we will assume that* $c = d = 0$ ‡*. In this example we have* $\mathcal{F}(x,u,u') = (1+u'^2)^{1/2}$ *and, therefore, the Euler-Lagrange equation reads*

$$0 - \frac{d}{dx}(1+u'^2)^{-1/2}u' = 0\ ,$$

or, after simplifications,

$$u'' = 0\ .$$

Its solutions are the linear functions $u(x) = Ax + B$ *and, after using the boundary conditions* $u(a) = u(b) = 0$*, we obtain* $u(x) \equiv 0$*. This confirms our obvious intuition that the straight line joining the points* $(a,0)$ *and* $(b,0)$ *should give a minimum for the functional length. Of course, in the case of the more general boundary conditions* $u(a) = c$*,* $u(b) = d$*, we will find that the function* $l(x) = c + \frac{d-c}{b-a}(x-a)$ *yields the minimum length.*

It is clear that our approach so far has been an informal, non-rigorous one. After all, we mentioned earlier that our main purpose in this section was to show how one could 'try to find' critical points of functionals using calculus. As we just saw, one

†This is a consequence of Du Bois-Reymond Lemma (see [4], pg. 200).

‡We note that the general case can be reduced to this case by writing $u(x) = v(x) + c + \frac{d-c}{b-a}(x-a)$ and considering instead the linear space of the corresponding functions v.

could go from a **Variational Problem for the Functional** $\Phi[u]$ to a **Differential Equation Problem for** u. This is precisely the approach that characterized the **Classic Period of the Calculus of Variations**.

On the other hand, we can say that the **Modern Period of the Calculus of Variations** initiated with the famous *Dirichlet Principle* mentioned in the Introduction. During this period, which continues on to our present days, the approach goes in the opposite direction. Namely, one starts from a given **Differential Equation Problem for** u (which is to be solved) and considers the corresponding (if any) **Variational Problem for the Functional** $\Phi[u]$, of which the given differential equation is the Euler-Lagrange equation. Then, the idea is to use solely *Variational Methods* (for finding minima, maxima and other critical points) in order to 'solve' the given differential equation problem. Our main goal in this monograph is precisely to illustrate this latter approach by means of some typical differential equation problems.

3 Five Ordinary Differential Equation Problems

Let us consider the following five boundary value problems:

$$(P_1) \qquad \begin{cases} u'' + \frac{1}{2}u = sint \ , & 0 < t < \pi \\ u(0) = u(\pi) = 0 \end{cases}$$

$$(P_2) \qquad \begin{cases} u'' + u = sint \ , & 0 < t < \pi \\ u(0) = u(\pi) = 0 \end{cases}$$

$$(P_3) \qquad \begin{cases} u'' + 2u = sint \ , & 0 < t < \pi \\ u(0) = u(\pi) = 0 \end{cases}$$

$$(P_4) \qquad \begin{cases} u'' + u^3 = 0 \ , & 0 < t < \pi \\ u(0) = u(\pi) = 0 \end{cases}$$

$$(P_5) \qquad \begin{cases} u'' - u^3 = 0 \ , & 0 < t < \pi \\ u(0) = u(\pi) = 0 \end{cases}$$

We would like to start by pointing out the following facts about these problems:

1) Problem (P_1) has the solution $u_0(t) = -2 \sin t$ (unique);

2) Problem (P_2) has no solution;

3) Problem (P_3) has the solution $u_0(t) = \sin t$ (unique);

4) Problem (P_4) has the solution $u_0(t) \equiv 0$ (not unique);

5) Problem (P_5) has the solution $u_0(t) \equiv 0$ (unique).

Using our knowledge of second order linear differential equations, it is not hard to verify statements 1) - 3), as well as the obvious fact that $u_0(t) = 0$ is a solution of problems (P_4) and (P_5). The fact that (P_5) has no other solution can be seen by multiplying the equation by u and integrating by parts once. Indeed, we obtain

$$0 = \int_0^\pi u''(t)u(t)\, dt - \int_0^\pi u(t)^4\, dt = -\int_0^\pi (u'(t))^2\, dt - \int_0^\pi u(t)^4\, dt\ ,$$

from which it clearly follows that $u(t) \equiv 0$. What is not so obvious, however (and the reader should take my word for it), is that (P_4) has many other solutions, in fact, infinitely many of them.

Question 1. What do problems $(P_1) - (P_5)$ have in common?

Obviously, such a general question could prompt a general, trivial response in return (such as, they all involve *ordinary* differential equations, they all have the same boundary conditions, etc., etc.). However, one answer which we would like to provide here is that these five problems are *variational* in the sense that each of them can be seen as the Euler-Lagrange problem of some suitable functional.

In order to do that, let us denote by $f : \mathbb{R} \longrightarrow \mathbb{R}$ any of the functions indicated in the left-hand side of the equations, and by $\rho : [0, \pi] \longrightarrow \mathbb{R}$ any of the right-hand side functions.

In other words,

$$f(s) = \begin{cases} \frac{1}{2}s \\ s \\ 2s \\ s^3 \\ -s^3 \end{cases} , \quad \rho(t) = \begin{cases} sint \\ sint \\ sint \\ 0 \\ 0 \end{cases} ,$$

for problems $(P_1), (P_2), (P_3), (P_4), (P_5)$, respectively. So, we write (P_j) $(j=1,\ldots,5)$ as

$$(P) \qquad \begin{cases} u'' + f(u) = \rho(t) \ , \quad 0 < t < \pi \\ u(0) = u(\pi) = 0 \end{cases}$$

Next, given a solution u_0 of (P), if we multiply the equation by an arbitrary function h in

$$C_0^1[0,\pi] := \{h : [0,\pi] \to \mathbb{R} \mid h \text{ is of class } C^1, \ h(0) = 0 \ , \ h(\pi) = 0 \ \} ,$$

we obtain, after an integration by parts:

$$-\int_0^\pi u_0'(t)h'(t) \ dt + \int_0^\pi f(u_0(t))h(t) \ dt = \int_0^\pi \rho(t)h(t) \ dt \ .$$

In other words, we get

$$\int_0^\pi u_0'(t)h'(t) \ dt - \int_0^\pi f(u_0(t))h(t) \ dt + \int_0^\pi \rho(t)h(t) \ dt = 0 \qquad (3.1)$$

for all $h \in C_0^1[0,\pi]$. Now, it is not hard to check that the above says that the *directional derivative of* Φ *at* u_0, *in the direction of* h, is equal to zero:

$$D\Phi[u_0] \cdot h := \lim_{\delta \to 0} \frac{\Phi[u_0 + \delta h] - \Phi[u_0]}{\delta} = 0 \ , \qquad (3.2)$$

where Φ is the functional defined by

$$\Phi[u] := \frac{1}{2}\int_0^\pi (u'(t))^2\, dt - \int_0^\pi F(u(t))\, dt + \int_0^\pi \rho(t)u(t)\, dt \tag{3.3}$$

and $F(s) = \int_0^s f(\sigma)d\sigma$ is an antiderivative of $f(s)$. We note that

$$F(s) = \begin{cases} \frac{1}{4}s^2 \\ \frac{1}{2}s^2 \\ s^2 \\ \frac{1}{4}s^4 \\ -\frac{1}{4}s^4 \end{cases}$$

for problems $(P_1), (P_2), (P_3), (P_4), (P_5)$, respectively.

We have 'shown' in (3.2) that a solution of (P) is a *critical point* of the functional $\Phi : C_0^1[0,\pi] \longrightarrow \mathbb{R}$ defined in (3.3) (whose Euler-Lagrange Problem is precisely problem (P)). Moreover, as we already pointed out in Section 2, it can also be 'shown' that any *critical point* of such a functional Φ turns out to be a *nice* function which is a solution of problem (P). Therefore, we have obtained the following answer to Question 1:

Answer 1. Problems $(P_1) - (P_5)$ are variational: its solutions are precisely the critical points of the associated functional Φ.

Remark 3.1. *We should note that the first term in the definition of $\Phi[u]$ can be regarded as the 'length squared' or 'norm squared' of the function u, provided we define the 'inner-product' of any two functions u, v in $C_0^1[0,\pi]$ by the formula*

$$\langle u, v\rangle := \int_0^\pi u'(t)v'(t)\, dt\ . \tag{3.4}$$

The first term in the r.h.s. of (3.3) *can then be written as* $\frac{1}{2}\|u\|^2$. *This definition of 'norm',*

$$\|u\| := \left(\int_0^\pi (u'(t))^2\, dt\right)^{1/2}, \tag{3.5}$$

is often called the 'mean-square norm'. *Such 'norms' are generalizations to the* *infinite-dimensional setting of the notion of 'length' of vectors, as defined in terms of the 'dot product'.**

Remark 3.2. *The space* $C_0^1[0,\pi]$ *is* <u>*not*</u> *'complete' with respect to the 'norm' above.*† *The 'completion' of* $C_0^1[0,\pi]$ *with respect to the norm* $\|\cdot\|$ *is the 'Hilbert space' (≡ 'complete inner-product space') denoted by* $X := H_0^1(0,\pi)$, *which is an example of a so-called 'Sobolev space'. The reader should not be overwhelmed by all the language and notation which we are bringing into play. It is enough to keep in mind that, in order to be successful in this variational setting, it is necessary to work with a space having an 'inner-product' (or a 'norm') that makes it 'complete'. And, for that, one needs to do some 'sophistication'. For all practical purposes, the reader may continue to think of all the given functions in these notes as being as 'nice' as he so wishes (or feels comfortable with).*

Remark 3.3. *Keeping Remarks 3.1 and 3.2 in mind, we can now write our functional* Φ *as*

$$\Phi[u] = \frac{1}{2}\|u\|^2 - \Psi[u] \ , \quad u \in X = H_0^1(0,\pi) \ ,$$

where

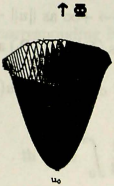
$$\Psi[u] := \int_0^\pi F(u(t))\, dt - \int_0^\pi \rho(t)u(t)\, dt \ .$$

Question 2. What 'type' of critical point is each solution u_0 presented in the beginning of this section? What is the 'geometry' of the functional Φ in each of the problems $(P_1) - (P_5)$?

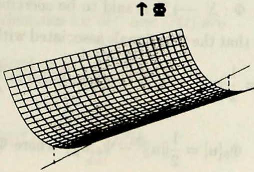
The rest of this monograph is devoted to 'answering' this question by analysing each problem. At this point, however, we provide a 'sneak preview' of the answer by means of the following 'pictures':

*Recall that $a \cdot b = \sum_{i=1}^n a_i b_i$ and $\|a\| = (\sum_{i=1}^n a_i^2)^{1/2}$ for given vectors $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$.

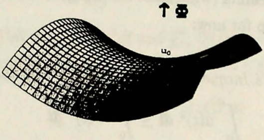
†There are sequences $\{u_m\}$ of functions in $C_0^1[0,\pi]$ satisfying $\|u_m - u_k\| \to 0$ as $m, k \to \infty$ (the *Cauchy Criterion*), but which *do not converge*; in other words, for <u>no</u> $v \in C_0^1[0,\pi]$ one has $\|u_m - v\| \to 0$.



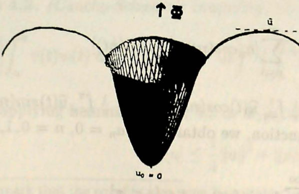
Problem (P_1)



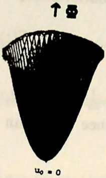
Problem (P_2)



Problem (P_3)



Problem (P_4)



Problem (P_5)

4 Problems (P_1) and (P_5)

The functional $\Phi : X \longrightarrow \mathbb{R}$ in each of these two problems belongs to the class of the so-called *coercive* functionals.

Definition. $\Phi : X \longrightarrow \mathbb{R}$ is said to be *coercive* if $\Phi[u] \longrightarrow +\infty$ as $\|u\| \to \infty$.

Let us check that the functionals associated with problems (P_1) and (P_5) are coercive:

$$\Phi_1[u] = \frac{1}{2}\|u\|^2 - \Psi_1[u] \ , \text{ where } \Psi_1[u] = \frac{1}{4}\int_0^\pi u(t)^2\, dt - \int_0^\pi \rho(t)u(t)\, dt \ ,$$

$$\Phi_5[u] = \frac{1}{2}\|u\|^2 - \Psi_5[u] \ , \text{ where } \Psi_5[u] = -\frac{1}{4}\int_0^\pi u(t)^4\, dt \ .$$

Since $\Psi_5[u] \le 0$, it is clear that $\Phi_5[u]$ is coercive, for $\Phi_5[u] \ge \frac{1}{2}\|u\|^2 \longrightarrow +\infty$ as $\|u\| \to \infty$. On the other hand, in order to show that $\Phi_1[u]$ is coercive we shall use the following important lemma (which also has a higher dimensional version), whose proof the reader may skip for now:

Lemma 4.1. *(Poincaré's Inequality) For any $u \in X$ one has*

$$\int_0^\pi u(t)^2\, dt \le \int_0^\pi u'(t)^2\, dt \ .$$

Proof. Let us consider the case in which $u \in C_0^1[0,\pi]$. We will need to use some basic knowledge of Fourier Series, including the Parseval's identity. Let us consider the *odd extension* $\widehat{u}(t)$ of $u(t)$ to the interval $[-\pi,\pi]$ ($\widehat{u}(t) = -u(-t)$ for $-\pi \le t \le 0$). It has the Fourier series

$$\widehat{u}(t) = a_0 + \sum_{n=1}^{\infty}[a_n cos(nt) + b_n sin(nt)]$$

where $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\widehat{u}(t)dt$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\widehat{u}(t)cos(nt)dt$, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\widehat{u}(t)sin(nt)dt$, $(n = 1,2,\ldots)$. Since $\widehat{u}(t)$ is an *odd* function, we obtain that $a_n = 0$, $n = 0,1,2,\ldots$ and, so,

$$\widehat{u}(t) = \sum_{n=1}^{\infty} b_n sin(nt) \ , \quad t \in [-\pi,\pi] \ .$$

From this, we conclude that the Fourier series of $\widehat{u}'(t)$ is

$$\widehat{u}'(t) = \sum_{n=1}^{\infty} nb_n cos(nt) \ , \quad t \in [-\pi, \pi] \ .^*$$

Now, the corresponding Parseval's identities for $\widehat{u}(t)$ and $\widehat{u}'(t)$ are

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \widehat{u}(t)^2 \ dt = \frac{1}{2}\sum_{n=1}^{\infty} b_n^2 \ , \tag{4.1}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \widehat{u}'(t)^2 \ dt = \frac{1}{2}\sum_{n=1}^{\infty} n^2 b_n^2 \ . \tag{4.2}$$

Therefore, it follows from (4.1) and (4.2) that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \widehat{u}(t)^2 \ dt = \frac{1}{2}\sum_{n=1}^{\infty} b_n^2 \leq \frac{1}{2}\sum_{n=1}^{\infty} n^2 b_n^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \widehat{u}'(t)^2 \ dt \ . \tag{4.3}$$

Finally, since $\int_{-\pi}^{\pi} \widehat{u}(t)^2 dt = 2\int_0^{\pi} u(t)^2 dt$ and $\int_{-\pi}^{\pi} \widehat{u}'(t)^2 dt = 2\int_0^{\pi} u'(t)^2 dt$, we conclude from (4.3) that Poincaré's inequality holds true for $u \in C_0^1[0,\pi]$, that is,

$$\int_0^{\pi} u(t)^2 \ dt \leq \int_0^{\pi} u'(t)^2 \ dt \ .$$

It can then be shown that Poincaré's inequality also holds true for an arbitrary $u \in X$.[†] □

Next, we recall the well-known inequality of *Cauchy-Schwarz* (see [6] for a proof):

Lemma 4.2. *(Cauchy-Schwarz's Inequality) For any $v, w \in C[0,\pi]$ one has*

$$\left|\int_a^b v(t)w(t) \ dt\right| \leq \left(\int_a^b v(t)^2 \ dt\right)^{\frac{1}{2}} \left(\int_a^b w(t)^2 \ dt\right)^{\frac{1}{2}} := \|v\|_2 \|w\|_2 \ .^{\ddagger}$$

Finally, applying Lemmas 4.1 and 4.2 to $\Psi_1[u]$ we obtain the estimate

$$\Psi_1[u] \leq \frac{1}{4}\|u\|^2 + \|\rho\|_2 \|u\| \ ,$$

[*] We remark that the equality sign must be suitably interpreted in the 'mean-square sense'.
[†] As you may guess, the proof of this statement uses the fact that X is the *completion* of $C_0^1[0,\pi]$.
[‡] In fact, the functions v, w can be more general than continuous functions.

hence

$$\Phi_1[u] \geq \frac{1}{2}\|u\|^2 - \frac{1}{4}\|u\|^2 - \|\rho\|_2\|u\| = \frac{1}{4}\|u\|^2 - \|\rho\|_2\|u\| ,$$

from which it is clear that $\Phi_1[u] \longrightarrow +\infty$ as $\|u\| \to \infty$. We can now state

Lemma 4.3. *Each of the functionals $\Phi_1[u]$ and $\Phi_5[u]$ is coercive on the space X.*

Remark 4.4. *It is somewhat intuitive (and, in fact, true) that, by using the notio of 'proximity' given by our definition of norm in (3.5), each of the functionals $\Phi_1[u$ and $\Phi_5[u]$ is continuous. In other words, denoting either Φ_1 or Φ_5 by Φ, we have*

$$\Phi[u+h] \longrightarrow \Phi[u] \text{ as } \|h\| \longrightarrow 0 .$$

The proof of this fact is not very hard, but it's not obvious either. It uses Lemma 4.1 and 4.2 together with the identity $(A^2 - B^2) = (A+B)(A-B)$ and the fact tha all the functions u in X turn out to be bounded[§]*. The reader should not worry abou it, if he feels comfortable in accepting it.*

Remark 4.5. *In case we were working with a finite-dimensional space like $\mathbb{R}^n$, the by having a* coercive *function $F : \mathbb{R}^n \longrightarrow \mathbb{R}$ which was also* continuous, *we would b able to conclude the existence of a point x_0 of* global minimum *for $F(x)$:*

$$F(x) \geq F(x_0) \text{ for all } x \in \mathbb{R}^n . \quad (4.4$$

Indeed, one first notices that, in view of the coercivity of the function $F(x)$, it sufficε to look for such a point x_0 in a a suitable set which is 'bounded' and 'closed' (≡ 'compact'), say a closed ball $B_R(0)$ of some appropriate radius $R > 0$ centered at th origin $x = 0$ (The coercivity of $F(x)$ allows one to pick $R > 0$ so that $F(x) > F(0$ for all $\|x\| > R$ and this insures that a point of minimum (should it exist) mu: necessarily lie inside $B_R(0)$).

Next, we point out the following two facts which the reader may (or may not know [¶]:

[§]See the proof of Lemma 5.1 (i).

[¶]These results are due to the mathematicians Bolzano and Weierstrass of the nineteenth centur

Fact 1. *A closed ball $B_R(0) \subset \mathbb{R}^n$ is a compact set, i.e., any sequence of points in $B_R(0)$ possesses a subsequence converging to a point in $B_R(0)$.*

Fact 2. *A continuous function $F : K \longrightarrow \mathbb{R}$ defined on a compact set $K \subset \mathbb{R}^n$ 'attains' a minimum value,*[||] *i.e., there exists x_0 in K such that $F(x) \geq F(x_0)$ for all x in K.*

Finally, using the above facts, we can readily conclude that (4.4) holds true. It is important to point out that the key ingredients in the above arguments are compactness of the closed ball $B_R(0) \subset \mathbb{R}^n$ and continuity of the restricted function $F : B_R(0) \longrightarrow \mathbb{R}$ (the coercivity of $F(x)$ in the whole space $\mathbb{R}^n$ was used to reduce the problem to the ball $B_R(0)$).

The unfortunate reality in our present situation is that we are no longer able to reach a conclusion analogous to (4.4) for our functionals $\Phi_1[u]$ and $\Phi_5[u]$ so readily as we have done above, simply because our underlying space X **is infinite-dimensional**! Indeed, in this infinite-dimensional setting it turns out that **the closed ball $B_R(0) \subset X$ is not a compact set!**[**]

Therefore, since we can no longer rely on the fact that a bounded sequence possesses a subsequence that converges (in the sense of the norm defined in (3.5)), our next task is twofold:

(a) We must define a suitable 'new notion of convergence' (of sequences) under which bounded sequences possess a convergent subsequence.

(b) We must define a suitable 'new notion of continuity' (for the functional $\Phi : X \longrightarrow \mathbb{R}$) which is 'compatible' with the new notion of convergence in the sense that $\lim \Phi[u_n] = \Phi[u]$ *whenever u_n 'converges' (in the new sense) to u: in fact, as we shall see, we will only need Φ to be 'lower-semicontinuous' with respect to the new notion of convergence.*

[||] It also attains a maximum value, but we are not concerned with that now.

[**] Intuitively, the infinitude of independent 'axes' in X allows one to pick an element of fixed norm $r \leq R$ in each of those 'axes' and form a sequence that goes 'around' and 'around' with no convergent subsequence.

Definition A. A sequence (u_n) in X **converges weakly** to u (which we write as $u_n \rightharpoonup u$) if we have $\langle u_n, h\rangle \longrightarrow \langle u, h\rangle$ for all h in X, that is,

$$\int_0^\pi u_n'(t)h'(t)\,dt \longrightarrow \int_0^\pi u'(t)h'(t)\,dt \quad \forall\, h \in X\ .$$

Definition B. A functional $\Phi : X \longrightarrow \mathbb{R}$ is called **weakly continuous** if

$$\Phi[u] = \lim \Phi[u_n] \text{ whenever } u_n \rightharpoonup u\ .$$

In fact, Φ is said to be **weakly lower-semicontinuous** (resp. **weakly upper-semicontinuous**) if $\Phi[u] \le \liminf \Phi[u_n]$ (resp. $\Phi[u] \ge \limsup \Phi[u_n]$) whenever $u_n \rightharpoonup u$. Thus, Φ is weakly continuous if and only if it is weakly lower-semicontinuous and weakly upper-semicontinuous.[††]

These definitions indeed allow us to accomplish the tasks (a), (b) we described above, since it is possible to prove the following results:

Lemma 4.6. The closed ball $B_R(0) \subset X$ is weakly compact, *i.e., given any sequence (u_n) with $\|u_n\| \le R$ there exists a subsequence (u_{n_k}) and $\tilde{u}$ with $\|\tilde{u}\| \le R$ such that $u_{n_k} \rightharpoonup \tilde{u}$.*

Lemma 4.7. *The* **norm** $N[u] = \|u\|$ $u \in X$ **is weakly lower-semicontinuous.**[‡‡]

Lemma 4.8. *Given a continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$,* **the functional** $\Psi[u] = \int_0^\pi f(u(t))\,dt$, $u \in X$ **is weakly continuous.**

We are now ready to state the main result of this section which guarantees the existence of a *global minimum* for our functionals $\Phi_1[u]$ and $\Phi_5[u]$. This result summarizes the so called 'Direct Method of the Calculus of Variations' and is the culmination of the efforts of many mathematicians of the second half of the nineteenth and first half of the twentieth centuries, notably, Hilbert, Lebesgue, Tonelli and Weierstrass.

[††]The reader should review/learn the definitions of *liminf* and *limsup* for sequences of real numbers.

[‡‡]N[u] is not weakly continuous, as it is not weakly upper-semicontinuous.

Theorem 4.9. (**Basic Minimization Theorem**) *Let X be a Hilbert space* and assume that a given functional $\Phi : X \longrightarrow \mathbb{R}$ is*

(*i*) *coercive,*

(*ii*) *weakly lower-semicontinuous.*

Then, $\Phi[u]$ is bounded from below and there exists $u_0 \in X$ such that

$$\Phi[u] \geq \Phi[u_0] \text{ for all } u \in X \ .$$

Proof. Using the coercivity of the functional Φ, pick $R > 0$ such that $\Phi[u] > \Phi[0]$ for all $\|u\| > R$. Then, the we may restrict our attention to the closed ball $B_R(0) \subset X$. We start by claiming that the *infimum* $m := inf_{u \in X} \Phi[u]$ is finite. Indeed, if we had $\lim \Phi[v_n] = -\infty$ for some sequence (v_n) in $B_R(0)$ then, by Lemma 4.6, there would exist $\tilde{v}$ in $B_R(0)$ and a subsequence (v_{n_k}) such that $v_{n_k} \rightharpoonup \tilde{v}$. But then, by the weak lower-semicontinuity of $\Phi[u]$, we would reach the absurd conclusion that $\Phi[\tilde{v}] \leq \liminf \Phi[v_{n_k}] = -\infty$. Therefore,

$$\Phi[u] \geq m > -\infty \quad \forall \ u \in X \ . \tag{4.5}$$

Now, let $u_n \in B_R(0)$ be a minimization sequence for Φ:

$$\lim \Phi[u_n] = m \ .$$

Again, using the weak compactness of $B_R(0)$ given by Lemma 4.6, there exists $u_0 \in B_R(0)$ and a subsequence (u_{n_k}) of u_n such that $u_{n_k} \rightharpoonup u_0$. And, by the weak lower-semicontinuity of Φ, we conclude that

$$\Phi[u_0] \leq \liminf \Phi[u_{n_k}] = m \ ,$$

hence $\Phi[u_0] = m$ and

$$\Phi[u] \geq \Phi[u_0] \text{ for all } u \in X \ ,$$

in view of (4.5). This completes the proof. □

*A Hilbert space is an 'inner-product' space which is 'complete'.

Remark 4.10. *We should observe that if a given functional $\Phi : X \longrightarrow \mathbb{R}$ ha* directional derivatives

$$D\Phi[\widehat{u}] \cdot h := \lim_{\delta \to 0} \frac{\Phi[\widehat{u} + \delta h] - \Phi[\widehat{u}]}{\delta} \quad , \quad \forall\, h \in X \ ,$$

at some $\widehat{u} \in X$ which is either a point of local minimum *or of* local maximum *then necessarily, $D\Phi[\widehat{u}] \cdot h = 0$ for all $h \in X$. The proof of this fact is exactly the sam as the one we saw in Calculus. From this observation, it follows that the point u_0 o* global minimum *obtained in the Basic Minimization Theorem is a* critical point *o the functional Φ in case Φ has directional derivatives at every point in X.*

Finally, we turn our attention back to the functionals $\Phi_1[u]$ and $\Phi_5[u]$. We recal that all the functionals $\Phi_1[u], \ldots, \Phi_5[u]$ are of the form

$$\Phi[u] = Q[u] - \Psi[u] \ ,$$

where $Q[u] = \frac{1}{2}||u||^2$ is *weakly lower-semicontinuous* (w.l.s.c.) by Lemma 4.7 anc $\Psi[u]$ is *weakly continuous* (w.c.) by Lemma 4.8. Therefore, it is not hard to see thal $\Phi[u]$ is w.l.s.c. And, since we have already seen in Lemma 4.3 that $\Phi_1[u]$ and $\Phi_5[u$ are coercive, we may use our basic minimization result, Theorem 4.9, to conclud that each of these functionals attains a global minimum u_0. Moreover, in view o Remark 4.10 above and our discussion in Section 3, any such u_0 is a solution of th corresponding problem (P_1) or (P_5). It is now clear that the corresponding (unique) solutions $u_0(t) = -2 \ sin\ t$ and $u_0(t) \equiv 0$ pointed out in the beginning of section 3 are precisely the global minima given by Theorem 4.9 .

5 Problem (P_4)

In this section we will see that the functional $\Phi_4 : X \longrightarrow \mathbb{R}$ corresponding to problem (P_4) is no longer a *coercive* functional. Moreover, we will describe some of its 'geometry' (cf. picture in section 3).

First, we set some useful notation which will be used in this and in the next sections.

Let $\phi_n(t) := sin(nt)$, $e_n(t) := \sqrt{\frac{2}{\pi}\frac{1}{n}}sin(nt)$, $n = 1, 2, \ldots$, and observe that

$$\int_0^\pi sin^2(nt)\,dt = \int_0^\pi cos^2(nt)\,dt = \frac{\pi}{2}\,, \tag{5.1}$$

$$\|\phi_n\|^2 = \int_0^\pi n^2 cos^2(nt)\,dt = \frac{\pi}{2}n^2\,,\quad \|e_n\|^2 = \int_0^\pi \frac{2}{\pi}cos^2(nt)\,dt = 1\,, \tag{5.2}$$

$$\langle \phi_m, \phi_n \rangle = 0 \text{ if } m \neq n\,. \tag{5.3}$$

In particular, keeping in mind that a function $u \in X$ has the Fourier expansion $u(t) = \sum_{n=1}^\infty b_n sin(nt)$ (cf. Lemma 4.1), we see that both the ϕ_n's and the e_n's form an *'orthogonal basis'* for the space X (Note that the e_n's are 'normalized').

Now, we recall that the functional corresponding to problem (P_4) is given by

$$\Phi_4[u] = \frac{1}{2}\int_0^\pi u'(t)^2\,dt - \frac{1}{4}\int_0^\pi u(t)^4\,dt = Q[u] - \Psi_4[u]\,. \tag{5.4}$$

In this case, since $\Psi_4[u]$ is *homogeneous of degree four*, we intuitively expect that the quadratic term $Q[u]$ will be the 'dominant' term for $\|u\|$ 'small' and, on the other hand, it will be somehow 'dominated' by the negative term $-\Psi_4[u]$ for $\|u\|$ 'large'. In fact, as we shall see in the next lemma, the (trivial) solution $u_0(t) \equiv 0$ of problem (P_4) turns out to be a *strict local minimum* of the functional $\Phi_4[u]$ and, on the other hand, along any *ray* $0 \leq \tau \mapsto \tau v$ (with $v \neq 0$), the functional $\Phi_4[u]$ tends to $-\infty$.*

Lemma 5.1. (i) *There exists $r > 0$ and $b_\rho > 0$ for $\rho \in (0, r]$ such that*

$$\Phi_4[u] \geq b_\rho > 0 \text{ for } \|u\| = \rho\,;$$

(ii) *For each $v \neq 0$, we have that $\Phi_4[\tau v] \longrightarrow -\infty$ as $\tau \to \infty$;*

(iii) *The functional Φ_4 is not bounded from above (i.e., for each $M > 0$, there exists $u \in X$ such that $\Phi_4[u] \geq M$).*

Proof. (i) Let $u \in X$ be arbitrary (for simplicity, you may assume that $u \in C_0^1[0, \pi]$, although the exact same argument works for $u \in X$). In view of the Fundamental

*Note that, since the space X is infinite-dimensional, this does not mean that the functional $\Phi_4[u]$ is bounded from above! Indeed, Lemma 5.1 (iii) will show that it is not!

Theorem of Calculus we can write

$$u(t) = \int_0^t u'(s)\ ds\ ,$$

so that Cauchy-Schwarz's Inequality (Lemma 4.2) and Poincaré's Lemma (Lemma 4.1) yield

$$|u(t)| \leq \left(\int_0^t 1^2\ ds\right)^{\frac{1}{2}} \left(\int_0^t u'(s)^2\ ds\right)^{\frac{1}{2}} \leq \pi^{\frac{1}{2}} \left(\int_0^\pi u'(s)^2\ ds\right)^{\frac{1}{2}},$$

hence

$$|u(t)|^4 \leq \pi^2 \|u\|^4 \quad \text{for all } u \in X\ .$$

This estimate implies the conclusion stated in (i) since we get

$$\Phi_4[u] \geq \frac{1}{2}\|u\|^2 - \frac{\pi^3}{4}\|u\|^4 \geq \frac{1}{4}\|u\|^2 \quad \text{if } \|u\| \leq \pi^{-\frac{3}{2}} := r\ .$$

(ii) Let $v \in X$ be given with $v \neq 0$. Then, it is easy to see that

$$\Phi_4[\tau v] = \frac{1}{2}A\tau^2 - \frac{1}{4}B\tau^4\ ,$$

where $A = \int_0^\pi v'(t)^2 dt > 0$ and $B = \int_0^\pi v(t)^4 dt > 0$. Therefore, we have that $\Phi_4[\tau v] \longrightarrow -\infty$ as $\tau \to \infty$.

(iii) Let $M > 0$ be given. Recalling the definition of $\phi_n(t)$, (5.1), and the fact that $sin^4(nt) = \frac{1}{4}(1 - cos(2nt))^2$ we calculate

$$\begin{aligned} \Phi_4[\phi_n] &= \frac{1}{2}\int_0^\pi n^2 cos^2(nt)\ dt - \frac{1}{4}\int_0^\pi sin^4(nt)\ dt \\ &= \frac{\pi}{4}n^2 - \frac{1}{16}\int_0^\pi [1 - 2cos(2nt) + cos^2(2nt)]\ dt\ , \end{aligned}$$

hence

$$\Phi_4[\phi_n] = \frac{\pi}{4}n^2 - \frac{3\pi}{32} \geq M\ ,$$

if $n \geq N$ for some N. This completes the proof of Lemma 5.1. □

We are now ready to answer **Question 0** which we posed in Section 2, namely, of whether there was an analogue of Rolle's Theorem in dimension $n > 1$. An affirmative answer can be given by the celebrated *Mountain-Pass Theorem* of Ambrosetti and Rabinowitz [2].

Theorem 5.2. (**The Mountain-Pass Theorem**) *Let X be a Hilbert space and let $\Phi : X \longrightarrow \mathbb{R}$ be a 'continuously differentiable functional'* † *such that $\Phi[0] = 0$. Assume that $\Phi[u]$ satisfies the so-called* Palais-Smale condition‡ *and that the following* geometric conditions *hold true:*

(*a*) *There exist $b > 0$ and $r > 0$ such that $\Phi[u] \geq b > 0$ for $||u|| = r$;*

(*b*) *There exists $e \in X$ with $||e|| > r$ such that $\Phi[e] \leq 0$.*

Then, $\Phi[u]$ has a critical point $\widehat{u} \in X$ with $\Phi[\widehat{u}] = c \geq b > 0$, where c is the 'minimax value'

$$c := \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} \Phi[\gamma(t)] \ , \tag{5.5}$$

and $\Gamma := \{\gamma : [0,1] \to X \mid \gamma$ is continuous, $\gamma(0) = 0$, $\gamma(1) = e\}$ is the class of all paths *joining* 0 *and* e.

Remark 5.3. *Let us try to 'visualize' the Mountain-Pass Theorem when $X = \mathbb{R}^2$ (cf. picture for problem (P_4) in section 3) and the functional $\Phi[u]$ represents the 'topography' of a certain terrain. Roughly speaking, in this situation the conditions of the Mountain-Pass Theorem say that the origin $u = (0,0)$ is at the 'level zero' and is surrounded by a 'mountain-range' located at a distance $r > 0$ with (possibly varying) heights of at least $b > 0$. In addition, there is a point e on the terrain 'outside the mountain-range' which is at the 'level zero' (or less). Therefore, if someone standing at the origin $u = (0,0)$ wishes to 'cross' the 'mountain-range' and travel toward the point e, he/she may want to do so by choosing the 'lowest possible passage in the mountain-range',* **the mountain-pass**, *which should be at the 'minimax' height $c \geq b$ defined in (5.5). The 'tangent plane' at a point $(\widehat{u}, \Phi[\widehat{u}])$ at such a level c must then be 'horizontal', in other words, $\widehat{u}$ must be a critical point of the functional $\Phi[u]$.*

†This means that all *directional derivatives* $D\Phi[u].\cdot h$ exist and depend continuously on $u, h \in X$.

‡This is a 'compactness condition' meaning that any sequence (u_n) such that $|\Phi[u_n]|$ is bounded and $|D\Phi[u_n] \cdot h| \leq \epsilon_n ||h|| \ \forall h \in X$, with $\epsilon_n \to 0$, necessarily has a convergent subsequence.

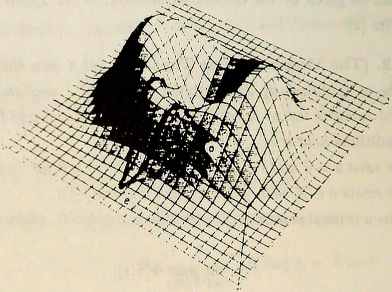


Figure 5

Remark 5.4. *In view of our discussion in Section 3, the functional* $\Phi_4[u]$ *(as well as any of the other* $\Phi_i[u]$*'s) is differentiable. It fact, it can be shown (but we will not do it in these notes) that each functional* $\Phi_i[u]$ *is* continuously differentiable. *Moreover,* $\Phi_4[u]$ *can be shown to satisfy the above mentioned* Palais-Smale condition *[8].*

Theorem 5.5. *Problem* (P_4) *has a nonzero solution* $\widehat{u}$ *.*

Proof. We will use the Mountain-Pass Theorem to find $\widehat{u}$. It is clear that $\Phi_4[0] = 0$ and, according to Remark 5.4, $\Phi_4[u]$ is continuously differentiable and satisfies the Palais-Smale condition . So, it remains to verify the 'geometric' assumptions (a) and (b) of Theorem 5.2. Indeed, (a) follows immediately from Lemma 5.1 (i) by (say) taking $b = b_r > 0$. On the other hand, Lemma 5.1 (ii) shows that there are many $e \in X$ satisfying assumption (b) (Indeed, given any 'direction' $v \neq 0$, we can take $e = \tau v$ with τ sufficiently large).

Therefore, the Mountain-Pass Theorem yields a critical point $\widehat{u}$ with $\Phi_4[\widehat{u}] = c \geq b > 0$. In particular, we have $\widehat{u} \neq 0$ because $\Phi_4[0] = 0$. And, according to our discussion in Section 3, we conclude that $\widehat{u}$ is a solution of Problem (P_4). □

6 Problem (P_3)

In this section we will see that the functional $\Phi_3 : X \longrightarrow \mathbb{R}$ corresponding to problem (P_3),

$$\Phi_3[u] = \frac{1}{2}\int_0^\pi u'(t)^2\,dt - \int_0^\pi u(t)^2\,dt + \int_0^\pi sin(t)u(t)\,dt\,, \tag{6.1}$$

is also *non-coercive*. And, we will describe its 'geometry' (cf. picture in section 3).

Let us denote by $V := span\{\phi_1\}$ the (one-dimensional) subspace spanned by ϕ_1, and by $W := \overline{span}\{\phi_n \mid n \geq 2\}$ the (closed, infinite-dimensional) subspace spanned by $\phi_2, \phi_3, \ldots$. Then, we can summarize some of *geometric properties* of $\Phi_3[u]$ through the following

Lemma 6.1. *We have the 'direct sum decomposition' $X = V \oplus W$ (i.e., each $u \in X$ can be written uniquely as $u = v + w$ with $v \in V$ and $w \in W$) with the property that:*

(i) $\Phi_3[v] \longrightarrow -\infty$ as $\|v\| \to \infty$, $v \in V$ (i.e., $\Phi_3[\tau\phi_1] \longrightarrow -\infty$ as $|\tau| \to \infty$);

(ii) $\Phi_3[w] \longrightarrow +\infty$ as $\|w\| \to \infty$, $w \in W$.

Proof. The 'direct sum decomposition' $X = V \oplus W$ results from the fact an arbitrary $u \in X$ has the (unique) Fourier expansion

$$u(t) = \sum_{n=1}^{\infty} b_n sin(nt)\,, \quad b_n = \frac{2}{\pi}\int_0^\pi u(t) sin(nt)\,dt\,.$$

In order to prove (i) we use (5.1), (5.2) to calculate

$$\Phi_3[\tau\phi_1] = \frac{1}{2}\int_0^\pi \tau^2 cos^2(t)\,dt - \int_0^\pi \tau^2 sin^2(t)\,dt + \int_0^\pi \tau sin^2(t)\,dt = -\frac{\pi}{4}\tau^2 + \frac{\pi}{2}\tau$$

and conclude that

$$\Phi_3[\tau\phi_1] \longrightarrow -\infty \text{ as } |\tau| \to \infty\,,$$

in other words, $\Phi_3[v] \to -\infty$ as $\|v\| \to \infty$, $v \in V$.

(ii) Now, the Fourier expansion of an arbitrary $w \in W$ is given by

$$w(t) = \sum_{n=2}^{\infty} b_n sin(nt)\,.$$

Therefore, the same *Parseval-type* arguments as in Lemma 4.1 give the estimates

$$\frac{1}{\pi}\int_0^\pi w(t)^2\,dt = \frac{1}{2}\sum_{n=2}^\infty b_n^2 \le \frac{1}{2}\frac{1}{4}\sum_{n=2}^\infty n^2 b_n^2 = \frac{1}{4\pi}\int_0^\pi w'(t)^2\,dt\,,$$

which imply the following *Poincaré Inequality* for $w \in W$:

$$\int_0^\pi w(t)^2\,dt \le \frac{1}{4}\int_0^\pi w'(t)^2\,dt\,. \tag{6.2}$$

Using the above inequality in the first two terms of (6.1) and the Cauchy-Schwarz's inequality (Lemma 4.2) in the third term, it follows that

$$\Phi_3[w] \ge \left(\frac{1}{2}-\frac{1}{4}\right)\int_0^\pi w'(t)^2\,dt - \left(\int_0^\pi sin^2(t)\,dt\right)^{\frac{1}{2}}\left(\int_0^\pi w(t)^2\,dt\right)^{\frac{1}{2}},$$

so that, by again using (5.1) and (6.2), we obtain

$$\Phi_3[w] \ge \frac{1}{4}\|w\|^2 - \sqrt{\frac{\pi}{2}}\frac{1}{2}\|w\|\,.$$

Therefore, $\Phi_3[w] \to +\infty$ as $\|w\| \to \infty$, $w \in W$, which completes the proof. □

Remark 6.2. *We should observe that the arguments of the above proof can be used to show the following more general* Poincaré Inequalities *for elements of the space X, where $V_k := span\{\phi_n \mid n \le k\}$ [the finite-dimensional subspace spanned by $\phi_1, \ldots, \phi_k$] and $W_{k+1} := \overline{span}\{\phi_n \mid n \ge k+1\}$ [the closed, infinite-dimensional subspace spanned by $\phi_{k+1}, \ldots$]:*

$$\int_0^\pi u'(t)^2\,dt \le k^2\int_0^\pi u(t)^2\,dt \text{ for all } u \in V_k\,, \tag{6.3}$$

$$(k+1)^2\int_0^\pi u(t)^2\,dt \le \int_0^\pi u'(t)^2\,dt \text{ for all } u \in W_{k+1}\,, \tag{6.4}$$

where $k = 1, 2, \ldots$. The coefficient k^2 appearing in the right-hand-side of (6.3) is precisely the k^{th}-eigenvalue λ_k of the problem

$$\begin{cases} -u'' = \lambda u \\ u(0) = 0\,,\ u(\pi) = 0 \end{cases} \tag{6.5}$$

Next, we introduce our second minimax result, the *Saddle-Point Theorem* of Rabinowitz [7].

Theorem 6.3. (**The Saddle-Point Theorem**) *Let X be a Hilbert space having the direct sum decomposition $X = V \oplus W$, with $\dim V < \infty$, and let $\Phi : X \longrightarrow \mathbb{R}$ be a continuously differentiable functional satisfying the Palais-Smale condition and the following geometric conditions:*

(*A*) *There exists $b > 0$ such that*

$$\Phi[w] \geq b > 0 \; \forall \; w \in W \; ;$$

(*B*) *There exists $a < b$ and $R > 0$ such that*

$$\Phi[v] \leq a < b \text{ for } v \in V, \; \|v\| = R \; .$$

Then, $\Phi[u]$ has a critical point $\widehat{u} \in X$ with $\Phi[\widehat{u}] = c \geq b > 0$, where c is the 'minimax value'

$$c := \inf_{h \in \Gamma} \; \max_{v \in V, \|v\| \leq R} \; \Phi[h(v)] \; , \tag{6.6}$$

and $\Gamma := \{h : B_R(0) \cap V \to X \mid h \text{ is continuous}, \; h(v) = v \;\; if \;\; \|v\| = R\}$ is the class of all deformations *of the closed ball $B_R \cap V$ in V which 'fixes' each point of its boundary.*

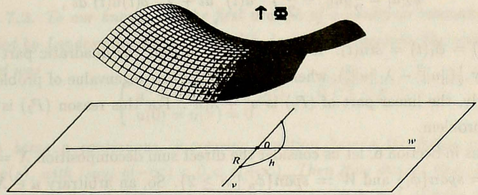


Figure 6

Theorem 6.4. *Problem* (P_3) *has a nonzero solution* $\widehat{u}$.

Proof. We will use the Saddle-Point Theorem to find $\widehat{u}$. According to Remark 5.4, $\Phi_3[u]$ is continuously differentiable. Also, since the quadratic part of $\Phi_3[u]$ in (6.1) is of the form $\frac{1}{2}(\|u\|^2 - \lambda\|u\|_2^2)$ where $\lambda = 2$ is not an eigenvalue of the problem (6.5), it is possible to show that $\Phi_3[u]$ satisfies the Palais-Smale condition. So, it remains to verify the 'geometric' assumptions (A) and (B) of Theorem 6.3. Indeed, Lemma 6.1 (ii) says that the functional Φ_3 restricted to the subspace W is coercive, so that an application of the Basic Minimization Theorem 4.9 implies (A). On the other hand, if we pick any $a < b$, an application of Lemma 6.1 (i) shows that we can find $R > 0$ satisfying assumption (B).

Therefore, the Saddle-Point Theorem yields a critical point $\widehat{u}$ with $\Phi_3[\widehat{u}] = c \geq b > 0$. Again, we have $\widehat{u} \neq 0$ because $\Phi_3[0] = 0$, and we conclude that $\widehat{u}$ is a nonzero solution of Problem (P_3). In fact, $\widehat{u}(t)$ is the unique solution $u_0(t) = sint$ pointed out in the beginning of section 3 . □

7 Problem (P_2)

In this last section, we consider problem (P_2) and show that it has no solution by analysing the geometry of its corresponding functional

$$\Phi_2[u] = \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_0^\pi u(t)^2\, dt + \int_0^\pi \rho(t)u(t)\, dt\ ,$$

where $\rho(t) = \phi_1(t) = sin(t)$. It should be noted that the quadratic part of $\Phi_2[u]$ is given by $\frac{1}{2}(\|u\|^2 - \lambda_1\|u\|_2^2)$, where $\lambda_1 = 1$ is the first eigenvalue of problem (6.5): equivalently, the linear part of (P_2) is $u'' + \lambda_1 u$. For this reason (P_2) is called a *resonant* problem.

Now, as in Section 6, let us consider the direct sum decomposition $X = V \oplus W$, where $V = span\{\phi_1\}$ and $W := \overline{span}\{\phi_n \mid n \geq 2\}$. So, an arbitrary $u \in X$ can be uniquely written as $u = \tau\phi_1 + w$ for some $\tau \in \mathbb{R}$ and $w \in W$. Then, observing that (5.3) implies

$$\int_0^\pi \phi_1'(t)w'(t)\, dt = \int_0^\pi \phi_1(t)w(t)\, dt = 0\ ,$$

a simple calculation shows that

$$\Phi_2[u] = \Phi_2[\tau\phi_1 + w] = \frac{\pi}{2}\tau + \Phi_2[w] \,. \tag{7.1}$$

In particular, we infer from (7.1) that $\Phi_2[u]$ has <u>no</u> critical point. Indeed, given an arbitrary $u = \tau\phi_1 + w$, we obtain

$$\begin{aligned} D\Phi_2[u]\cdot\phi_1 &= \lim_{\delta\to 0}\frac{1}{\delta}\left(\Phi_2[\tau\phi_1 + w + \delta\phi_1] - \Phi_2[\tau\phi_1 + w]\right) \\ &= \lim_{\delta\to 0}\frac{1}{\delta}\left(\frac{\pi}{2}(\tau+\delta) - \frac{\pi}{2}\tau\right) = \frac{\pi}{2}\,. \end{aligned}$$

Remark 7.1. *Linear* resonant *problems such as problem* (P_2) *do not always have a solution. Using Fourier expansions, it is not very hard to show that problem*

$$(P_2)_\rho \qquad \begin{cases} u'' + u = \rho(t) \quad , \quad 0 < t < \pi \\ u(0) = u(\pi) = 0 \end{cases}$$

has a solution if and only if

$$\int_0^\pi \rho(t)sin(t)dt = 0 \,.$$

This is a special case of the so-called Fredholm Alternative. *The case we considered was precisely the one with* $\rho(t) = sin(t)$*, which clearly violates the above orthogonality condition.*

Remark 7.2. *To our knowledge, the first version of a nonlinear* resonant *problem was studied by Landesman and Lazer in [5]. In its ODE version, the problem they considered was*

$$(P)_\rho \qquad \begin{cases} u'' + u + g(u) = \rho(t) \quad , \quad 0 < t < \pi \\ u(0) = u(\pi) = 0 \end{cases}$$

with $g(s)$ *being a continuous, increasing function having the limits* $g(\pm\infty) = \lim_{s\to\pm\infty} g(s)$*, with (say)* $g(-\infty) < 0 < g(+\infty)$*. Then they showed that* $(P)_\rho$ *has a solution provided that the given r.h.s. continuous function* $\rho(t)$ *satisfies the condition*

$$(LL) \qquad g(-\infty)\int_0^\pi sin(t)\, dt < \int_0^\pi \rho(t)sin(t)\, dt < g(+\infty)\int_0^\pi sin(t)\, dt \,.$$

An alternative proof of this result using the Saddle-Point Theorem *and which works for the more general situation where*

(*i*) $(\rho(t) - g(u))$ *is replaced by a bounded continuous function* $g(t,u)$,

(*ii*) (LL) *is replaced by* $\lim_{|u|\to\infty} G(t,u) = +\infty$,

was given by Rabinowitz [7] in 1978. In fact, Ahmad, Lazer and Paul [1] had studied this more general resonant situation by a different method in 1976. But it was precisely this problem that motivated Rabinowitz to announce and prove his celebrated Saddle-Point Theorem *in [7] and apply it first-hand to the Ahmad-Lazer-Paul situation.*

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