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Hamiltonety and automorphims group of graph preserved by substitution. *

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Abstract.
The substitution is a graph operation. This operation consists in replacing a vertex by a graph. The aim of this work is to analize the preservation of certain properties in the substitution of a graph. Specifically, these properties are: (i) hamiltonety and (ii) group of automorphisms of a given graph G.

1 Introduction

The graphs to be considered will be in general simple and finite, with a nonempty set of edges. For a graph G, $V(G)$ denote the set of vertices and $E(G)$ denote the set of edges. The cardinality of $V(G)$ is called order of G and the cardinality of $E(G)$ is called size of G. A (p, q) graph has p order and q size. Two vertices u and v are called **neighbors** if $\{u, v\}$ is an edge of G. For any vertex v of G, denote by N_v the set of neighbors of v. To simplify the notation, an edge $\{x, y\}$ is written as xy (or yx). Other concepts used in this work and not defined explicitly can be found in the references.

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2 The substitution

Assume that G and K are two disjoint graphs by vertices. For a vertex v in $V(G)$ and a function $S: N_v \to V(K)$ it will be defined the substitution [9] of the vertex v by the graph K, as the graph $M = G(v,s)K$ such that:
(1) $V(M) = (V(G) \cup V(K)) - \{v\}$ and
(2) $E(M) = (E(G) - \{vx : x \in N_v\}) \cup \{xs(x) : x \in N_v\}$

The vertex v is said to be the vertex substituted by K in G under the function s and this function is called of substitution.

Now let $v_1, \cdots, v_n$ be the vertices of a graph G and $H_1, \cdots, H_n$ a sequence of graphs with no common vertices among themselves or with G. It will be denoted by $M_k = M_{k-1}(v_k, s_k)H_k$ the graph which is obtained by substitution of k vertices of G by graphs H_i, $1 \leq i \leq k$, where $M_0 = G$. In other words, M_1 denotes a graph obtained by substitution of only one vertex of G, M_2 denotes a graph obtained by substitution of only one vertex of M_1, and so on. Note that every vertex substituted must belong to $V(G)$.

It can be said that an edge of the substitution M_p is an **internal edge** if it is of the forms $s_i(x)s_i(y)$. The edge in M_p that are not internal edge will be nominated **external edge**. Let G be a graph whithout isolated vertices. If each vertex v of G is substituted by a graph complete with val(v) vertices, through an injective function, it will be said that the graph G has been **expanded**. It will be noted such graph by $\tilde{G}$. To be worth while to observe that the type of graphs constructed by Sabidussi [4] will be isomorphic with the expanded graph constructed by substitution.

3 The problem

Let G be a (p,q) graph, connected, hamiltonean and $\delta(G) \geq 3$.

If $\{S_i\}$ is a sequence of graphs with no common vertices among themselves nor with G, then

(i) Is $M_p(G)$ hamiltonean? and

(ii) $Aut(M_p(G)) \approx Aut(G)$?

3.1 Complete graph case

In this case G is a copy of the complete graph K_p, $p \geq 3$, and $\{S_i\}$ is a sequence of graphs with no common vertices among themselves nor with G where each S_i is a copy of a complete graph (or each S_i is a copy of a cycle).

Theorem 3.1 *Let G be a copy of the complete graph K_p, $p \geq 3$, whose vertices are labeled $v_1, \cdots, v_p$ and $\{S_i\}$ is a sequence of graphs with no common vertices among themselves nor with G, where each $S_i \approx K_{p-1}$, then*

(i) $M_p(G)$ is a hamiltonean and

(ii) $Aut(M_p(G)) \approx Aut(G)$.

Proof

(i) Since G is a complete graph then it has a generator cycle denoted by $C(G)$. Also each S_i have a generator cycle denoted by $C^{(i)}(G)$. Through an suitable selection of the substitution functions, the cycle $C(M_p(G))$ defined by $C(M_p(G) \approx \cup_{i=1}^{p} C^{(i)}(G)$ is a generator of $M_p(G)$.

(ii) The only one admissible movement in $M_p(G)$, through a symmetry of its vertices that preserve edge, are the induced by symmetry of the vertices of G that preserve edge. In fact the internal edge of a block of $M_p(G)$ only may be interchanged by internal edges of other block of $M_p(G)$ [12]. By this reason G and $M_p(G)$ have the same group of automorphisms, S_p. ■

Theorem 3.2 *Let G be a copy of the complete graph K_p, $p \geq 3$, whose vertices are labeled $v_1, \cdots, v_p$ and $\{S_i\}$ is a sequence of graphs with no common vertices among themselves nor with G, where each $S_i \approx C_{p-1}$, then*

(i) $M_p(G)$ is a hamiltonean and

(ii) $Aut(M_p(G))$ not always is isomorphic with $Aut(G)$.

Proof

(i) Since G is a complete graph then it has a generator cycle denoted by $C(G)$. Through an suitable selection of the substitution functions, the cycle $C(M_p(G))$ defined by $C(M_p(G)) \approx \cup_{i=1}^{p} S_i$ is a generator of $M_p(G)$.

(ii) By example, if G is the complete graph K_5 then $Aut(M_5(G))$ is the dihedral group of order 10 [Mont.,Scientia], while $Aut(G) \approx S_5$. ■

3.2 Regular graph case

Theorem 3.3 *Let G be a hamiltonean graph, r-regular, $r > 2$, whose vertices are labeled $v_1, \cdots, v_p$. If $\{S_i\}$ is a sequence of graphs with no common vertices among themselves nor with G, where each $S_i \approx K_{val(v_i)}$, then*

(i) $M_p(G)$ is hamiltonean and

(ii) $Aut(M_p(G)) \approx Aut(G)$.

Proof

(i) Since G is a hamiltonean graph then it has a generator cycle denoted by $C(G)$. Also each S_i have a generator cycle denoted by $C^{(i)}(G)$. Through a suitable selection of the substitution functions, the cycle $C(M_p(G))$ defined by $C(M_p(G)) \approx \cup_{i=1}^{p} C^{(i)}(G)$ is generator of $M_p(G)$.

(ii) The only one admissible movement in $M_p(G)$, through a symmetry of its vertices that preserve edges, are the induced by symmetry of the vertices of G that preserve edge. In fact the internal edge of a block of $M_p(G)$ may be interchanged only by internal edges of other block of $M_p(G)$ [12]. For this reason G and $M_p(G)$ have the same group of automorphisms. ■

Theorem 3.4 *Let G be a hamiltonean graph, $r-$regular, $r > 2$, whose vertices are labeled $v_1, \cdots, v_p$. If $\{S_i\}$ is a sequence of graphs with no common vertices among themselves nor with G, where each $S_i \approx C_{val(v_i)}$, then*

(i) $M_p(G)$ is a hamiltonean and

(ii) $Aut(M_p(G))$ not always is isomorphic with $Aut(G)$.

Proof

(i) Since G is a hamiltonean graph then it has a generator cycle denoted by $C(G)$. Through a suitable selection of the substitution functions, the cycle $C(M_p(G))$ defined by $C(M_p(G)) \approx \cup_{i=1}^{p} S_i$ is generator of $M_p(G)$.

(ii) Is obvious according with Theorem 3.2,(ii). ■

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