Cubo Matemática Educacional
Vol., 4. N°2, Junio 2002

THE COMPANION CURVES OF GILLES PERSONNE DE ROBERVAL

Barnabas Hughes
Dept. of Mathematics
California State University,
Northridge-California
91330-8313 USA

In the study of the mathematics created years, moreso centuries ago, the historian is ever alert to possible development of ideas which their creators might not have noticed. Such happened while I was analyzing a tract on the conic sections created by Roberval some three hundred and sixty-five years ago. There is more to his creation than he wrote about. Let us understand what he created; then how he might have gone farther.

* ** *** ** *

Gilles Personne de Roberval (1602-1675) was among the first mathematicians, if not the first, to construct an introductory analytic geometry, To *Find the Analytic Equation of a Given Geometric Curve*[1].His general method was to take the standard geometric definition of each conic section (circle, ellipse, parabola, and hyperbola) and translate it into an algebraic equation. Each equation consists of one or more constants and two variables, the latter being forerunners of rectangular coordinates. Underlying his work is the geometric principle that line-segments are constructed by a moving point, much as the end of a piece of chalk draws a line-segment on a chalkboard. As the hand holds the other end of the piece of chalk, so this end of Roberval's line-segment is moving along a track. By substituting the sum or difference of another constant and variable for one or both of the given variables, he proceeded to create many new equations for each of the conic sections. However, as he created new equations by varying values in the principal equations, Roberval seemed not to have noticed the total effect of substituting, say, a new constant. It can be interpreted as a line-segment. Hence, while one end of it is tracing out the desired curve, the other end is also curve-tracing. The purpose of my investigation is to explain how these additional curves, which I call companion curves, arose. Perhaps of equal interest, given the change of variables made by Roberval, is the apparent impossibility of graphing on a computer these equations by a single input. An historical overview brings this report to a close.

Opening a window to the beginning of his work assists understanding how he created such a variety of equations for each conic section. He began the work with an analysis of the circle. What appears hereafter in italics is my translation of the Latin text, in which as the context demands I translated *linea* as line-segment. For the moment the points G, F, and H together with the broken line-segments may be ignored. The boxed diagram is a copy of his, identifying a rectangle by three letters, BEC; that is, BE is the length and EC is the width. Note however that $2be - e^2$ is not the area of the rectangle but rather the product of two segments of the diameter of the circle.

Given a circle whose center is A, circumference BDC, and one of its diameters BC to which all points on the circumference are necessarily related through some analytic equation. The foundation of this relationship is the property that every straight line-segment, say DE, falling from the circumference to the diameter at right angles is the mean proportional between BE and EC, the parts of the diameter.

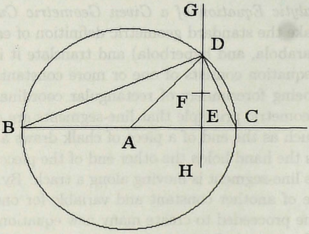

figure 1

This specific property is one among the particular ways of regarding the circle. From that way innumerable equations will be deduced, among which *are the following.*

The First Equation

Let AB be b,	*Likewise let AB be* b,
DE a,	*DE* a,
DE squared a^2	*DE squared* a^2
BE e	CE e
EC $2b - e$	*BE* $2b - e$
BEC the rectangle $2be - e^2$	*BEC the rectangle* $2be - e^2$
Therefore the equation is $2be - e^2 = a^2$ *or* $2be - e^2 - a^2 = 0$	*Hence, the equation will be as above* $2be - e^2 - a^2 = 0$

And so, given curved line BDC and a perpendicular DE dropped from it to some straight line-segment BC: if such an equation is found as we have discovered, then we may announce that this curve is the circumference of a circle. The property is reversible and the proposition created from it can simply be turned around, as is easily enough apparent to anyone considering it. The length of every given straight line-segment, however, can be referred to as 2b.

If Roberval were lecturing on the creation of the first equation, he most probably would have reminded his students how geometric figures are constructed: "The movement of a point describes a line-segment, the movement of a line-segment describes a surface, the movement of a surface describes a solid. The motion is easily understood."[2] This principle governs the generation of the circle. The end point D of perpendicular DE describes the circle as the other end point E moves along the track BC. The first equation of the circle, $2be - e^2 - a^2 = 0$, describes the completed action where b is the radius of the circle, e the distance from the left (or right) end of the diameter to the foot of the perpendicular E, and a the length of the moving perpendicular as shown in his chart.

Although Roberval wrote that the curved line BDC is the circumference of a circle, in reality it is the semicircumference. If the upper half is generated by moving perpendicular DE from B to C, then the lower half is formed by reflecting DE about BC and letting it move from C back to B. Or, the entire upper half may be reflected about BC. Regardless, the curve is generated by the end point D of the perpendicular DE as the other end point E moves along the track BC. As will become clear, it is important to remark that in the discussion which follows, the end point E always moves along BC but the point generating the curve is not always end point D.

The creation of the second equation, knowingly or not, laid the ground work for the companion curves. Roberval would introduce point F, as shown in the diagram above, which permitted him to set a equal to the sum of a constant length and a varying length, $c + i$, the crucial substitution. As before, his text is in italics.

The Second Equation

Let the same things be set down: from DE let the given [line-segment] *EF be called c, and DF be called i. Therefore the square of DE will be* $c^2 + 2ci + i^2$. *And then by inserting this into the same pattern, the equation will be*

$$2be - e^2 = +c2 + 2ci + i^2, or - c2 + 2be - e^2 - 2ci - i^2 = 0.$$

Thus from this or a similar equation we may argue to the circumference of a circle. Moreover, if $c^2 + 2ci + i^2$ *may be referred to one species, a2 (for the species is that of i squared), then we will have returned entirely to the first equation, as is obvious. In turn, it is easy to move from the first to this second* [equation].

Roberval used the word *species* where we would use *variable* or *unknown.* Hence, if i^2 varies, then a^2 varies in a corresponding way. Further, no diagram accompanies these statements, perhaps because the data are clear: segment FE is of fixed length or constant, segment DF varies. The *construction* begins with FE $= c$ perpendicular to the left endpoint of diameter BC. Additionally, point D which would supposedly create the locus of the circumference is atop point F because $i = 0$. Then as FE moves along the diameter, point D rises to begin an arc that will lead to the circumference of a semicircle, as seen on the left side of the figure below.

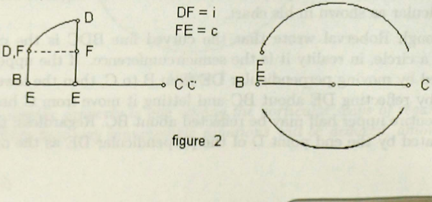

figure 2

However, when the semicircle is completed and reflected about the diameter BC to form the required circle, the strange figure to the right above appears. Since Roberval had stated, *Thus from this or a similar equation we may argue to the circumference of a circle*, I would say that either he never tried to construct the figure given these conditions, or there is a typographical error in the text. Since he focused on geometric figures, negative values of i would not have been appropriate. We may consider what would happen by exchanging values of c and i and by setting DF $= c$ and FE $= i$. This exchange permits point F on DE to become the point generating the semicircle as FE grows from zero to whatever the radius may be. The figure to the left below depicts the scenario, the creation of the curve for $2be - e^2 = (c+i)^2$.

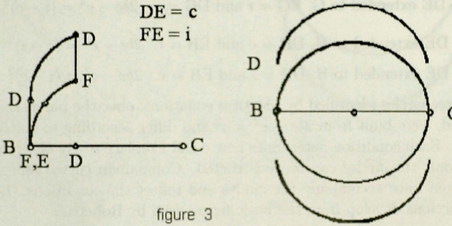

figure 3

When the semicircle has been completed, it is reflected about BC to form the desired circle. The points on the circumference of the circle can be constructed from the equation, despite the surrounding border (the broken line). My interpretation suggests a typographical error in the text.

Unexpectedly as the figure illustrates, two additional semicircles (the broken lines) have appeared. These were generated by end point D of perpendicular DE as FE grew from zero to radius length and then back to zero as E moved along its track, BC. The creation of the additional semicircle is consonant with Roberval's principle that a line-segment is generated by a moving point, in this case end point D. I would call these semicircles companion or shadow curves, curves which he apparently never considered. Further, a word of caution seems advisable. As is well known, computer graphing programs produce displays that reflect input. As will be seen in the next paragraph, different data can produce the same equation. Conversely, the same equation will not suggest the different data.

Roberval continued to create more equations by extending ED to G and DE to H. For instance, by extending DE to G, he set EG $= c$ and DE $= i$. Thereby $c - i$ takes the place of a and a new equation appears:

$$2be - e^2 = (c - i)^2 \tag{1}$$

The reader can try for oneself to produce a companion curve. I would look at three other variations, all of which produce the same equation yet in the concrete of their respective data exhibit different companion curves; specifically,

$$\text{DE extended to G, EG} = i \text{ and DG} = c\text{:}\ \ 2be - e^2 = (i - c)^2 \tag{2}$$

$$\text{DE extended to H, DH} = c \text{ and EH} = i\text{:}\ \ 2be - e^2 = (c - i)^2 \tag{3}$$

$$\text{DE extended to H, DH} = i \text{ and EH} = \text{c:}\ \ 2be - e^2 = (i - c)^2 \tag{4}$$

The three circles identified by identical equations, once the parentheses are removed, were built from $2be - e^2 = a^2$ and differ according to stated conditions. Each condition determines how c and i replace a. By observing the conditions, the circles can be constructed. Companion curves represented by broken lines accompany the circles and reflect the conditions. Let the constructions develop from the basic figure used by Roberval:

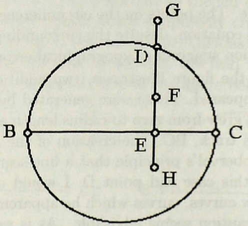

figure 4

With ED extended to G, EG $= i$, and DG $= c$, equation (2) appears: $2be - e^2 = (i - c)^2$. The construction begins below in the figure on the left with the perpendicular EG upright at B where c is a constant and $i = c$.

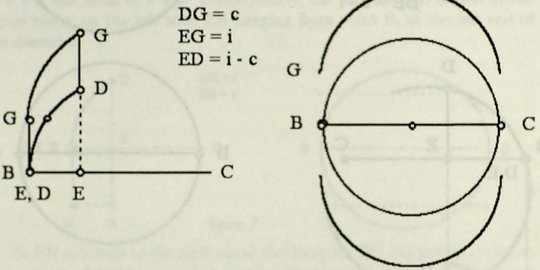

figure 5

As the perpendicular EDG moves to the right, ED takes on values which added to DG change the length of EG or i, because EG = ED+DG. A maximum value for i is reached when EG reaches the center of the diameter BC. Afterwards the values of ED decrease to 0 at the right end of the diameter where c again equals i. The overall effect, assuming symmetry about the diameter, is of a circle hedged in above and below by two companion semicircles. On the other hand, if we let $b = 4$ and $c = 6$, the resulting Cartesian equation $8x - x^2 = (i - 6)^2$ can be graphed on a scientific calculator showing none of this construction nor the accompanying companion curves. There is, however, a shift of the diameter of the circle off the x-axis, something which Roberval could not have conceived. Finally to be noted is that despite the difference between this equation and that of the second equation above, their circles and companion curves are identical.

In equation (3), $2be - e^2 = (c - i)^2$, DE is extended to H, DH $= c$, and EH $= i$. The construction begins as shown below on the left with the perpendicular DH *hanging down from the endpoint* B where $c = i$ and the measure of DE $= 0$.

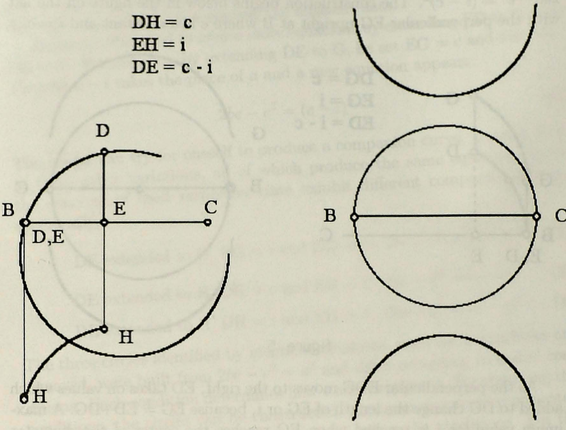

figure 6

Under these conditions, the following scenario constructs a circle. As DH moves to the right, point D rising above the diameter BC and point E moving along it, the difference between DH and EH, or $(c - i)$, appears above the diameter so that the endpoint D traces out the locus, a semicircle. The maximum difference is reach at the midpoint of BC; thereafter the differences decrease until $c = i$ at the right endpoint C. A circle is formed by rotating the curve BDC about the diameter as with equation (2). Perhaps more interesting is the companion locus formed by the moving point H; it creates another semicircle opening downward. Upon rotation of semicircle BDC to create the circle, semicircle H becomes another semicircle over and above the circle and opening upwards. Again, if we let $b = 4$ and $c = 6$, the Cartesian equation $8x - x^2 = (6 - i)^2$ can be graphed on a scientific calculator showing none of this construction.

Equation (4), $2be - e^2 = (i - c)^2$, with DE extended to H, DH = i, and EH = c, is a variation on equation (1). Since line-segment EH is given, it must remain constant as it moves along the diameter BC. Line-segment DH, however, which is composed of parts DE and EH, varies from $0 + c$ to $e + c$ and back to $0 + c$. Consequently, the construction begins in the figure below on the left with EH hanging from point B, at the left end of the diameter.

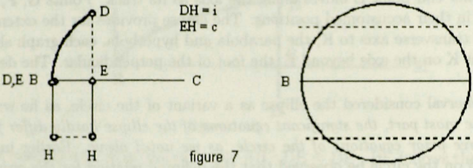

figure 7

As EH proceeds to the right along the diameter BC, the points D rise on the nascent circumference of the circle; likewise, the segments DE appear as the extension of EH to make i. When the diameter has been traversed, a reflection of the figure about the it produces the circle together with the companion loci, two parallel lines. Once again, if we let $b = 4$ and $c = 6$, the Cartesian equation $8x - x^2 = (i - 6)^2$ can be graphed on a scientific calculator showing none of this construction.

Not all the circular equations have companion curves. For instance, consider the case where the diameter is extended beyond the circumference to some arbitrary point K. Consequently, BA = b, DE = a, EK = i, CK = c, EC = $i - c, andBE = 2b - (i - c)$. The equation, therefore, is $[2b - (i - c)](i - c) = a^2$. The construction begins, as suggested in the figure at the right below, with EK = CK; that is, $i = c$. Then, as the perpendicular DE moves to the left, EK becomes longer by the measure of EC.

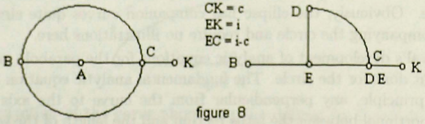

figure 8

Hence, in order to maintain the proportion upon which all circle equations are built, (BE)(EC)= De^2, c must be subtracted from i. Once the semicircle has been completed, it is reflected to form the desired circle. This graph is not accompanied by a companion.

Roberval developed algebraic equations for the other conic sections in much the same way as he fashioned equations for the circle. A perpendicular DE to the axis is chosen that initially cuts the curve at end point D while the other end point E moves along the axis as its track. Points G, F, and H are in their accustomed positions. The ellipse provides for the extension of the transverse axis to K; the parabola and hyperbola, each graph shows a point K on the axis beyond E, the foot of the perpendicular. The details follow.

Roberval considered the ellipse as a variant of the circle, as he wrote: *For the most part, the significant equations of the ellipse hardly differ from the three prior equations of the circle, as we noted above.* Earlier in the section on the circle he observed that the defining relation for the circle is (BE)(EC) = De^2. For the ellipse the relationship is the ratio of (BE)(EC) : De^2 which will be more or less than the given ratio of the transverse axis to the latus rectum.

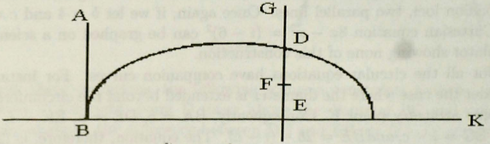

figure 9

Farther on in the analysis of the circle where he finished the first division of DE at F which produced the first set of companion curves, he remarked that the same sort of operation with consequent results can be done with the ellipse. Obviously, the ellipse has companion curves quite similar to those accompanying the circle and require no illustrations here.

Roberval's development of analytic equations for the parabola parallels closely that done for the circle. The fundamental analytic equation follows from the principle, any perpendicular from the curve to the axis is the mean proportional between the latus rectum and the length of the segment between the foot of the perpendicular and the curve. The latus rectum is AB, along the axis is the length of the segment BE, and the perpendicular DE must be parallel to the latus rectum.

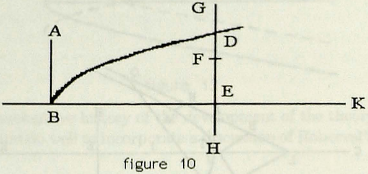

figure 10

Consequently, from the principle AB : DE = DE : BE follows the equation $be = a^2$, where b = AB, e = BE, and a = DE. (Substituting x for e and y for a produces the familiar Cartesian equation of the parabola.)

Mindful of how Roberval created new equations by cutting and extending certain line-segments, a glance at the figure of the parabola shows the same letters of segmentation (I and F) and letters of extension (C, H, K, and G). For instance, by cutting DE at F, the two parts may be described thus: DF $= i$ and FE $= c$. Substituting $c + i$ for y, we find $be = (c + i)^2$. Since i will vary from zero to however so much, c stands perpendicular to the vertex of the nascent parabola. As E moves along the axial track BC, the upper arm of the parabola is traced by F as end point D draw the companion curve as shown below.

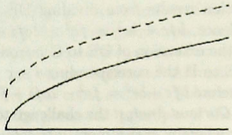

figure 11

A reflection of the curve about the line defined by C and K produces the other arm of the parabola and its companion curve. Other variations on y^2 produce their own companion curves which are easily drawn. By extending DE to H, we obtain $be = (c-i)^2 = (i-c)^2$. And so on. Roberval concludes the section on the parabola with the remark, *Let us move on to the hyperbola.*

He begins the analysis from the diameter relationship, utilizing the figure below[3],

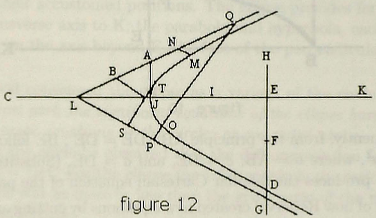

figure 12

Hyperbola BD has vertex B, latus rectum AB = b, transverse axis CB = f, L is the center of CB, perpendicular DE = a, and BE = e. The segments CE and BE represent a rectangle whose area is $e(f+e)$. Now, in every hyperbola the ratio of this area to a square on the ordinate DE is the same as the ratio of the transverse axis to the latus rectum: (CE)(BE) : DE^2 = BC : AB which leads to $e(f+e) : a^2 = f : b$. Consequently, the first of seven analytic equations arises: $bfe + be^2 = fa^2$ which Roberval called *the universal equation proper to every hyperbola.*

The second equation results from dividing DE at F; that is, $c+i$ is substituted for a. Hence, $bfe + be^2 = fc^2 + 2cfi + fi^2$. Next, as might have been expected, the extension of DE to G introduces line-segment $c-i$, or by extending DE to H the corresponding $i-c$ arises. The respective third equation is therefore $bfe + be^2 = fc^2 - 2cfi + fi^2$ for either extension. Roberval left to *The Curious Analyst* the challenge to divide and/or extend DE and BE together whereby *many more intricate equations will be born.* This third equation, after setting $b = .5$, $f = 4$, and $c = 6$ to produce $2e + .5e^2 = 4(6-i)^2$, will be used to exemplify hyperbolic companion curves, below.

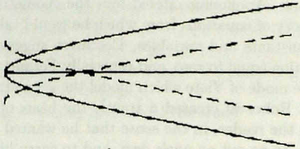

figure 13

Any comprehensive history of the development of the theory of analytic geometry would do well to incorporate a discussion of Roberval's companion curves.

+ +++ + +++ + +++ +

Historical Remarks

Gilles Personne de Roberval[4] was an extremely talented work-a-holic. He was one of the five geniuses who founded the French Royal Academy of Sciences. For the last twenty-four years of his life, he held three outstanding teaching professorships, one at the College of Maitre Gervais and two (!) at the Royal College. At the latter was a chair in honor of Pierre de Ramée (1515-1572), a distinguished polymath. The appointment to the Ramée Professorship was for just three years, at the end of which the holder had to enter into competition with anyone else who might wish to claim the chair for the next triennium. Since the holder, Roberval, had the right to pose the problems that decided who earned the chair, Roberval always won!

Among his many works[5] is the development of the early algebraic theory of conic sections on which this investigation was based, the aforementioned *Propositum locum geometricum ad aequationem analyticam revocare, & qui simpliciores sint loci, aut secus, explicare.*[6] Assuming separate geometric principles for each of the four conic sections, Roberval crafted an algebraic equation that matches the geometric principle of the respective conic. By varying the data as described above, Roberval offered his reader some forty equations as starters towards even further development.

What was Roberval's own data base? He was well acquainted with four books: *Conics of Apollonius, Introduction to the Analytic Art* (1591) by

François Viète, *Géométrie* (1637) of René Descartes, and *Introduction to Plane and Solid Loci* (read in 1637 but published posthumously in 1679) by Pierre de Fermat. Apollonius offered him the geometric principles, Viète explained a theory of equations from which he would take the use of letters to represent constants and variables, Descartes suggested exponents and setting an equation equal to zero, and-especially-Fermat presented algebraic equations in the mode of Viète which model the geometric principles. With this equipment Roberval created a tract[7], the basis of this investigation, that challenges the reader, in the sense that he wanted his reader to think along with him, to go out on one's own, and to carry his ideas farther.

---

[1] Roberval's work was not published until after his death; see footnote 6 below. Further, Jan de Witt had developed a complete theory of analytic geometry by 1646; but his work, *Elementa curvarum linearum*, was not published until 1659. Hence, if priority belongs to whomever published first, then John Wallis probably deservess the credit (De sectionibus conicis 1657).

[2] "Nam primum, motus ille, sive sit puncti alicujus ad lineam aliquam describendam, sive sit alicujus lineae ad describendam superficiem, sive superficiei ad solidum describendum, est simpliciter intelligibilis" in Roberval, *De geometrica planarum et cubicarum aequationum resolutione*, in *Divers ouvrages de mathématique et de physique, par messieurs de l'Académie Royale des Sciences*, a Paris: De l'Imprimerie Royale 1693, p. 171. Isaac Newton made the same remark perhaps fifty years later in *Lexicon Technicum*(1710), p. 141.

[3] From the 1693 edition, p. 223; see footnote 6 below.

[4] Kokiti Hara, Roberval, Gilles Personne, in Charles G. Gillipsie (ed.) *Dictionary of Scientific Biography* XI, (New York: Charles Scribner's Sons, 1975), 486-91. *See also* Léon Auger, *Un savant méconnu: Gilles Personne de Roberval (1602-1675): son activité intellectuelle dans les domaines mathématique, physique, mécanique et philosophique*, Paris: Blanchard, 1962.

[5] Of considerable interest is Vincent Jullien's *éléments de géométrie de G. P. de Roberval*. Paris: J.Vrin, 1996, a critical annotated text which offers a development of Euclid's Elements, together with an extensive bibliography

of Robervaliana.

[6] *Propositum locum geometricum ad aequationem analyticam revocare, & qui simpliciores sint loci, aut secus, explicare*, in *Mèmore de l'Académie Royale des Sciences Depuis 1666 jusqu'à 1699.* Tome VI, a Paris, par La Compani des Libraires, M.DCCXXX, pp. 212-30. While the two printings (editions ?) differ in size, the earlier is larger, the plates from the 1693 issue were clearly used in the later printing.

[7] I am preparing a critical analysis of the tract.