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The Solution of the Cubic Equation: Renaissance Genius and Strife

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The Problem

To solve a linear equation $ax + b = 0$, just take

$$x = \frac{-b}{a}.$$

To solve a quadratic equation $ax^2 + bx + c = 0$, take

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

In each case x can be found from an algebraic formula involving only the coefficients of the equation.

This paper tells of the discovery of similar formulas for the cubic equation $ax^3+bx^2+cx+d = 0$ and for the quartic equation $ax^4+bx^3+cx^2+dx+e = 0$.

The Ancients

How nice it would be if we could start the story of algebra by identifying who first solved linear equations, then who worked out quadratic equations, then proceed to the solvers of the cubic equation, the main topic of this paper. There is a great satisfaction in putting a name and a face to those who developed the ideas we study. However, in the very earliest mathematical sources we have from Egypt and Babylon, dating from perhaps 1800

BC, anonymous solutions to quadratic and linear problems appear. They are stated entirely in words, except for number symbols, and we will likely never know whose words they are.

It is actually a bit anachronistic to call these writings solutions to equations, since equations as we would recognize them did not appear until the evolution of algebraic notation in the sixteenth and seventeenth centuries AD. But this language will serve our purposes here, and the interested reader can pursue the finer points in historical books. Our subject, the solution of the cubic equation, was itself a major force in the expansion and refinement of algebraic notation.

The ancient Greek mathematicians, such as Euclid (c. 300 BC), Archimedes (c. 287-212 BC), and Apollonius (250-c. 174 BC) mostly emphasized geometry. Their version of linear and quadratic problems related to line segments and squares. Our phrase "x squared" for x^2, which we usually think of as "x multiplied by itself," reflects the Greek idea of the area of a square having a side of length x.

The geometric investigations of Archimedes and Apollonius led them to take results about squares into the third dimension, thus obtaining results about cubes. Looking at their conclusions and questions, modern mathematicians can interpret them algebraically as involving cubic equations, while still recognizing the exclusively geometric character of the originals.

Mathematicians in China were also working on similar problems, though lack of communications left both the West and China ignorant of the other until around 1300 AD. There are no documents remaining of ancient Chinese mathematics. In works from the eleventh century, Chinese scholars had found excellent methods for approximating the solution for polynomials of any degree. They do not seem to have been greatly interested in the problem of finding a formula to give an exact solution, however.

The Islamic World

Beginning around 1000 AD, mathematicians of the Islamic world took the Greek work (and perhaps some sources from India) and began to develop powerful techniques. Beside using geometry, they developed a kind of reasoning that led to our modern algebraic approach. In fact the very word *algebra* comes to us from the work of al-Kwarizmi (c. 780-805), who used the term *al-jabr* to represent the operation of moving a subtracted quantity from one side of an equation to the other side as an added quantity. It is easier for us to see this in symbols than in words:

$$\begin{aligned} &7z + 3 = 5 - 2x \quad \text{becoming} \\ &9x + 3 = 5 \end{aligned}$$

is an example of *al-jabr* in modern notation.

Except for numerals, all this Islamic work was done in words. Importantly, although subtraction was well understood, the free use of negative numbers was not. This directly affected the way problems were considered. For example, we know the single general quadratic equation

$$ax^2 + bx + c = 0.$$

But an early mathematician, such as al-Kwarizmi, would assume all quantities to be positive (unless he was Indian, but that's another story for the interested reader to pursue). So anyone who presented this expression would seem absurd. How can three positive numbers, ax^2, bx, and c, ever add up to 0? The early mathematician would surely think we were quite ignorant to suggest this, despite our clever notation!

Let us see how al-Kwarizmi viewed linear and quadratic problems.* He saw six types of problems, involving numbers, roots(x) and *mal*, or the square of the root(x^2):

1. *mal* equal to roots ($ax^2 = bx$)
2. *mal* equal to numbers ($ax^2 = c$)
3. roots equal to numbers ($ax = b$)
4. *mal* and roots equal to numbers ($ax^2 + bx = c$)
5. *mal* and numbers equal to roots ($ax^2 + c = bx$)
6. roots and numbers equal to *mal* ($bx + c = ax^2$)

Interestingly, we will see a similar analysis later in the study of cubic and higher equations.

Al-Kwarizmi developed different approaches to handle each of the six types. Those involving squares are calculationally equivalent to our use of the quadratic formula. He was even able to handle the cases where the

*Modern interpretations are given in parentheses. After this, most equations and formulas will be in modern form. The historical evolution of notation is a fascinating subject, but using each author's version would make this paper unreadable. For more on notation see the references, especially Cajori.

quadratic has two positive roots. The Babylon sources, of which he seems to be aware, never gave more than one answer.

Successors of al-Kwarizmi in the Islamic world were able to solve a number of equations of higher degree. Some were quadratics in disguise, such as $x^4 + 2x^2 = 1$, but deep analyses of cubics were also undertaken. Around 1100 AD, Umar al-Khayyami wrote the most influential treatise on the cubic equation until the problem was finally settled in the sixteenth century. It has long been assumed he was also the author (known in the West as Omar Khayyam) of the poem the *Rubaiyat*; but modern scholars have cast doubt on this.

Al-Khayyami classified cubic equations into twenty-five types, allowing only positive coefficients; this is something like al-Kwarizmi's list of cubics. His study proceeded geometrically, discussing how many solutions each type would have and giving methods of intersecting conic sections to obtain solutions. For the case "cube and side equals a number" ($x^3 + ax = b$), he obtained a solution by intersecting a parabola and a semi-circle. From our point of view, al-Khayammi considered only solutions that occur in the first quadrant, not any that involve negative numbers. His influential work was studied by the Italian Scipione del Ferro, of whom we will hear later, since he was the first to find an algebraic solution (c. 1510). Al-Khayammi was also studied by René Descartes and Isaac Newton.

The Islamic algebraic work was further refined by Sharif al-Din al-Tusi (c. 1200), who also showed how to find approximate solutions to equations. Unfortunately it does not seem that later European mathematicians had access to al-Tusi's insights.

Medieval Europe

The Middle Ages in Europe saw the beginnings of the great universities. Paris, Oxford, and Bologna had all been founded by the early 1200s and many others grew up in the next centuries. Translations of some Greek and Islamic mathematical texts into Latin were made in the twelfth and thirteenth centuries. In the fourteenth century Italian mathematicians, beginning with Leonardo of Pisa (better known as Fibonacci), introduced the Hindu-Arabic numerals and the calculation techniques devised by Islamic mathematicians to Western Europe. This led them to a closer study of al-Kwarizmi and other Islamic algebraists.

In 1494, Luca Pacioli published *Summa de Arithmetica, Geometrica,* *Proportioni et Proportionalita.* This attempted to summarize all that was known at the time about mathematics. It even included a detailed introduction to double entry bookkeeping for which Pacioli has been called "The Father of Accounting" . In fact little, if any, of the *Summa* is original work of Pacioli's, but it served as a very important compendium for the next generation of mathematicians. Pacioli gave solutions for linear and quadratic problems and for special cases of equations of higher degree. He seems to have concluded that the general problem of the cubic would never be solved.

Of course, it is always dangerous to say something can't be done! Soon after, somewhere between 1500 and 1515, Scipione del Ferro, a professor at Bologna, was able to find a solution to the case $x^3 + ax = b$.

Today a mathematician making such a great discovery would rush to publish it, partly to share the excitement and partly to ensure proper credit for the discovery. For del Ferro, however, the opposite was the case. Not only did he not publish the solution, but he told only a few trusted students that he had even succeeded. We shall see he had a good reason for such secrecy.

At this time, large sums of money and even keeping a university position often depended on defeating a rival in a mathematical duel. Each competitor would set the other a certain number of problems, and whoever solved the most was declared the winner. Knowing how to solve a type of equation your opponent could not solve was thus a decisive advantage. So del Ferro's discovery was best kept as a secret weapon, only to be used in case a strong opponent emerged who threatened his prestige and academic position.

Del Ferro died in 1526 without having needed to use his solution to defeat a challenger. Before his death, he confided his solution to his students Antonio Fiore and Annibale della Nave. While they too kept the solution secret-for the same reason as del Ferro-they must have let slip some hints, for rumors began to spread in Italy that a solution of a cubic had been achieved.

Tartaglia

Now enters on the scene the brash and brilliant Nicolo Tartaglia (1449-1557), a remarkable character even for the colorful scene of Renaissance Italy. At the age of twelve in his native Brescia, which is near Venice, he had been seriously disfigured during an invasion by French troops. A sabre wound to his face left a large ugly scar and caused enough damage to his mouth that ever after he had trouble speaking clearly. His neighbors, probably unkindly, called him *tartaglia*, "the stammerer." It is a sign of his pugnacious character that young Niccolo defiantly took the slur and used it in place of his last name, the bland Fontana.

Tartaglia was so poor he had almost no schooling. He reports that at the age of fourteen he quit primary school after only a few weeks due to lack of money. But instead of returning the schoolbooks he had borrowed, Tartaglia used them to learn to read on his own and to master the elements of mathematics. Thus he was entirely self-taught. By dint of hard work, persistence, and a touch of megalomania, the untutored young man became an excellent mathematician and scientist. Although he eventually became a teacher of mathematics in Verona and Venice, as well as at home in Brescia, Tartaglia was unable to achieve the kind of academic success and recognition he dreamed of. A loner and an outsider, rather rough in his manners, Tartaglia could not bring himself to flatter and please those who held power in the academic world.

Around 1530, Tartaglia boasted that he had solved a version of the cubic. (It was, in fact, $x^3 + ax^2 = b$). Could it be true, scholars wondered? Knowing that the bold Tartaglia sometimes claimed more knowledge than in fact he had, del Ferro's student Fiore challenged him to a contest in 1535. Each man was to set the other thirty problems with thirty days allowed to solve them. The stakes were somewhat unusual, and certainly expensive: the loser would buy the winner and his friends thirty banquets! But it was the unspoken stakes-professional renown for the winner-that mattered most to Tartaglia.

The Duel

Here are three of Fiore's problems, numbered as they appeared on his list [Fauvel and Gray]. First a purely mathematical question:

1. Find me a number such that when its cube root is added to it the result is six.

Let x be the cube root of the desired number and you can see this leads to $x^3 + x = 6$.

Now a practical problem, though I've yet to meet a merchant who uses this particular method for setting prices:

15. A man sells a sapphire for 500 ducats, making a profit of the cube root of his capital. How much is his profit?

As in the previous problem, this leads to $x^3 + x = 500$.

Next comes a scientific application. The loggers in those days must have had a remarkably good mathematics background! Consider:

17. There is a tree 12 *braccia* high, which was broken in two parts at such a point that the height of the part standing was the cube root of length of the part that was cut away. What is the height of the part that was left standing?

This leads to $x^3 + x = 12$.

The cubic solution given him by del Ferro seems to be the only trick Fiore knew. Each of his problems reduced to the solution of an equation of the form $x^3 + x = b$. Wily Tartaglia's problems were much more varied, coming from different areas of mathematics, including cubics of the form he had solved, namely $x^3 + ax^2 = b$.

Despite his boast, Tartaglia had no idea how to solve the version of the cubic Fiore handed him. Days of frantic effort turned to weeks with no success. At last, just before the deadline, he cracked it and was able to solve all thirty of Fiore's problems in short order. Fiore proved himself a poor mathematician, as he solved almost none of Tartaglia's wide-ranging problems. Thus Tartaglia won the contest handily.

Perhaps believing this victory would assure him an attractive university offer, Tartaglia did not insist on Fiore's paying for the promised thirty banquets. In the end, disappointingly, he got neither a chair at dinner nor one in a university. Still, the competition broke open del Ferro's secret, and word spread that some cubics had been solved.

Cardano

Gerolamo Cardano (1501-1576) is another of the remarkable characters that peopled Renaissance Italy. Born in Pavia, he was educated there, including study at the University of Pavia. He then studied medicine at the University of Padua. His attempts to secure a medical position in the great city of Milan met with rejection, allegedly due to his illegitimate birth, but also due to powerful enemies he had made by his public criticism of senior doctors. He returned to Padua to practice and study, and made himself one of the most famous physicians of Europe. Cardano made a tour of Europe during which he treated many notable people. He cured the Archbishop of

Scotland of a serious, chronic difficult in breathing by banishing all feathers from the Archbishop's bedroom. On returning to Italy, his fame was now so great (and his tact sufficiently improved) that he was accepted into the College of Physicians in Milan, where he settled down to enjoy his new wealth and status.

Cardano wrote extensively on scientific and philosophical topics, including mathematics. Like many other mathematicians of the time he often calculated horoscopes, believing that his superior mathematical ability ensured better accuracy than others could obtain. He is rumored to have run into trouble with Church officials after casting a horoscope for Jesus, and another tale has it that Cardano committed suicide in order to confirm his own astrological prediction of his demise. His life was full of troubles as well. For example, Cardano's beloved, but wayward son, poisoned his spouse and was put to death. Cardano's very readable autobiography, *De propria vita*, tells his version of many of the feuds, escapades, trials and triumphs of his packed life. Disappointingly for us, it mentions his mathematical work only slightly.

Cardano also loved to gamble. When young he squandering much of his small store of money. Older and wiser, he turned his powerful mind to analyzing the chances of winning in the games he had played so eagerly. This work was published after his death as *Liber de ludo aleae (The Book on Games of Chance)*. It marks the beginning of the scientific study of probability.

Cardano's interest in algebra was of long standing. His *Practica arithmetica*, published in 1539 (though written earlier), was an overview of arithmetic and algebra, based on the great *Summa* of Luca Pacioli mentioned above. In it Cardano agreed with Pacioli that the cubic could not be solved purely algebraically. How embarrassing then to hear of the cubic contest between Tartaglia and Fiore!

Intrigued, in early 1539 Cardano invited the victorious Tartaglia for a visit during which he implored Tartaglia to reveal the solution. Despite this flattering admiration, the proud and contrary Tartaglia declined repeatedly. At last he offered Cardano several poems that gave, in somewhat obscure terms, the recipes for solving the versions of the cubic known to him. In return the grateful Cardano swore a solemn oath not to publish the solution before its discoverer Tartaglia did.

No proof or explanation of the poetic solutions was provided, so Cardano eagerly set to work to fill in the details. In a short time, he had not only proved that Tartaglia's solutions were correct but extended the solutions to all the other cases of the cubic. One of Cardano's important observations was that the square term call always be eliminated. As we would say, if we substitute $x = y - \frac{a}{3}$ into $x^3 + ax^2 + bx + c$, we obtain an expression of the form $y^3 + sx + t$, where s and t are expressed in terms of a, b and c. (Those who have not seen this before will find it worthwhile to take a few moments to work this out.) Cardano's work was not so neatly expressed, but it was effective in reducing his number of cases significantly.

Not content to stop here, Cardano also set his talented pupil Ferrari to work on the quartic equation, and in a few months Ferrari succeeded in taking care of all the cases for this problem as well.

Now Cardano was eager to make public his results. Why keep them secret? He was already a prominent physician and had no need to win mathematical competition Tartaglia, however, had not yet published and seemed increasingly unlikely to do so. He had now become absorbed in ballistics, the mathematical study of the motion of artillery shells, an important practical and theoretical subject as cannons came to be more accurate and powerful. Indeed, over time Tartaglia received much more financial reward for his firing tables and analysis of the trajectory of projectiles than he ever would expect for his solution to cubics. Perhaps reflecting his painful childhood experience, his books on military subjects indicate that he had ethical doubts about applying his scientific expertise to improving destructive engines of war.

Chafing at the delays this impasse was causing, Cardano now learned of the rumors that del Ferro had solved a cubic well before Tartaglia. Journeying from Milan to Bologna to investigate, he and Ferrari unearthed del Ferro's solution among his papers. Elated by this discovery and eager to publish, they decided that the solution was close enough to Tartaglia's as to invalidate Cardano's oath. Having come so far in his own research Cardano now proceeded to write everything up in a book called *Ars Magna (The Great Art)*. His generous preface gives full credit to del Ferro and Tartaglia for their pathbreaking work as well as acknowledging his industrious pupil Ferrari. Cardano published the book in 1545.

Outrage! Tartaglia was furious In 1546 he accused Cardano of the spiritual and civil offenses of oath-breaking and plagiarism. Writing his version of the 1539 visit with Cardano, he cast himself as the victim of an unscrupulous schemer. Ferrari, who was now seeking an academic position, took it upon himself to make a lengthy reply in defense of himself and his teacher, disputing Tartaglia's story about the meeting, pointing out the clear credit given Tartaglia in the *Ars Magna*, and injudiciously noting Tartaglia's own professional lapses. Among other misdeeds recounted by Ferrari, Tartaglia had in 1543 published as his own a Latin translation of Archimedes done by William of Moerbeke in 1269.

How would two Renaissance mathematicians resolve a dispute? As you might guess, Ferrari challenged Tartaglia to a duel-a mathematical duel. In 1547 each set thirty-one problems for the other. Neither solved all, but Ferrari solved more. Not content with victory, the younger Ferrari publicly criticized some of Tartaglia's solutions. Continuing the battle, the two met the following year for a public disputation, which ended with the hot-tempered Tartaglia leaving town in a hurry.

It is not known precisely what happened, but the outcome was bad for Tartaglia. He lost the academic position he had finally obtained in Brescia and died in poverty a few years later. Ferrari, on the other hand, was invited to lecture in Venice and went on to wealth and comfort, a career in government, a post in Church administration, and finally a university position. Alas, his life was cut short at age 43, in suspicious circumstances: possibly poisoned for his money by his sister. Note to any aspiring writer of mathematical melodrama: there are plots aplenty for you in the Renaissance!

The Complex Numbers Appear

What were the actual solutions of cubic equations that inspired so many hot words and high emotions? Deriving them in Cardano's own style would be lengthy and tedious for us. For versions of his derivations in modern dress see [Katz] and [Calinger], among others. However, some aspects of these solutions are worth nothing, since they led to the development and ultimate acceptance of the complex numbers.

For equations of the form $x^3 = ax + b$, Cardano's solution would be

$$x = \sqrt[3]{\frac{b}{2} + \sqrt{\frac{b^2}{4} - \frac{a^3}{27}}} + \sqrt[3]{\frac{b}{2} - \sqrt{\frac{b^2}{4} - \frac{a^3}{27}}}.$$

Do you see the family resemblance to the quadratic formula? Note at the same time that the level of complexity has shot up significantly!

For example, to solve $x^3 = 6x + 40$, substitute to see that

$$x = \sqrt[3]{20 + \sqrt{392}} + \sqrt[3]{20 - \sqrt{392}}.$$

Notice, however, that $x = 4$ solves the equation. (Check by substituting 4 into $x^3 = 6x + 40$.) If you work on it a bit you can see that, in fact,

$$x = \sqrt[3]{20 + \sqrt{392}} + \sqrt[3]{20 - \sqrt{392}} = 4.$$

So Cardano's formula gives the right answer, but it does so in a complicated form.

In fact, it is far more than merely complicated. Those of us who worry about such things (we call ourselves mathematicians, others call us fussy) would notice that in Cardano's formula

$$\frac{b^2}{4} - \frac{a^3}{27}$$

appears inside a square root. What if this quantity were negative?

For instance, consider another equation of the same form: $x^3 = 15x + 4$. Its solution, from the formula, is

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}.$$

Again, notice that $x = 4$ works. Now, Cardano knew that $\sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}} = 4$, though we are not sure exactly how he thought about this or calculated it. No appropriate vocabulary or notation existed. Yet this construction forced mathematicians to grapple with complex numbers.

Negative numbers under the square root can occur in the quadratic formula, but only if the resulting solutions are themselves complex numbers, so until complex numbers were accepted mathematicians could dismiss them. In the case of the cubic, we see that the negative under the square root can appear even in expressing a solution that is a real number, in fact an integer.

Bombelli

Cardano's *Ars Magna* is extraordinary, but like many original works it is not easy to read. For one thing it assumes that the reader already has a strong algebraic background. Only a few years after Cardano's work appeared, an engineer from Bologna, Rafael Bombelli (1526-1572), who greatly admired Cardano's mathematics, decided to write his own book which would take the reader through all the algebraic prerequisites and then present the solution of the cubic and quartic equations systematically and completely. This became *l'Algebra*, written in five parts, the first three of which were published in 1572 and 1576. The fourth and fifth parts, which were not complete at the time of his death, dealt with geometric topics that were not directly needed for the solution of the cubic-and did not appear in print until 1929!

For us, the most interesting part of Bombelli's book is his careful treatment of quantities involving the square root of negative numbers. His notation is quite clumsy to our eyes. He uses the phrase *piu di meno* (plus of minus abbreviated *p.* di *m.*) for what we would call i or $\sqrt{-1}$; *meno di meno* (minus of minus abbreviated *m.* di *m.*) stands for $-i$ or $-\sqrt{-1}$. Bombelli then developed in great detail the rules for calculating with quantities involving these terms, starting with the crucial calculation piu di meno times *piu di meno* is -1; that is, $i^2 = -1$. He did not can these quantities numbers, but we would: the complex numbers.

With these rules in hand, Bombelli turned to expressions such as our previous example $\sqrt[3]{2+\sqrt{-121}} + \sqrt[3]{2-\sqrt{-121}}$. He assumed that there must he numbers t and s such that $\sqrt[3]{2+\sqrt{-121}} = t+\sqrt{-s}$ and $\sqrt[3]{2-\sqrt{-121}} = t-\sqrt{-s}$. Using his rules and judicious guesswork, Bombelli was able to find that $t = 2$ and $s = 1$. Thus

$$\sqrt[3]{2+\sqrt{-121}} + \sqrt[3]{2-\sqrt{-121}} = 2+\sqrt{-1}+2-\sqrt{-1} = 4,$$

as we suspected.

Now remember, at this time negative numbers were yet to be accepted as real and even zero was handled with suspicion. So it is no surprise that Cardano himself was not happy at all with this type of calculation, which he viewed as a flaw in the formula. Even Bombelli did not seem to consider these quantities numbers, but rather useful fictions. Later mathematicians slowly expanded the idea of number, but full acceptance of the complex numbers did not come until the 1800s.

Beyond the Fourth Power

The remarkable solution of the cubic and quartic equations inspired mathematicians to set out to solve the quintic-equations involving fifth powers-and even higher. Despite great efforts and much improvement in algebraic notation and technique, only some special cases could be solved. Some began to say the solution was not possible algebraically. Recall we heard that before, in the case of the cubic!

Alas, this time the doubters were right. In 1798 Paolo Ruffini claimed to give a proof that the quintic was unsolvable, but no one could quite understand his explanation. Around 1820 Niels Abel gave a complete proof for all equations of fifth degree and higher: There can be no general solution above the fourth power. The bright side of this mathematical disappointment was that it led to the beginnings of the modern algebra of groups, rings, and fields. But that is a story for another time.

Here we conclude our journey into the past. As in the present, we find mathematicians soaring blissfully in the lofty realms of pure thought, apparently above the small details of daily life. But when they land, they often show themselves just as capable of petty jealousy and inflated pride as anyone else.

Acknowledgements

The story told in this paper is fairly well known among historians of mathematics. More details and further references can be found in Victor Katz's excellent technical survey, which is particularly good on the contribution of the Islamic world. Another excellent survey by Ronald Calinger includes much more of the general cultural context than is typical in such books.

I thank my dear wife, Ellen, who greatly improved the writing in this paper, then rendered it in TeX. John Fauvel was a great scholar and a generous supporter of all of us interested in the history of mathematics. Word of his untimely death reached me just as I finsihed this paper. He will be greatly missed.

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