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ASYMPTOTIC BEHAVIOR OF THE SOLUTION OF A NONLINEAR DIFFERENCIAL EQUATION

by

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1.- INTRODUCCION. In this paper we study continuation, non-oscillation and asymptotic behavior of solutions of an equation

$$x'' - x = f(t,x), \tag{1}$$

where $f = f(t,x)$ is a continuous function in $[t_0,\infty) \times \mathbb{R}$ with values in $\mathbb{R}$.

We will prove that certain solution $x(t)$ of (1), whose initial conditions do not exceed an amount which will be specified, are defined on all of the interval $[t_0,\infty)$ and satisfy

$$\lim_{t\to\infty} x(t)/e^t = \delta, \tag{2}$$

where δ is a constant (see Theorem 1).

Among these solutions of (1), there exist solutions x for which $\delta \neq 0$, that is, which are non-oscillatory.

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In theorem 2, the results on continuation and asymptotic behavior are extended to

$$(p(t)x')' - q(t)x = f(t,x), \quad t \in [t_0,\infty)$$

where f satisfies (5) below, proving that all the solutions x such that

$$|x'(t_0)z_2(t_0)-x(t_0)z_2'(t_0)| + |x(t_0)z_1'(t_0)-x'(t_0)z_1(t_0)| \le c,$$

where c can computed, are continuable to $[t_0,\infty)$ and they satisfy for t ∞

$$x(t)=(\delta_1+o(1))z_1(t)+(\delta_2+o(1))z_2(t), \tag{3}$$

where $\{z_1,z_2\}$ is a fundamental system of solution of

$$(p(t)y')' - q(t)y = 0. \tag{4}$$

These theorems extend and improve some results of Atkinson [1], Taliaferro [2], Waltman [3], Trench[4] and Hille [5].

In general, the function $f=f(t,x)$ is a continuous function on $[a,\infty)\times\mathbb{R}$, $a\in\mathbb{R}$ fixed, satisfying

$$|f(t,x)| \le \sum_{i=1}^{m} \lambda_i(t)w_i(|x|), \quad m \in \mathbb{N} ;$$

where $\lambda_i=\lambda_i(t)$, $(1\le i \le m)$ are suitable continuous functions on the interval $[a,\infty)$ and the function w_i verify the following condition:

H)The functions w_i $(1\le i\le m)$ are continuous and mondecreasing on $[0,\infty)$, positive on $(0,\infty)$ and $\frac{w_{i+1}}{w_i}$ are nondecreasing on $(0,\infty)$.

The proofs of our results are based on a theorem which gives an explicit pointwise estimate, independent on u, for a funtion $u = u(t)$ which satisfies the inequality

$$u(t) \le c + \sum_{i=1}^{m} \int_a^t \lambda_i(s)w_i(u(s))ds,$$

for $t\in[a,b]$, where $c>0$ is a constant and the funtions λ_i

$(1 \le i \le m)$ are continuous and nonnegatives on the interval $[a,b]$.

We define:

I_1) The functions

$$W_k(u)=\int_{u_k}^{u} \frac{ds}{w_k(s)} \ ; \ u \ge u_k > 0, \ 1 \le k \le m$$

and W_k^{-1} their inverse functions.

I_2) The functions $\varphi_0(U) = u$ and

$\varphi_k = o\, g_{k-1} o \dots g_1; \quad 1 \le k \le m$

where

$$g_k(u) = W_k^{-1}[W_k(u) + \alpha_k(a, b_1)],$$

$$\alpha_k(a, b_1) = \int_a^{b_1} \lambda_k(s)ds, \quad b_1 \le b.$$

Now, for any $k=1,2,\dots,m$ the functions g_k, and hence φ_k, dependend on u, a and b_1.

Theorem A ([6]). Assume that the functions w_i $(1 \le i \le m)$ satisfy (H), the function u and λ_i $(1 \le i \le m)$ are continuos and nonnegative on the interval $[a,b]$ and the constant c is positive. If

$$u(t) \le c + \sum_{i=1}^{m} \int_a^t \lambda_i(s) w_i(u(s))ds, \quad \text{for} \quad t \in [a,b].$$ Then for any $t \in [a,b_1)$ we have

$$u(t) \le W_m^{-1}[W_m(\varphi_{m-1}(c) + \int_a^t \varphi_m(s)\, ds],$$

where b_1 is a number in the interval $[a, b]$ such that

$$\int_a^{b_1} \lambda_k(ds) \le \int_{\varphi_{k-1}(c)}^{\infty} \frac{ds}{w_k(s)}, \quad 1 \le k \le m.$$

2.- THE MAIN RESULTS. Now, we can establish our results about continuation and nonoscillation mainly, and also on asymptotic integration of the solution of equation (1). We begin by considering f such that

$$|f(t, x)| \le \sum_{i=1}^{m} \lambda_i(t)|x|^{\gamma_i}, \ \gamma_i > 1. \tag{5}$$

Theorem 1. Let $t_0>0$ and $1<\gamma_1\le\gamma_2\le\ldots\le\gamma_m$. Suppose

H_1) $f=f(t,x)$ is a continuous function on $[t_0, \infty)\times R$ and satisfies (5), where $\lambda_i= \lambda_i(t)$, $1\le i\le m$, are continuous and monnegative functions on the interval $[t_0,\infty)$ such that

$$e^{t\gamma_i}\cdot\lambda_i \in L_1([t_0, \infty)).$$

H_2) There exists a positive constant c such that

$$\alpha_i(t_0) = \int_{t_0}^{\infty} e^{s(\gamma_i-1)}\cdot\lambda_i(s)ds \le \frac{2^{\gamma_i-1}}{\gamma_i-1}\varphi(c)^{1-\gamma_i} \ ; \quad 1\le i\le m-1$$

$$\alpha_m(t_0) = \int_{t_0}^{\infty} e^{S(\gamma_m-1)}\cdot\lambda_m(s)ds < \frac{2^{\gamma_m-1}}{\gamma_m-1}\varphi(c)^{1-\gamma_m},$$

where $\varphi=\varphi(u, t_0)$ is the continuons and positive function on $(0,\infty)\times(0, \infty)$ given by

$$\varphi=g_{m-1}\circ g_{m-2}\circ\ldots\circ g_1; \quad g_i(u)=W_i^{-1}[W_i(u)+2^{1-\gamma_i}\cdot\alpha_i(t_o)], \tag{6}$$

with W_i is the primitive defined in (I_1) for the functions $w_i(s) = s^{\gamma_i}$. Then every solution $x = x(t, t_0), x'(t_0))$ of (1) such that $e^{-t_0}[|x'(t_0)+x(t_0)|+|x(t_0)-x'(t_0)|]\le c$ is defined on all of $[t_0, \infty)$ and satisfies

$$\lim_{t\to\infty} x(t)/e^t = \delta > 0. \tag{7}$$

Proof. The functions $z_1(t) = \frac{1}{2}e^t$ and $z_2(t) = \frac{1}{2}e^{-t}$ are linearly independent solutions of the equation

$$x'' - x = 0.$$

Let

$A(t)= x'(t)z_2(t)-x(t)z_2'(t)$, and

$B(t)= x(tz_t'(t)-x'(t)z_1(t)$. Then

$x(t)= A(t)z_1(t) + B(t)z_2(t)$,

$A'(t)z_1(t) + B'(t)z_2(t) = 0$.

Hence A and B are solutions of the differential system

$A'(t)= 2z_2(t)f(t, A(t)z_1(t)+B(t)z_2(t))$

$B'(t)= -2z_1(t)f(t, (A(t)z_1(t)+B(t)z_2(t))$

which integrated on $[t_o, t]$ $(t \geq t_o)$ yields

$$A(t) = A(t_O) + 2\int_{t_o}^{t} z_2(s)f(s, (Az_1+ Bz_2)(s))ds$$

$$B(t) = B(t_O) - 2\int_{t_o}^{t} z_1(s)f(s, (Az_1+Bz_2)(s))ds.$$

Thus, by (5) we have

$$|A(t) \leq |A(t_O)| + \sum_{i=1}^{m}\int_{t_o}^{t} 2^{-\gamma_i} e^{S(\gamma_i-1)}\lambda_i(s)[|A(s)|+\frac{|B(s)|}{e^{2S}}]^{\gamma_i}ds$$

$$|B(t)\leq \frac{|B(t_o)|}{e^{2t_o}} + \sum_{i=1}^{m}\int_{t_o}^{t} 2^{-\gamma_i} e^{S(\gamma_i-1)}\lambda_i(s)[|A(s)|+\frac{|B(s)|}{e^{2S}}]^{\gamma_i}ds$$

, and $u=|A(t)|+|B(t)|/e^{2t}$ satisfies

$$u(t) \leq u(t_O)) + \sum_{i=1}^{m}\int_{t_o}^{t} 2^{1-\gamma_i} e^{S(\gamma_i-1)}\lambda_i(s)u(s)^{\gamma_i}ds. \qquad (8)$$

Since for $w_i(u)=u^{\gamma_i}$, $\gamma_i \leq \gamma_{i+1}$ is equivalent to w_{i+1}/w_i nondecreasing; if $u(t_o)\leq c$, we can apply the Theorem A to obtain

$$u(t) \leq W_m^{-1}[W_m(\varphi_{m-1}(c))+2^{1-\gamma_m}\int_{t_0}^{t}\lambda_m(s)e^{S(\gamma_m-1)}ds].$$

By (H_2), this last estimate es valid for every $t\in [t_o, \infty)$.

Moreover, for any $t \in [t_0, \infty)$,

$$u(t) \le W_m^{-1}[W_m(\varphi_{m-1}(u(t_0)) + 2^{1-\gamma_m}\int_{t_0}^{\infty} \lambda_m(s)e^{s(\gamma_m - 1)}ds] =$$

$$\varphi_m(u(t_0), t_0). \tag{9}$$

where $\varphi_m = \varphi_m(u, t_0)$ is given by (I_2).

Thus, if $u(t_0) = \frac{e^{-t_0}}{2}[|x'(t_0)+x(t_0)| + |x(t_0) - x'(t_0)| \le c$, then u is bounded on the interval where it is defined.

Let us assume now that u(t) is defined on a maximal interval $[t_0, t_1)$, then u(t) is a bounded solution on $[t_0, t_1)$. Thus $z_i(s)\cdot f(s,(Az_1+Bz_2)(s)) \in L_1([t_0, t_1))$, $(i=1,2)$ and $\lim_{t \to t_1^-} u(t)$ exists. Then we can continue the solution beyond t_1. Therefore ü(t) is defined and bounded on all of $[t_0, \infty)$. Then A and B are defined on $[t_0, \infty)$ and hence $\lim_{t\ \infty} A(t)$ and $\lim_{t\ \infty} B(t)/e^{2t}$ exist. So, the same is true for

$$V(t) = A(t) + B(t)/e^{2t} = 2x(t)/e^t.$$

Therefore, the corresponding solution x(t) is defined on all of $[t_0, \infty)$ and $\lim_{t\ \infty} V(t) = 2\lim_{t\ \infty} x(t)/e^t$ exists. Thus we have proved (2). Now we will prove (7):

Let $0<\epsilon<c(1)$ and $\bar{x} = \varphi_m(c(1), 1)$ and let $t_0 > 1$ be a number such that

$$\sum_{i=1}^{m} \int_{t_0}^{\infty} 2^{-\gamma_i}\lambda_i(s)e^{s\gamma_i}\bar{x}^{-\gamma_i}ds < \epsilon \tag{10}$$

Moroever, let $V_0(t) = A_0(t) + B_0(t)/e^{2t}$ be the solution with initial conditions

$$A_0(t_0) = \varepsilon, \qquad B_0(t_0) = 0. \text{ Then}$$

$$V_0(t) = \varepsilon + \int_{t_0}^{t} \frac{(t-s)}{e^t} x(s)ds + \int_{t_0}^{t} \frac{(t-s)}{e^t} f(s, \frac{e^s}{2}V_0(s))ds, \text{ hence}$$

$$|V_0(t)|\ge\varepsilon-\int_{t_0}^{t}\frac{(t-s)}{e^{t}}|x(s)|ds-\int_{t_0}^{t}\frac{(t-s)}{e^{t}}|f(s,\frac{e^{s}}{2}V_0(s))|ds$$

$$\ge\varepsilon-\int_{t_0}^{t}\frac{(t-s)}{e^{t-s}}\frac{|x(s)|}{e^{s}}ds-\sum_{i=1}^{m}\int_{t_0}^{t}\frac{(e^{t}-s)}{e^{t}}\cdot 2^{-\gamma_i}\cdot\lambda_i(s).e^{s\gamma_i}\bar{x}^{-\gamma_i}ds.$$

From (9), we have

$$|V_0(t)|\le\varphi_m(V_0(t_0)t_0)=\varphi_m(\epsilon,t_0)\le\varphi_m(\epsilon,1)\le\varphi_m(c(1),1)=\bar{x};\qquad(11)$$

because $t_0>1$, $\varepsilon<c(1)$ and $\varphi_m(u,t_0)$ is nondecreasing in the variable u and nonincreasing in the variable t_0.

Finally we obtain

$$V_0(\infty)>\varepsilon-\sum_{i=1}^{m}\int_{t_0}^{\infty}2^{-\gamma_i}\cdot\lambda_i(s)\cdot e^{s\gamma_i}\cdot\bar{x}^{-\gamma_i}ds>0.$$

Then any solution $x=x(t,t_0,\ \varepsilon e^{t_0}/2,\varepsilon)$ of (1) as it is indicated in (10) satisfies (7).

Remark 1. Reordering the sub-index i, theorem 1 is valid for any usual order between the γ_i.

Remark 2. Since, for $\gamma_i>1$, $1\le i\le m$, it is easy to prove that $\varphi(0^+)=0$, then for any $t_0>0$, there exists, $c=c(t_0)$ which satisfies (H_2).

Corollary 1. Under the conditions of Theorem 1, equation (1) has nonoscillatory solutions.

The results on continuation and asymptotic integration are extended to

$$(px')'-qx=f(t,x),\qquad(12)$$

Where $p=p(t)$ is a positive and continuosly differentiable funtion on $[t_0,\infty)$ such that $p(t_0)=1$ and $q=q(t)$ is a continuous function on $[t,\infty)$. (13)

Theorem 2. Let $t>0$ and $1<\gamma_1\le\gamma_2\le\ldots\le\gamma_m$, and assume that

H_1^*) The functions $p=p(t)$ and $q=q(t)$ satisfy the conditions (13).

H_2^*) $f=f(t,x)$ is a continuous funtion on $[t_0,\infty)\times\mathbb{R}$ and verifies (5), where $\lambda_i=\lambda_i(t)$, $1\le i\le m$, are continuous and nonnegative functions on the interval $[t_0,\infty)$ such that $\lambda_i\tilde{z}_i\in L_1([t_0,\infty))$, and $\tilde{z}_i(t)=\max(|z_1(t)|^{\gamma_{i+1}},|z_2(t)|^{\gamma_{i+1}})$, (z_1,z_2) is a funamental system of solutions of (4).

H_3^*) There exists a positive constant c such that

$$\beta_i(t_0)=\int_{t_0}^{\infty}\lambda_i(s)\tilde{z}_i(s)ds\le\frac{1}{2}\int_{\varphi(c)}^{\infty}\frac{ds}{W_i(s)};\quad 1\le i\le m-1$$

$$\beta_m(t_0)=\int_{t_0}^{\infty}\lambda_m(s)\tilde{z}_m(s)ds<\frac{1}{2}\int_{\varphi(c)}^{\infty}\frac{ds}{W_m(s)},$$

where the function φ is defined in (6), with β_i instead of α_i.

Then any solution $x=x(t,t_0,x(t_0),x'(t_0))$ of (12) such that $|x'(t_0)z_2(t_0)-x(t_0)z_2'(t_0)|+|x(t_0)z_1'(t_0)-x'(t_0)z_1(t_0)|\le c$ is defined on all $[t_0,\infty)$ and satisfies (3).

Proof. In order to establish (3), it is enough to do it for the fundamental system of solution $(z_1(t),z_2(t))$ so that the wronskian $z_1(t)z_2'(t)-z_2(t)z_1'(t)$ is equal to -1.

Let $A(t)=x'(t)z_2(t)-x(t)z_2'(t)$ and $B(t)=x(t)z_1'(t)-x'(t)z_1(t)$. Then

$$\left.\begin{aligned} x(t) &= A(t)z_1(t) + B(t)z_2(t) \\ A'(t)z_1(t) &+ B'(t)z_2(t) = 0 \end{aligned}\right\} \qquad (14)$$

We have

$$(px')' = p(A'z_1' + B'z_2') + q(Az_1 + Bz_2),$$

then

$$(px')' = qx = p(A'z_1' + B'z_2'),$$

thus

$$A'z_1' + B'z_2' = p^{-1}f(t, Az_1 + Bz_2). \qquad (15)$$

Since $p(z_1'z_2 - z_1z_2') = 1$, by solving equations (14) and (15) we get $A' = -z_2 \cdot f(t, Az_1 + Bz_2)$, and $B' = z_1 \cdot f(t, Az_1 + Bz_2)$, where the integration on $[t, t]$, $t \geq t_o$, yields

$$\left.\begin{aligned} A(t) &= A(t_o) - \int_{t_o}^{t} z_2(s)\cdot f(s, (Az_1 + Bz_2)(s))ds \\ B(t) &= B(t_0) + \int_{t_0}^{t} z_1(s) f(s, (Az_1 + Bz_2)(s))ds. \end{aligned}\right\} \qquad (16)$$

Thus, by (5) we get

$$|A(t)| + |B(t)| \leq |A(t_o)| + |B(t_o)|$$
$$+ \sum_{i=1}^{m} \int_{t_0}^{t} (|z_1(s)| + |z_2(s)|)\cdot \lambda_i(s) |(Az_1 + Bz_2)(s)|^{\gamma_i} ds,$$

and hence, using

$|Az_1 + Bz_2|^{\gamma_i} \leq (|A| + |B|)^{\gamma_i} \max(|z_1|^{\gamma_i}, |z_2|^{\gamma_i})$, for $1 \leq i \leq m$, we have that $v = |A| + |B|$ satisfies

$$v(t) \leq v(t_o) + 2 \sum_{i=1}^{m} \int_{t_0}^{t} \lambda_i(s) z_i(s) v(s)^{\gamma_i} ds,$$

where z_i, $(1 \leq i \leq m)$ are given by (H_2^*).

So, if $|A(t_o)| + |B(t_o)| \leq c$, where c satisfies (H_3^*), we obtain

$$v(t)\le c+2\sum_{i=1}^{m}\int_{t_0}^{t}\lambda_i(s)z_i(s)v(s)^{\gamma_i}ds.$$

Now, in this last inequality, the application of Theorem A allows to deduce

$$v(t)\le W_m^{-1}[W_m(\varphi_{m-1}(c))+2\int_{t_0}^{t}\lambda_m(s)z_m(s)ds]$$

which by (H_3^*) is valid for every $t\in[t_0,\infty)$. Moreover, for all $t\in[t_0,\infty)$ we get

$$v(t)\le W_m^{-1}[W_{m-1}(\varphi_{m-1}(c))+2\int_{t_0}^{\infty}\lambda_m(s)\tilde{z}_m(s)ds]= k,$$ where

$k>0$ is a constant.

Hence v is bounded on the interval where it is defined. Finally, by proceeding as in the proof of Theorem 1, conclusions of Theorem 2 are achieved.

Now, we apply Theorem 2 to obtain an asymtotic formula:

Theorem 3. Assume that $f=f(t,x)$ satisfies the hypothesis (H_2^*) of Theorem 2, where the continuous and nonnegative functions λ_i $(1\le i\le m)$ verify that $e^{\gamma_i t}\lambda_i(t)\in L_1([t_0,\infty))$. If c verifies (H_3^*) of theorem 2, then any solution x of equation (12) such that $e^{-t_0}[|x'(t_0)+x(t_0)|+|x(t_0)-x'(t_0)|]\le c$ is defined on $[t_0,\infty)$ and there exist constants δ_i; $i=1,2$ such that $x(t)=\delta_1e^{t}+\delta_2e^{-t}+O(1)$, as $t\ \infty$. (17)

Proof. We take $p(t)=q(t)=1$ in equation (12) and the fundamental system of solutions of $z''-z=0$; $z_1(t)=e^{t}/2$ and $z_2(t)=e^{-t}/2$. The system (z_1,z_2) verify $z_1(t)z_2'(t)-z_2(t)z_1'(t)$ $= -1$. Then, by Theorem 2, every solution x of (12) such that $e^{-t_0}[|x'(t_0)+x(t_0)|+|x(t_0)-x'(t_0)|]\le c$ may be written as $x(t)=A(t)e^{t}+B(t)e^{-t}$, where $A(\infty)=\delta_1$ and $B(\infty)=\delta_2$; δ_1, δ_2 constants. But $t(A(\infty)-\delta_1)$ and $t(B(\infty)-\delta_2)$ tends to zero as

$t \infty$. In fact, from (16) we obtain:

$$e^{t}|A(t)-\delta_1|\le\sum_{i=1}^{m}\int_t^{\infty}2^{-\gamma_i}e^{s\gamma_i}(|A(s)|+|B(s)|)^{\gamma_i}ds$$

and

$$e^{-t}|B(t)-\delta_2|\le\sum_{i=1}^{m}\int_t^{\infty}2^{-\gamma_i}e^{s\gamma_i}(|A(s)|+|B(s)|)^{\gamma_i}ds$$

Then $e^{t}|A(t)-\delta_1|=o(1)$ and $e^{-t}|B(t)|)-\delta_2|=o(1)$, therefore, (17) follows.

Some related results can be found in N.Naito [7], Medina Pinto [8], F.Dannan [9] and M.Eastham [10].

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