CUBO 6, 13-18 (1990)
Recibido: Agosto 1989

ON NUCLEAR BERNSTEIN ALGEBRAS

by

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0.-Abstract. In [3] P.Holgate proved that the core of any orthogonal Bernstein algebra is a special train algebra and consequently a genetic algebra. Let A be a Bernstein algebra. Then A is a nuclear algebra if and only if the core of A is A. The present paper proves that the core of a Bernstein algebra is a special train algebra.

1.-Preliminaries. In the following let K be an infinite commutative field whose characteristic is neither 2 nor 3. Let A be a commutative nonassociative K algebra. For every sequence $a_1, a_2, \ldots, a_k$ of k elements of A we define the principal product $a^i = a^{i-1}a_i$, with $a^1 = a_1$. B^k is the set of all finite sums of products a^k of k elements in $B \subseteq A$. B^k is called principal power of B. We say that $a \in A$ is nilpotent if there exists $n \in \mathbb{N}$ such that $a^n = 0$. A is nilpotent if there exists $t \in \mathbb{N}$ such that $A^t = 0$. If all elements of A are

1) FONDECYT 227-89; DIUFRO 710-88; CCINT-USP-BRAZIL
2) TO APPEAR IN L.A.A.

nilpotents we say that $\mathcal{A}$ is a nilalgebra. $\mathcal{A}$ is called a Jordan algebra if $xy=yx$ and $x^2(xy)=x(x^2y)$ $\forall x,y\in\mathcal{A}$. A.A.Albert proved that all finite dimensional Jordan nilalgebras of characteristic $\neq 2$ are nilpotent.

Lemma 1. Let $\mathcal{A}$ be a commutative algebra ,char$(\mathcal{A})\neq 2$ and $x^3=0$ $\forall x\in\mathcal{A}$. Then $\mathcal{A}$ is a Jordan algebra.

Proof. The identity $x^3 = (x+y)^3 = (x-y)^3 = 0$ implies that $0 = (x+y)^3-(x-y)^3 = 2(2x(xy)+x^2y)$. But char$(\mathcal{A}) \neq 2$, hence $x^2y=-2x(xy)$. Replacing y by xy we obtain $x^2(xy)=-2x(x(xy)) = x(-2x(xy)) = x(x^2y)$, i.e. $\mathcal{A}$ is a Jordan algebra. ∎

Let $(\mathcal{A},\omega)$ be an $(n+1)$ dimensional commutative non-associative baric K-algebra where $\omega:\mathcal{A} \longrightarrow K$ is a weight function. $(\mathcal{A},\omega)$ is called a **Bernstein** algebra iff $(x^2)^2=\omega(x)^2x^2$, $\forall x\in\mathcal{A}$. In any Bernstein $(\mathcal{A},\omega)$ algebra the nontrivial homomorphism ω is uniquely determined, and $\mathcal{A}$ possesses at least one nontrivial idempotent element $e\in\mathcal{A}$, (see [5]).The e-canonical decomposition of $\mathcal{A}$ is $Ke\oplus U\oplus V$ where $U=\{y\in\ker(\omega):ey=\frac{1}{2}y\}$ and $V=\{y\in\ker(\omega):ey=0\}$.The subspaces U and V satisfy the fundamental relations $U^2\subseteq V$, $UV\subseteq U$, $V^2\subseteq U$, $UV^2=0$, $U^2V^2=0$ and the fundamental identities $u_1^3=0$, $u_1(u_1v_1)=0$, $u_1(u_2u_3)+u_2(u_3u_1)+u_3(u_1u_2) = 0$ (Jacobi's identity); $\forall u_1\in U$, $v_1\in V$, and $(xy)(zt)+(xz)(yt)+(xt)(yz) = 0$ $\forall x,y,z,t\in N = \ker(\omega) = U\oplus V$.

Lemma 2. Let $(\mathcal{A},\omega)$ be a Bernstein algebra. Then $N=\mathrm{Ker}(\omega)$ and its principal powers are ideals of $\mathcal{A}$.

Proof. N is an ideal of $\mathcal{A}$, since the kernell of a homomorphism is an ideal. P.Holgate proved [3,p 615], that all $y\in N$ satisfy $e(y_1y_2) = (\frac{1}{2}y_1)y_2-(2ey_1)(ey_2)$.Thus if $y_1\in Y_1$

and $y_2 \in Y_2$ where Y_1 and Y_2 are L_e- invariant subspaces of N, then the product Y_1Y_2 is L_e-invariant. For all $(\alpha e+u+v)\in A$ we have $(\alpha e+u+v)N^{k+1} = \alpha e(NN^k) + (u+v)(NN^k)$. For the induction hypothesis, let us suppose that N^k is an ideal of A. But we know that N is also an ideal of A, hence they are L_e-invariant subspaces of A. Finally, it is concluded that $(\alpha e+u+v)N^{k+1} \subseteq N^{k+1}+(u+v)N^k = N^{k+1}$. ∎

2.- Bernstein Special-train algebras. Let (A,ω) be a baric K-algebra. (A,ω) is called a Special-train algebra when the principal powers of $\ker(\omega)$ are ideals of A and $\ker(\omega)$ is nilpotent.

Proposition 1. Let (A,ω) be a Bernstein algebra. If $x^3=0$, $\forall x\in N$ then A is a special train algebra.

Proof. $N=\ker(\omega)$ is a subalgebra of B, hence N is a finite dimensional commutative algebra with $x^3=0$ $\forall x\in N$. By lemma 1, N is a Jordan algebra, and $x^3=0$. Thus N is a finite dimensional Jordan nilalgebra and by Albert's Theorem quoted on p. 96 of [4], N is nilpotent. When Char(K) = 0 we can use the result of Gerstenhaber " All nilalgebra of finite nilindex is nilpotent" (p 53,5), for establishing the fact that N is nilpotent. By lemma 2 the principal powers of N are ideals of A, hence A is a special train algebra. ∎

Corollary . If A is a Bernstein-Jordan algebra then it is a special train algebra.

Proof. It will be proved that $x^3=0$ $\forall x\in N$. Let $x=u+v$ be in $N=U\oplus V$. Then $x^3=u^3+u^2v+2u(uv)+2v(uv)+uv^2+v^3$. By the fundamental relations and identities in Bernstein algebras we can write $x^3=u^2v+v^3+2v(vu)$. In [1] we proved that A is a Bernstein-Jordan algebra iff $V^2=0$ and $v(vU)=0$ $\forall v\in V$. Hence $x^3=0$, $\forall x\in N$ and A is a special train algebra. ∎

Lemma 3. Let $C=Ke\oplus U\oplus U^2$ be a Bernstein algebra, $N=U\oplus U^2\neq 0$ and $D = Ann(N) = \{x\in C:xN=0\}$, then C/D is a Bernstein algebra.

Proof. If $x=\alpha e+u+u_1u_2$ be in D, then $xU=0$ implies $\alpha eU+uU+(u_1u_2)U=0$ Hence $uU = 0$, being the only term on the left that lies in U^2. By Jacobi's identity and $uU=0$ we have $uU^2=0$; thus $u\in D$. Consequently $\alpha e + u_1u_2\in D$, and $\alpha eu_3+(u_1u_2)u_3=0$ for $u_3\in U$. But $\frac{1}{2}\alpha u_3=-u_3(u_1u_2)$ implies $\frac{1}{2}\alpha u_3^2=-u_3(u_3(u_1u_2)) = 0$. If $\alpha \neq 0$ then $u_3^2=0$ for $u_3\in U$. Thus $0 =(u_1 + u_2)^2= u_1^2 + 2u_1u_2+u_2^2$ and $u_1u_2= 0$. Hence $\alpha e \in D$. But $e \notin D$ because $U \neq 0$, and $eu = \frac{1}{2}u$ for $u \in U$. That is, $\alpha = 0$ and $x=u+u_1u_2\in N$ with u, $u_1u_2\in D$. Having, $x=u+u_1u_2\in D$, $y=\alpha e+n\in C$, $n\in N$, we obtain $xy=(u+u_1u_2)\alpha e = \alpha eu = \frac{1}{2}\alpha u$. But $u\in D$; hence $xy\in D$ and D is an ideal of C. That is, $(C/D,\omega')$ is a Bernstein algebra, since it is a homomorphic image of C. Naturally $\omega'(c+D) = \omega(c)$ defines the weight function in C/D. Let us note that $x=c+D \in ker(\omega')$ iff $\omega(c) = 0$ iff $c\in ker(\omega)$; thus $ker(\omega') = \{c+D:c\in N\} = \overline{N} = \overline{U}\oplus\overline{V}$. ∎

Proposition 2. Let $A = Ke\oplus U\oplus V$ be a Bernstein algebra, and $C=Ke\oplus U\oplus U^2\neq K$ be its core. Then C is a special train algebra.

Proof. First, we shall prove that C/D is a special train algebra. By lemma 3 we know that C/D is a Bernstein algebra. It only remains to prove that $x^3 = \overline{0}$ $\forall x\in ker(\omega')$ and then use Proposition 1. Let $x=n+D$ be an element of $ker(\omega')$, thus $n = u+u_1u_2$. Using the fundamental relations and identities in the Bernstein algebra C, we obtain $n^3= u^2(u_1u_2)+2(u_1u_2)[u(u_1u_2)]\in U$. $V^2\subseteq U$, $V^2U=0$ and Jacobi's identity imply $V^2\subseteq D$ and, consequently $u^2(u_1u_2)\in D$. Furthermore the second and the last fundamental identities imply that

$$2u_3[(u_1u_2)[u(u_1u_2)]] = -2[u(u_1u_2)][u_3(u_1u_2)]$$
$$= (uu_3)[(u_1u_2)(u_1u_2)]\in(uu_3)V^2=0 \quad \forall u_3\in U,$$

because $V^2\subseteq D$. Then we have $(u_1u_2)[u(u_1u_2)] \in U\cap Ann(U)$, and by Jacobi's identity it is in **D**. That is $x^3=n^3+D=D$ and C/D is a special train algebra. This shows that $r \in \mathbb{N}$ exists such that $\overline{0}= (\ker(\omega'))^r = (\overline{\mathbf{N}})^r= \mathbf{N}^r+D$ i.e. $\mathbf{N}^r\subseteq D$. Then $\mathbf{N}^{r+1}=\mathbf{N}\mathbf{N}^r\subseteq \mathbf{N}D = 0$. By lemma 2 we know that $\mathbf{N}^i$ is an ideal of C for all integer i>0. But **N** is nilpotent, hence C is a special train algebra. ∎

Remark 1. It was proved in [2] that the orthogonality is not a necessary condition to be a special train algebra. The present work proves that the orthogonal hypothesis can be removed from Proposition 4 of [3].

Remark 2. It is proved in [5] that the core C of a Bernstein algebra **A** satisfy $C = \mathbf{A}^2$. Let $A = Ke\oplus U\oplus V$ be a Bernstein algebra, then $A^2=A$ implies $V=U^2$, and the meaning of our proposition 2 is that "Every nuclear Bernstein algebra is a special train algebra".

References.

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