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Nonlinear Impulsive Differential Systems*

Manuel Pinto

Departamento de Matemáticas.
Facultad de Ciencias
Universidad de Chile. Casilla 653.
Santiago - Chile.

Abstract

Using Schauder fixed point theorem we prove the asymptotic equilibrium of nonlinear differential equations with impulse action.

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1 Introduction

Let $\nu = \{t_i\}_{i=1}^{\infty} \subset [0,\infty) = I$, $t_i < t_{i+1} \to \infty$ as $i \to \infty$ the fixed moments of impulsive effects of the system:

$$\begin{aligned} x'(t) &= F(t, x(t)), \quad t \neq t_i \\ \Delta x(t_i) &= G_i(x(t_i)), \quad t = t_i \end{aligned} \tag{1}$$

where $\Delta x(t_i) = x(t_i^+) - x(t_i^-)$. As usual, $x(t_i^+)$ and $x(t_i^-)$ denote respectively, the right and the left lateral limit of $x(t)$ as $t \to t_i$.

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The theory of impulsive differential equations has developed over the last ten years (see [1, 4-8]). These process appear as a natural description of many models in Medicine, Biology, optimal control models in Economics, etc.

Let U be an open subset of $\mathbb{C}^n$ containing the origin. Assume the following hypotheses

(F) The function $F : [0,\infty) \times U \to \mathbb{C}^n$ satisfies

$$|F(t,x)| \leq \lambda(t)|f(x)|,$$

$$|F(t,x_1) - F(t,x_2)| \leq \lambda(t)|f(x_1) - f(x_2)|$$

where $f : U \to \mathbb{C}^n$ is a continuous function.

(G) The functions $G_i : U \to \mathbb{C}^n$ $(i = 1,2,\cdots)$ satisfy

$$|G_i(x)| \leq \beta_i|g(x)|,$$

$$|G_i(x_1) - G_i(x_2)| \leq \beta_i|g(x_1) - g(x_2)|$$

where $g : U \to \mathbb{C}^n$ is a continuous function.

(I) The function $\lambda : [0,\infty) \to [0,\infty)$ is an integrable function and $\beta : \mathbb{N} \to [0,\infty)$ is an absolutely summable sequence.

Among those equations we have the interesting systems

$$\begin{aligned} x'(t) &= A(t)F(x(t)), \\ \Delta x(t_i) &= B_i g(x(t_i)). \end{aligned} \tag{2}$$

Let $B_r = B[0,r] \subseteq U$ a closed ball. Define

$$\|f\|_{B_r} = \sup_{x\in B_r} |f(x)| \tag{3}$$

and for $t_0 \geq 0$

$$\alpha(t_0) = \|f\|_{B_r} \cdot \int_{t_0}^{\infty} \lambda(s)ds + \|g\|_{B_r} \cdot \sum_{(t_0,\infty)} \beta_i, \tag{4}$$

where

$$\sum_{(t_0,t)} \beta_i := \sum_{i,t_i \in (t_0,t)} \beta_i.$$

Let be (μ, t_0) satisfying

$$\mu + \alpha(t_0) \le r \tag{5}$$

We will prove that for $|x_0| \le \mu$, any solution $x = x(t, t_0, x_0)$ of Eq. (1) is defined and bounded on $[t_0, \infty)$ and it satisfies

$$x(t) = \xi + 0(\Lambda(t)) \tag{6}$$

where $\xi \in \mathbf{C}^n$ is constant and

$$\Lambda(t) = \int_t^\infty \lambda(s)ds + \sum_{(t,\infty)} \beta_i. \tag{7}$$

Moreover, $x(t_0) \neq 0$ implies $\xi \neq 0$ whenever t_0 is sufficiently large. Conversely, given $\xi \in U$ there exist t_0 sufficiently large and a solution x defined on $[t_0, \infty)$ a neighborhood of infinity such that (6) holds. Moreover, if $\xi \neq 0$ then there exists such a solution x of Eq.(1) satisfying $x(t) \neq 0$ for any $t \in [t_0, \infty)$.

2 Preliminary Facts

Let $C_\nu^+(\mathrm{I})$, $I = [0, \infty)$, be the vectorial space formed by the continuous functions $x : I - \nu \to \mathbf{C}^n$ such that $x(t_i^-) = x(t_i)$ and $x(t_i^+)$ exist for $t_i \in \nu$. Consider $\mathcal{V}_{\nu^+}(I)$ the bounded functions x in $C_\nu^+(I)$. $\mathcal{V}_{\nu^+}$ is a Banach space with the supremun-norm:

$$\|x\| = \sup_{t \in I} |x(t)|.$$

Lemma 1 *Any $S \subset \mathcal{V}_{\nu^+}([a,b])$ bounded and equicontinuous in $t \neq t_i$ is relatively compact in $\mathcal{V}_{\nu^+}([a,b])$.*

Proof: Let $\{t_1, t_2, \cdots, t_m\}$ be a finite number of $t_i \in \nu$ contained in $[a, b]$. Next, the result will follow from Arzela-Ascoli theorem in this way. i) By applying it on any $I_i = [t_i, t_{i+1}]$ $(1 \le i \le m)$ where we consider $x(t_i) := x(t_i^+)$ ii) $N_x = max_{1 \le i \le m} |\Delta x(t_i)|$ is uniformly bounded and hence they forme a totally bounded set in $\mathbf{R}$.

Definition 1 *$S \subset \mathcal{V}_{\nu^+}([a,\infty))$ is called an equiconvergent set if any $x \in S$ converges to x_∞ as $t \to \infty$ and for any $\varepsilon > 0$ there exists T (big enough) such that*

$$|x(t) - x_\infty| \leq \varepsilon \quad \textit{for} \quad t \geq T$$

for every $x \in S$.

Lemma 2 *If $S \subset \mathcal{V}_{\nu^+}([a,\infty))$ is bounded, equicontinuous in $t \notin \nu$ and equiconvergent, then S is relatively compact.*

Proof: Given $\varepsilon > 0$, there exists $T = T(\varepsilon)$ such that $|x(t) - x_\infty| \leq \varepsilon$ for $t \geq T$ and every $x \in S$. The set $S_\infty = \{x_\infty / x \in S\}$ is contained in a ball $B(0,\rho) \subset \mathbb{C}^n$ and consequently S_∞ is totally bounded. Finally on $[a,T]$ we apply Lemma 1. So the proof is complete.

Consider the operator $\mathcal{H}$ given by

$$(\mathcal{H}x)(t) = x_0 + \int_{t_0}^{t} F(s,x(s))ds + \sum_{(t_0,t)} G_i(x(t_i)).$$

Lemma 3 *Under conditions (F), (G) and (I), the operator $\mathcal{H}$ is completely continuous.*

Proof: Let $D = D(O,r)$ a ball in $\mathcal{V}_{\nu^+}$. Assume that $\mathcal{H}x$ is defined for $x \in D$. By conditions (F), (G) and (I), the operator $\mathcal{H}$ is well defined and

$$\mathcal{H} : D \to \mathcal{V}_{\nu^+}([t_0,\infty)).$$

Now, we will prove that $\mathcal{F} = \mathcal{H}(D)$ is equiconvergent. In fact, for any $x \in D$, $y = \mathcal{H}x$ is convergent to ξ as $t \to \infty$, where

$$\xi = x_0 + \int_{t_0}^{\infty} F(s,x(s))ds + \sum_{(t_0,\infty)} G_i(x(t_i)). \tag{8}$$

Moreover,

$$|y(t) - \xi| \leq \int_{t}^{\infty} |F(s,x(s))|ds + \sum_{(t,\infty)} |G_i(x(t_i))| = 0(\Lambda(t)), \tag{9}$$

where

$$\Lambda(t) = \int_{t}^{\infty} \lambda(s)ds + \sum_{(t,\infty)} \beta_i$$

So, $\mathcal{F}$ is equiconvergent, and by Lemma 2 $\mathcal{F}$ is relatively compact. On the other hand, $\mathcal{H} : D \to \mathcal{V}_\nu$ is continuous. Suppose $x_n \to x$ in $\mathcal{V}_\nu$. We get

$$\begin{aligned} |(\mathcal{H}x_n)(t) - (\mathcal{H}x)(t)| &\leq \int_{t_0}^t |F(s, x_n(s)) - F(s, x(s))| ds \\ &+ \sum_{(t_0,t)} |G_i(x_n(t_i)) - G_i(x(t_i))| \\ &\leq \int_{t_0}^\infty \lambda(s) |f(x_n(s)) - f(x(s))| ds \\ &+ \sum_{(t_0,\infty)} \beta_i |g(x_n(t_i)) - f(x(t_i))| \end{aligned}$$

and the continuity of the operator $\mathcal{H}$ follows at once from the Lebesgue's theorem.

Remark 1: If F and G_i $(i = 1, 2, \cdots)$ are continuous functions then the second inequalities in (F) and (G) are not necessary to prove the continuity of the operator $\mathcal{H}$.

3 Main Results

Theorem 1 *Assume that conditions (F), (G) and (I) are fulfilled. Let be* (μ, t_0) *satisfying (5), where* r *will be especified below. Then for* $|x_0| \leq \mu$ *any solution* $x = x(t, t_0, x_0)$ *of Eq. (1) is defined and bounded on* $[t_0, \infty)$ *and it satisfies (6). Moreover,* $x(t_0) \neq 0$ *implies* $\xi \neq 0$ *whenever* t_0 *is sufficiently large.*

Proof: Since $0 \in U$ there exists $r > 0$ such that $B = B(0, r) \subset U$. For this r, let be μ, t_0 satisfying (5). We define

$$D = D(0, r) = \{x \in \mathcal{V}_{\nu^+}([t_0, \infty))/\|x\|_\infty \leq r\}.$$

For $x \in D$ we define the operator

$$(\mathcal{H}x)(t) = x_0 + \int_{t_0}^t F(s, x(s)) ds + \sum_{t_i \in (t_0, t)} G_i(x(t_i))$$

for $t \geq t_0$. For $x \in D$ and $|x_0| \leq \mu$ by (F) and (G) we have $\|\mathcal{H}x\| \leq \mu + \alpha(t_0) \leq r$ and hence $\mathcal{H} : D \to D$. The operator $\mathcal{H}$ is continuous by Lemma 3. Moreover, $\mathcal{H}(D)$ is compact by Lemma 3. Then the hypothesis of Schauder-Tichonov fixed point theorem is satisfied and hence the equation $\mathcal{H}x = x$ has a solution x in D. Since $y = \mathcal{H}x$ satisfies

$$y'(t) = F(t, x(t)), \quad t \neq t_i$$

$$\Delta y(t_i) = G_i(x(t_i)), \quad t = t_i$$

this fixed point is a solution of Eq (1) and it satisfies (6) by (9).

Finally, we will prove that for any x with $x(t_0) \neq 0$ there exists t_0 sufficiently large such that (6) is satisfied with $\xi \neq 0$. This follows from (8) taking $x(t_0) = x_0 \neq 0$ and t_0 large enough so that

$$\left| \int_{t_0}^{\infty} F(x, x(s))ds + \sum_{(t_0,\infty)} G_i(x(t_i)) \right| \leq |x_0|/2.$$

Theorem 2 *Assume that conditions* $(F), (G)$ *and* (I) *are fulfilled. Then for any* $\xi \in U$ *there exist* t_0 *sufficiently big and a solution* x *of (1) defined on* $[t_0, \infty)$ *such that (6) holds. Moreover, if* $\xi \neq 0$ *then there exists such a solution* x *of Eq (1) verifying* $x(t) \neq 0$ *for any* $t \in [t_0, \infty)$.

Proof: Let $\xi \in U$, then there is $r > 0$ so that the closed ball $B = B[\xi, r] \subset U$. For this r let be t_0 such that

$$\alpha(t_0) = \|f\|_r \cdot \int_{t_0}^{\infty} \lambda(s)ds + \|g\|_r \cdot \sum_{i=1}^{\infty} \beta_i \leq r, \tag{10}$$

where $\|f\|_r = max\{|f(s)|/x \in B\}$.

Let

$$D = D(\xi, r) = \{x \in \mathcal{V}_\nu([t_0, \infty))/\|x - \xi\|_\infty \leq r\}.$$

For $x \in D$, we define the operator

$$(Ax)(t) = \xi - \int_t^{\infty} F(s, x(s))ds - \sum_{t_i \geq t} G_i(x(t_i)), \quad t \geq t_0$$

where t_0 verifies (10). Since

$$|(Ax)(t) - \xi| \leq \alpha(t_0) \leq r, \quad t \geq t_0 \tag{11}$$

we get $A : D \to D$. We will now prove that A is continuous. The functions f and g are uniformly continuous on B, hence given $\varepsilon > 0$ there exists $\delta > 0$ such that $|x_1 - x_2| < \delta$ implies $|f(x_1) - f(x_2)| \leq \varepsilon$ and $|g(x_1) - g(x_2)| \leq \varepsilon$. Let $x_n \to x$ in D. Then there is N such that

$$sup_{t\in[t_0,\infty)}|x_n(t)-x(t)|\leq\delta$$

for $n\geq N$. Thus $|x_n(t)-x(t)|\leq\delta$ for every $t\geq t_0$ and $n\geq N$. Therefore $|f(x_n(t))-f(x(t))|\leq\varepsilon$ for any $t\geq t_0$ and $n\geq N$. The same is true for the function g.

Thus for $n\geq N$, we get

$$\begin{aligned}\|\mathcal{A}x_n-\mathcal{A}x\| &= sup_{t\in[t_0,\infty)}|\mathcal{A}x_n(t)-\mathcal{A}x(t)|\\ &\leq \int_{t_0}^{\infty}\lambda(s)|f(x_n(s))-f(x(s))|ds\\ &+ \sum_{i=1}^{\infty}\beta_i|g(x_n(t_i))-g(x(t_i))|\\ &\leq \varepsilon(\int_{t_0}^{\infty}\lambda(s)ds+\sum_{i=1}^{\infty}\beta_i)\end{aligned}$$

from where the continuity of $\mathcal{A}$ follows. Furthermore, D is a bounded, closed and convex set in $\mathcal{V}_{\nu^+}$. Since $\mathcal{A}(D)$ is equiconvergent by Lemma 2, $\mathcal{A}$ is a compact operator in $\mathcal{V}_{\nu^+}$.

Then the Schauder-Tichonov fixed point theorem implies that there exists a solution $x\in D$ of the equation $x=\mathcal{A}x$. This function x is solution of Eq.(1) on $[t_0,\infty)$.

Finally, if $\xi\neq 0$ then taking r small enough, as for instance $r=|\xi|/2$, from (11) we obtain that $x(t)\neq 0$ for every $t\geq t_0$.

Remark 2: The second inequalities in (F) and (G) are used in the last proof to prove the continuity of the operator $\mathcal{A}$. Since we can use also the method of the proof of Lemma 1, by Remark 1, Theorems 1 and 2 are true if F and G_i $(i=1,2,\cdots)$ are continuous functions satisfying only the first inequalities in (F) and (G).

References

[1] **D. Bainov, V. Lakshmikantham and P. Simeonov**, *Theory of impulsive differential systems*. World Scientific, 1989.

[2] **P. González and M. Pinto**, *Asymptotic behavior of impulsive differential equations*. (Submitted).

[3] **M. Pinto**, *Impulsive delay differential equations*. (To appear).

[4] **S. Pandit**, *Differential systems with impulsive perturbations*, Pacific J. Math. 86 (1980), 553-560.

[5] **S. Leela**, *Stability of measure differential equations*. Pac. J. Math. 55 (1974), 489-498.

[6] **D. Bainov and N, Milev**, *Stability of linear inpulsive differential equations*, Int. J. Systems SCI. 1990, vol. 21, N^o 11, 2217-2224.

[7] **T. W. Stallard**, *Functions of bounded variations as solutions of differential systems*, Proc. Amer. Math. Soc. 13 (1963), 366-372.