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GRAPH WITH GIVEN AUTOMORPHISM GROUP AND GIVEN CHROMATIC INDEX.

by

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Abstract. In 1938, Frucht [2] proved that every finite group may be represented by a graph, that is to say, given any finite group H, there is a graph G whose automorphism group is isomorphic to H.

This paper pretends to prove that for every finite group H and for every positive integer number n≥3, there exists a graph G, that represents H and whose chromatic index is n.

1. Introduction. Every graph has an automorphism group (permutations of their vertices that preserve adjacency). A famous result of Frucht [2] states that for any finite group H there is a graph G whose automorphism group is isomorphic to H. In this direction it is said that the group H is represented by the graph G. Later, in 1949, Frucht [3], proved the existence of 3-regular graphs with group isomorphic to a given finite group. Gert Sabidussi [6] was a pioneer in studying the existence of such graphs with given properties. For this reason it is usual to name these results as Sabidussi's Theorems. The object of this paper is a prove that for every finite group H and for every positive integer number n≥3, there exists a graph G, that represents H and whose chromatic index is n.

2. General Terminology. A graph G is a system (V,E) where V is a finite non empty set and E is a set of pairs {x,y} where x and y are distinct elements of V. Each element of V is called a **vertex** and each element of E is called an **edge**. The set V and the set E may be denoted by V(G) and E(G) respectively. Two vertices u and v it are called **neighbors** if {u,v} is an edge of G. For any vertex v of G, denote by N_v the set of neighbors of v. To simplify the notation, an edge $\{x,y\}$ is written as xy. When refering to disjoint graphs we mean graphs whose sets of vertices are pairwise disjoint. Other concepts not defined explicitely in this work can be found in the texts [1], [4] or [5].

3. Proof of the Theorem. An operation, introduced in [5] called substitution is performed by replacing a vertex by a graph. A more precise description is the following:

Assume that G and K are two graphs with no common vertices. For a vertex v in V(G) and function $s:N_v \dashrightarrow V(K)$ we define the substitution of the vertex v by the graph K, as the graph $M=G(v,s)K$ such that:

(1) $V(M)=(V(G)\cup V(K))-\{v\}$ and

(2) $E(M)=(E(G)-\{vx/x\in N_v\})\cup E(K)\cup\{xs(x)/x\in N_v\}$.

The vertex v is the vertex substituted by K in G, under the function s. In the Figure 1, the vertex v is the vertex substituted by G_2 in G_1, under the function $s:N_v \longrightarrow V(G_2)$ indicated.

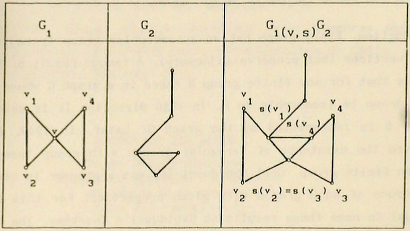

FIGURE 1

Now let $v_1,\ldots,v_n$ be the vertices of a graph G and $H_1,\ldots,H_n$ a sequence of disjoint graphs such that each H_i is disjoint wwith G. We will denote by $M_k=M_{k-1}(v_k,s_k)H_k$, $1\le k\le n$, the graph which is obtained by substitution of k vertices of G by graphs H_i, $1\le i\le k$, where $M_0=G$. In other words, M_1 denotes a graph obtained by substitution of only one vertex of G, M_2 denotes a graph obtained by substitution of only one vertex M_1, and so on. Note that every substituted vertex must belong to V(G).

Figure 2 shows a diagram of a graph $K_4(v_i, s_i)S_i$, $1\le i\le 4$, where the functions s_i are biyectives and the graphs S_i are isomorphic a C_3.

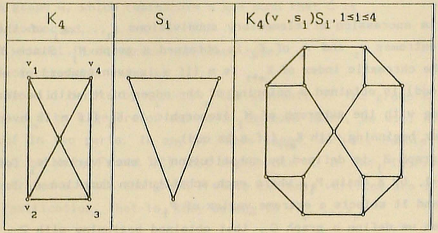

FIGURE 2

A necessary concept by this work is the chomatic index. A more precise description is the following:

Let G be a nonempty graph. G is r-edge colorable if there is a epijective function $f:E(G) \longrightarrow \{1,\dots,r\}$ so that if e and e' are incident edges then $f(e) \neq f(e')$. The minimun r for which a graph G is r-edge colorable is its **chromatic index** and is denoted by $\chi_1(G)$.

LEMMA 1: If H is the trivial group and $m\ge 3$, then there exist infinite not homeomorphic graphs G, such that G represents H and its chromatic index is m.

Proof. Case (1): m=3. For each positive integer number j take a (4j+1)-cycle with vertices $v_1, v_2, \dots, v_{4j+1}$ and substitute each vertex with a path P_i (of length i-1) so that one extreme vertex of P_i is identified with v_i. A graph G_j is formed. Clearly the group of G_j is trivial and its chromatic number is 3. In Figure 3 it's illustrated the graph G_1.

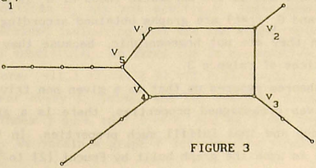

FIGURE 3

Case (2): $m \geq 4$. We consider now an arbitrary natural number $j \geq 2$, the complete graph K_m with $V(K_m)=\{V_1,\ldots,V_m\}$ and m chains P_i, $1 \leq i \leq m$, with no common vertices between themselves and without vertices in $V(K_m)$.

By a succession of elementary subdivisions $g_1,\ldots,g_j$ of the edge with extremes v_1 and v_m of K_m is obtained a graph M_j. Since $\Delta(M_j)=m$ and the chromatic index of K_{m+1} is m (if m is even number) or m+1 (if m is odd) is obtained a coloring of the edges of M_j with mcolors beginning with the subgraph of M_j isomorphic to K_m (if n is even number) or beginning with K_{m+1} (if n is odd).

A graph Z_j is defined by substitution of each vertex v_i (of M_j), $1 \leq i \leq m+1$, by a chain P_i, where each substitution function s_i is constant and it selects a extreme vertex of P_i.

Then we define a graph G_j, that obtained beginning with Z_j, substituting each one of the j vertices of subdivision $g_1,\ldots,g_j$ by a copy of K_2 according to a constant substitution, as illustrated in the Figure 4, for j=3 and m=4.

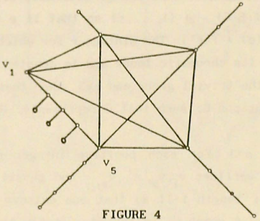

FIGURE 4

As the chromatic index of M_j is n and $\Delta(Z_j)=m+1$, then the chromatic index of Z_j is m+1 and so the chromatic index of G_j is m+1.

Finally, if G_r and G_t $(r \neq t)$ are graphs obtained according the previous method, then they are not homeomorphic, because they defer in the number of vertices of valence 3.

The following theorem assures us that to a given non trivial finite group H and given preassigned properties, there is a graph that represents the group and that fulfill such properties. In the proof of this Theorem it is used the graph built by Frucht [2] to the given group H. Thus, we will not prove the properties (a) and (b) of the next theorem.

Theorem . Given a finite group H of order >1 and integer number $n \geq 3$, exist a graph G, which represents H and such that G is

a) connected,

b) without fixed vertex nor fixed edge,

c) of chromatic index n.

Proof. We will denote by $v_1, \ldots, v_p$ the elements of the group H. Let $C=\{h_1, \ldots, h_m\}$ be a generator set of H. Because $|C| \geq 1$, we will divide the proof in two parts. In any of these cases, we will denote by F the graph built by Frucht [2] that represents H. In the Figure 5 it is illustrated a chain the graph F whose extremes vertices are vertices of ramification, that is, they belong to $\{v_1, \ldots, v_p\}$.

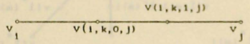

FIGURE 5

First part : $|C|>1$. Here it's distinguished between $n>3$ or $n=3$.

Subcase 1: $n>3$.

To a fixed natural number $j \geq 2$, we considerate the G_j built in the proof of Lemma 1 (Case(2)), and for each μ, such that $1 \leq \mu \leq pm$, we define M_μ as a copy of G_j separated by vertex of F and so that each pair of such copies do not have vertices in common.

Beginning with F, it is built a graph Y, substituting each ramification vertex v_i, $1 \leq i \leq p$, of F by a 2m-cycle C(i,2m).

That substitution will have to be performed by a inyective substitution function

$$s_i : N_{v_i} \longrightarrow V(C(i,2m))$$

These functions s_i are not arbitrary. Their description remains totally determined indicating what are the edges impinging on the copies C(i,2m) substituted. we label the 2m consecutive vertices in the following form: v(i,1,1), v(i,1,2), v(i,2,3), v(i,2,4), ..., v(i,m,2m-1), v(i,m,2m). For every i, $1 \leq i \leq p$, and for every k, $1 \leq k \leq m$, we denote by $i^{*}(k)$ that index j such that $v_i = v_j h_k$ and by $i^{\wedge}(k)$ that index t such that $v_t = v_i h_k$. We obtain a graph Y defining the substitutions so that for every k, $1 \le k \le m$, the vertex v(i,k,2k-1) is adjacent to the vertex v(i,k,0,i^(k)) and the vertex v(i,k,2k) is adjacent to the vertex v(i•(k),k,1,i).

In the Figure 6 (a) is illustrated the substitution of the 2m directed edges impinging on a vertex v_i by 2m chains of length 3 and in the Figure 6 (b) it is shown the subgraph induced by C(i,2m) in Y together with the adjacent vertices to each vertex of C(i,2m).

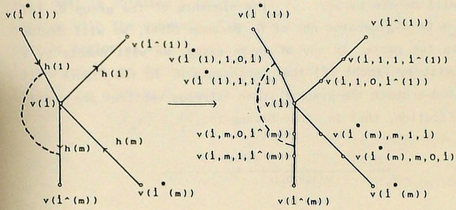

FIGURE 6 (a)

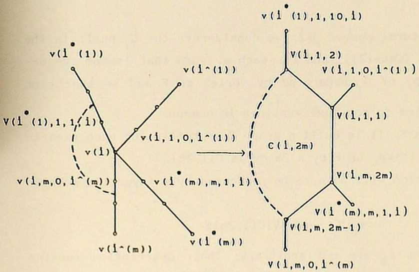

FIGURE 6 (b)

Now, we will prove that all the cycles of Y are even, that is, they have even length.

Since each cycle C(i,2m) is a pair and each directed edge (v_i, v_j), with color h_k, appears represented by a chain of length 3 whose vertices are v(i,k,2k-1), v(i,k,0,j), v(i,k,1,j), v(j,k,2k), we can define the following coloring of the vertices of Y.

Let f be such that:

$$f: V(Y) \longrightarrow \{0,1\},$$

$$f(v(1,k,2k))=f(v(1,k,0,j))=0,$$

$$f(v(1,k,2k-1))=f(v(1,k,1,j))=1.$$

and in the rest of the vertices of Y (which are in copies of chains of length k or 2k+1) f assigns, alternatively, 0 or 1. So Y is bicolorable and accordingly all its cycles are even, that is to say, the graph Y is a bipartite graph.

Beginning with Y is obtained a connected graph H_j, substituting a vertex of valence 1 of each chain of length 2k+1 ($1 \le k \le m$) of Y, by one and only one graph M_μ, $1 \le \mu \le pm$. This substitution is effectuated throughout an inyective function and throughout the only vertex of valence n of M_μ.

Subcase (2): n=3.

For $j \in \mathbb{N}$ let G_j be the graph obtained in the Case (1) of the Lemma 1 for H and for each μ, $1 \le \mu \le pm$, we define M_μ as a copy of G_j disjoint by vertex with F and between them.

Beginning with the graph Y, obtained in the previous Case (1), it is built a graph H_j substituting each extreme vertex (chosen of valence 1) of the chains of length 2k+1 of Y, by one and only one graph M_μ, $1 \le \mu \le pm$. It is observed that this last substitution is fulfilled thoughout a constant function that chooses a vertex of valence 2 and in a cycle of M_μ.

Afterwards we will prove that the graph H_j, built to $m \ge 3$, has the preassigned properties.

As the construction of H_j was fulfilled the color and the direction of the directed edge of $D_c(H)$ and the vertices of ramification v_i were substituted by cycles C(1,2m) (mutually isomorphic) we have that $Aut(G_j) \cong Aut(F) \cong H$.

Moreover H_j is prime graph [6], having vertices of valence one (such vertices do not belong to any cycle of H_j), and conserve the properties of the graph G_j, i.e., H_j is connected graph and has chromatic index n.

Finally, if H_r and H_t, with $r,t \in \mathbb{N}$ and $r \ne t$, are graphs obtained according to the previous method, then they are not homeomorphic, because they defer in the number of vertices of valence 3.

Second part: $|C|=1$. In this case we will do the proof to $|H|=2$, since if $|H|>2$ we can choose a generator set C such that $|C|>1$ and we can built graphs with the prescribed properties according to the exposed method in the first part.

Subcase (1): n=2.

It's obtained a graph H_j, $j\in\mathbb{N}$, with the required properties substituting the vertices of valence 1 of the chains of length 3 of F by graphs isomorphic to G_j graph, built in the Case (1) of the Lemma 1 to H. In this case the substitution are inyectives and chooses a vertex of valence 2 in a cycle of such graph. In the Figure 7 it is diagramed H_1 and cyclic group $H=\{1,a\}$ with generator $C=\{a\}$.

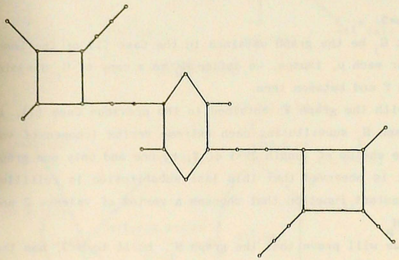

FIGURE 7

Subcase (2): n>2

For each natural number $j\geq 2$ it's obtained a graph H_j, by substitution of each vertex of valence 1 of F (whose distance to the cycle of F is 3), by a graph isomorphic to the graph G_j built in the case (2) of the Lemma 1.

In both cases , H_j fulfills the required properties. Moreover H_i is not homeomorphic to H_j if $i\neq j$, by having different number of vertices of valence 3.

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