ON GENETIC ALGEBRAS WITH PRESCRIBED DERIVATIONS

R.COSTA

1- INTRODUCTION

The terminology and notations of this paper are those of [1]. We recall that a baric algebra over the real field R is an ordered pair (A,ω) where A is a (finite dimensional non associative, non commutative) real algebra and $\omega: A \to R$ is a non zero homomorphism of algebras.
An important class of baric algebras is the class of genetic algebras (in Gorshor's sense). A real algebra A, of dimension n+1, is genetic if it has a basis $C_0, C_1, \ldots, C_n$ such that, if

$$C_i C_j = \sum_{K=0}^{n} \gamma_{ijK} C_K \quad (i, j=0, 1, \ldots, n) \text{ then:}$$

(1) $\gamma_{ooo} = 1$
(2) $\gamma_{ojK} = \gamma_{joK} = 0$ if $K < j$
(3) $\gamma_{ijK} = 0$ when $1 \leq i, j$ and $K \leq \max\{i, j\}$

In this case, $C_0, C_1, \ldots C_n$ is called a canonical basis of A. It can be proved ([14], Chapter 5) that the real numbers γ_{0ii} and γ_{i0i} $(i = 0, 1, \ldots, n)$ are, in fact, independent of the canonical basis of A.
They are called left (resp.rigth) train roots, in short, t-roots, of A. When A is commutative, $\gamma_{i0i} = \gamma_{i0i}$ $(i=1, \ldots, n)$.

Departamento de Matemática y Estadística. Universidad de Sao Paulo, Brasil.

Profesor visitante Universidad de la Frontera (Abril 1986) Visita financiada por UNESCO, Programa de Participación N 2126.

Every genetic algebra may be equipped with a unique non zero homomorphism ω. This function is defined on a canonical basis by $\omega(C_0)=1$ and $\omega(C_i)=0$, $1\leq i\leq n$. The Kernel of ω, which is the n-dimensional ideal generated by $C_1,\ldots,C_n$ will be indicated N.

In [1], we have stutied the derivation algebra of the gametic algebra for a n+1-allelic and 2m-ploid population, denoted by G(n+1,2m). In particular we have proved (§3) that all algebras G(2,2m) have the same derivation algebra, namely, the non Abelian Lie algebra of dimension 2. We have proved also that the derivation algebra of G(n+1,2) has dimension n(n+1), which is the maximum dimension of derivation algebras of genetic algebras of dimension n ([1],th1,cor.1).

We recall, for further use, some facts about G(2,2m). This algebra is commutative and has a canonical basis C_0, $C_1,\ldots,C_m$ such that

$$C_iC_j\begin{cases}\binom{2m}{i+j}^{-1}\binom{m}{i+j}C_{i+j} & \text{if } i+j\leq m\\ 0 & \text{if } m<i+j\end{cases}$$

The t-roots of G(2, 2m), denoted by $t_0, t_1,\ldots,t_m$ are the real numbers $t_k=\binom{2m}{k}^{-1}\binom{m}{k}$ $(k=0,1,\ldots,m)$ and so

$$1=t_0>t_1=\tfrac{1}{2}>t_2>\ldots>t_m=\binom{2m}{k}^{-1}>0$$

Consider now the linear mappings $\partial,\eta\colon G(2,2m)\rightarrow G(2,2m)$ given by

$\partial(C_i)=iC_i$ $(i=0,1,\ldots,m)$ and

$$\eta(C_i)=\frac{t_{i+1}}{t_i-t_{i+1}}C_{i+1}\quad (i<m),\ \eta(C_m)=0$$

We have proved ([1],th 3) that ∂ and η form a basis for the derivation algebra of G(2,2m). We observe that ∂ has proper values $0,1,\ldots,m$, η is nilpotent of index m+1. Moreover $\omega\circ\partial=\omega\circ\eta=0$ and $\partial\circ\eta-\eta\circ\partial=\eta$.
In this paper we construct a large class of non

commutative genetic algebras of dimension m+1 having two derivations satisfying the above conditions. We prove some results concerning the derivation algebra of these algebras.

2.-THE CLASS Ω_{m+1}

Let A be a real vector space of dimension m+1 and $\omega: A \to R$ a non zero linear form. Take a basis $C_0, C_1, \ldots, C_m$ of A such that $\omega(C_0)=1$ and $\omega(C_i)=0$ $(1 \leq i \leq m)$. We have a direct sum decomposition $A=RC_0 \oplus N$ where N=Ker. Let now $\partial: A \to A$ be given by $\partial(C_i) = iC_i (i=0, 1, \ldots, m)$ and $n: A \to A$ be given by $n(C_i) = C_{i+1}$ $(i<m)$ and $n(C_m)= 0$. It is easy to see that $\partial o \eta - \eta o \partial = \eta$, $\omega o \partial = \omega o \eta = 0$, ∂ has proper values $0, 1, \ldots, m$ and η is nilpotent of index m+1.

We show now that this process of construction of ∂ and η is essencially unique.

LEMMA 1

Let A a be real vector space of dimension m+1. Suppose $\partial, \eta: A \to A$ and $\omega: A \to R$ are linear mappings such that:

(1) ∂ has proper values $0, 1, \ldots, m$
(2) η is nilpotent of index m+1
(3) $\omega \neq 0$ and $\omega o \partial = \omega o \eta = 0$
(4) $\partial o \eta - \eta o \partial = \eta$

Then, there exists a unique $C_0 \in A$ such that $\omega(C_0)=1$, $C_0, \eta(C_0), \ldots, \eta^m(C_0)$ is a basis of A and $\partial(\eta^i(C_0))=i\eta^i(C_0)$ for all $i=0, 1, \ldots, m$.

PROOF

Let A_i be the proper subspaces of ∂, that is,

$$A_i=\{x \in A : \partial(x)=ix\}, \quad 0 \leq i \leq m$$

If $1 \leq i$, we have for $x \in A_i$:

$$0=(\omega o \partial)x=\omega(\partial(x))=\omega(ix)=i\omega(x)$$

Hence $\omega(x)= 0$, $x \in \text{Ker}\, \omega$. So $A_i \subset \text{Ker}\, \omega$. This implies $A_1 \oplus \ldots \oplus A_m \subset \text{Ker}\, \omega$. As they have the same dimension m, we have $\text{Ker}\, \omega = A_1 \oplus \ldots \oplus A_m$.
On the other hand, η has a cyclic vector z (its minimal polynomial has degree m+1). Necessarily, $z \notin \text{Ker}\, \omega$ because

otherwise $\{z, \eta(z), \ldots, \eta m(z)\} \subset \mathrm{Ker}\, \omega$, and so $\omega = 0$, contrary to (3). It is also clear that a scalar multiple of a cyclic vector is again a cyclic vector. Hence we may suppose $\omega(z)=1$. Decompose z as $z=C_0+C_1+\ldots+C_m$ with $C_i \in A_i, 0 \leq i \leq m$. We prove now that C_0 is also a cyclic vector for η. We have for $x \in A_i$:
$\partial(\eta(x)=(\eta \circ \partial+\eta)(x)=\eta(ix)+\eta(x)=(i+1)\eta(x)$.

So $\eta(A_i) \subset A_{i+1}$ $(i<m)$ and $\eta(A_m) = 0$. We have the set of equations in triangular form:

$$\left\{\begin{array}{lllll} z = C_0 + & C_1 + & \ldots + C_{m-1} & + C_m \\ \eta(z) = & \eta(C_0) + & \ldots + \eta(C_{m-2}) & + \eta(C_{m-1}) \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \eta^m(z) = & & & \eta^m(C_0) \end{array}\right.$$

from which it is clear that $C_0, \eta(C_0), \ldots, \eta^m(C_0)$ is also a basis of A.

The unicity of C_0 is clear: If C'_0 satisfies $\partial(C'_0) = 0$ then $C'_0=\mu C_0$ $(\mu \in R)$ because 0 is a simple proper value of ∂. Hence $1 = (C'_0) = \mu\,(C_0) = \mu$ so $C'_0 = C_0$.

From now on, A will be a fixed real vector space of dimension m+1, equipped with a non zero linear formω, two linear mappings ∂, $\eta: A \to A$ such that $\omega\circ\partial=\omega\circ\eta=0$, $\partial\circ\eta-\eta\circ\partial=\eta$, η is nilpotent of index m + 1 and ∂ has proper values $0, 1, \ldots, m$. By Lemma 1, these assumptions determine the (unique) basis $C_0, \eta(C_0), \ldots, \eta^m\,(C_0)$. We shall denote this basis by $C_0, C_1, \ldots, C_m$ that is, $C_i=\eta^i(C_0)$.

Recall that each bilinear mapping $\mu: A \times A \to A$ defines on the vector space A a structure of algebra(non associative, non commutative, in general) denoted (A, μ). We say μ is admissible for ω, ∂ and η if $\omega(\mu(a, b)=\omega(a)\omega(b)$, $\partial(\mu(a, b)= =\mu(\partial(a), b) + \mu(a, \partial(b))$, $\eta(\mu(a, b))=\mu(\eta(a), b) + \mu(a, \eta(b))$ for all $a, b \in A$. These conditions mean that when A is equipped with the multiplication μ, ω becomes a homomorphism of algebras and ∂ and n are derivations of this algebras (A, μ).

We denote by Ω_{m+1} the set of all admissible(for ω, ∂ and η) bilinear mappings $\mu: A \times A \to A$. As usual, we indicate $\mu(a, b)$ by ab and we omit the mapping μ when refering to the algebra (A, μ).

We shall denote the derivation algebra of A by Der A. By the own definition of Ω_{m+1}, we have $2 \leq \dim \operatorname{Der} A$, for any $A \in \Omega_{m+1}$. By([1], th. 1), we have also $\dim \operatorname{Der} A \leq m(m+1)$. Any member $A \in \Omega_{m+1}$ such that $\dim \operatorname{Der} A = 2$ will be called a minimal element of Ω_{m+1}. This is the case of $G(2, 2m)$. Conversely, if $\dim \operatorname{Der} A=m(m+1)$, we say A is a maximal element of Ω_{m+1}, as it happens to $G(m+1, 2)$.

PROPOSITION 1:

All members of Ω_{m+1} are genetic algebras relative to the same basis $C_0, C_1, \ldots, C_m$, where $C_i=\eta^i(C_0)$. Moreover, C_0 is an idempotent for each one of these algebras. The real number ½ is a t-root for all commutative algebras in Ω_{m+1}.

PROOF:

Take the cyclic basis $C_0, C_1, \ldots, C_m$ of A, given by Lemma 1.

Given now $0 \leq i, j \leq m$:

$$\partial(C_iC_j)=\partial(C_i)C_j+C_i\partial(C_j)=iC_iC_j+jC_iC_j=(i+j)C_iC_j$$

If $m < i+j$ we must have $C_iC_j=0$ because $i+j$ is not a proper value of ∂. When $i+j \leq m$, C_iC_j is a proper vector of ∂(or the zero vector) corresponding to the proper value $i+j$ of ∂. Hence $C_iC_j= \alpha_{ij}C_{i+j}$ for some real number α_{ij}. In particular, $C^2{}_0=\alpha_{00}C_0$, so $\alpha_{00}=1=\omega(C^2{}_0)$ and $C^2{}_0=C_0$. We have proved that $C_0, C_1, \ldots, C_m$ is a canonical basis of A. Moreover the left (resp.right) t-roots are $1, \alpha_{01}, \ldots, \alpha_{0m}$ (resp. $1, \alpha_{10}, \ldots, \alpha_{m0}$). Taking again the equality $C_iC_j=\alpha_{ij}C_{i+j}$ $(i+j \leq m)$ we get, applying η:

$$\eta(C_iC_j)=\eta(C_i)C_j+C_i\eta(C_j)=C_{i+1}C_j+C_iC_{j+1} \text{ or}$$

$$(\alpha_{i+1,\, j}+\alpha_{i,\, j+1})C_{i+j+1}=\alpha_{ij}C_{i+j+1}$$

If $i+j+1 \leq m$ then we may cancel and obtain $\alpha_{i+1,\, j}+\alpha_{i,\, j+1}=\alpha_{ij}$. In particular $\alpha_{00}=1=\alpha_{01} + \alpha_{10}=2\alpha_{01}$ if A is commutative, and ½ is a t-root .

We will describe more accurately the class Ω_{m+1} in the following way: Each A in Ω_{m+1} has the cyclic basis $C_0, C_1, \ldots, C_m$ of proper vectors of ∂, with $\omega(C_0)=1$. Moreover, $C_iC_j = \alpha_{ij}C_{i+j}$ $(i+j \leq m)$, $C_iC_j = 0$ $(m < i+j)$ and $\alpha_{ij}=\alpha_{i+1,\, j}+\alpha_{i,\, j+1}$ if $i+j+1 \leq m$.
From this, we can associate to A the matrix

$$\tilde{A} = \begin{bmatrix} \alpha_{oo} & \alpha_{o1} & \dots \alpha_{o,m-1} & \alpha_{om} \\ \alpha_{10} & \alpha_{11} & \dots \alpha_{1,m-1} & 0 \\ \alpha_{m-1,0} & \alpha_{m-1,1} & 0 & 0 \\ \alpha_{mo} & 0 & 0 & 0 \end{bmatrix}$$

where $\alpha_{oo} = 1$. The recurrence relation $\alpha_{ij} = \alpha_{i+1,j} + \alpha_{i,j+1}$ $(i+j+1 \leq m)$ shows that each α_{ij} $(i+j \leq m)$ can be expressed as a linear combination of $\alpha_{om}, \alpha_{1,m-1}, \dots, \alpha_{m-1,1}, \alpha_{mo}$ with integral coefficients, in the following way :

We have $\alpha_{ij} = \alpha_{i+1,j} + \alpha_{i,j+1} = \alpha_{i+2,j} + \alpha_{i+1,j+1} + \alpha_{i,j+2} =$
$= \alpha_{i+3,j} + 3\alpha_{i+2,j+2} + 3\alpha_{i+1,j+3} + \alpha_{i,j+3}$ and so on.

We obtain

$$\alpha_{ij} = \alpha_{i+r,j} + \binom{r}{1}\alpha_{i+r-1,j+1} + \dots + \alpha_{i,j+r} = \sum_{k=o}^{r} \binom{r}{k}\alpha_{i+r-k,j+k}$$

for all r such that $i+j+r \leq m$. In particular, taking $i+j+r=m$, that is, $r=m-i-j$, we get

$$\alpha_{ij} = \sum_{k=o}^{m-i-j} \binom{m-i-j}{k} \alpha_{m-j-k,j+k},$$ our desired relation. This

formula can be replaced by

$$(1)\quad \alpha_{ij} = \sum_{l=j}^{m-i} \binom{m-i-j}{l-j} \alpha_{m-l,l} \text{ (calling } j+k=l\text{)}$$

and by

$$\alpha_{ij} = \sum_{l=j}^{m-i} \binom{m-i-j}{m-i-l} \alpha_{m-l,} \quad \text{(because } \binom{m}{k} = \binom{m}{m-k}\text{)}.$$

In particular the left (resp.right) train roots are given by

$$(2)\quad \alpha_{oj} = \sum_{l=j}^{m} \binom{m-j}{l-j} \alpha_{m-l,l} = \sum_{l=j}^{m} \binom{m-j}{m-l} \alpha_{m-l,l} = \sum_{i=O}^{m-j} \binom{m-j}{i} \alpha_{i,m-i}$$

$$(3)\quad \alpha_{jo} = \sum_{l=O}^{m-j} \binom{m-j}{l} \alpha_{m-l,l} = \sum_{l=O}^{m-j} \binom{m-j}{m-j-l} \alpha_{m-l,l} = \sum_{k=j}^{m} \binom{m-j}{m-k} \alpha_{k,m-k}$$

(4) $\alpha_{00}=\sum_{K=0}^{m}\binom{m}{K}\alpha_{K,m-K}=1$

We consider now the affine hyperplane H_1 of R^{m+1} define by $H_1=\{(x_0,x_1,\ldots,x_m)\in R^{m+1}:\sum_{K=0}^{m}\binom{m}{K}x_K=1\}$.

We have shown that, to each member A of Ω_{m+1} we can associate, via matrix $\tilde{A}$, the point $P=(\alpha_{0m},\alpha_{1,m-1},\ldots,\alpha_{m-1,1},\alpha_{m0})$ of H_1. Suppose conversely, we have a point P of H_1, denoted for convenience by $P=(\alpha_{0m},\alpha_{1,m-1},\ldots,\alpha_{m0})$. There existe one and only one matrix $\tilde{A}$, of order m + 1, whose elements α_{ij} satisfy:

(a) $\alpha_{ij}=0$ if $m<i+j$;
(b) $\alpha_{ij}=\alpha_{i+1,j}+\alpha_{i,j+1}$ if $i+j+1\leq m$;
(c) $\alpha_{00}=1$.

Whit this matrix in hand, we define a bilinear mapping on the vector space A by putting $C_iC_j=\alpha_{ij}C_{i+j}$ $(i+j\leq m)$ and $C_iC_j=0$ otherwise. It is routine to verify that this bilinear mapping is ω, ∂, η-admissible.
The coordinates of the point $P=(\alpha_{0m},\alpha_{1,m-1},\ldots,\alpha_{m0})\in H_1$, corresponding to a given A in Ω_{m+1}, are called the H_1-coordinates of A. We have proved:

PROPOSITION 2:

The correspondence associating to $A\in\Omega_{m+1}$ its H_1-coordinates is a one-to-one correspondence between Ω_{m+1} and H_1. In particular, Ω_{m+1} is a m-parametric family of genetic algebras.

As it is well known, H_1 has a natural affine basis, the set of points $P_0=(1,0,\ldots,0),\ldots,P_K=(0,\ldots,\binom{m}{K}^{-1},\ldots,0)$ $,\ldots,P_m=(0,0,\ldots,0,1)$. This means that every point $P=(x_0,x_1,\ldots,x_m)\in H_1$ can be written as

$P=\sum_{K=0}^{m}x_K\binom{m}{K}P_K$ where $\sum_{K=0}^{m}x_K\binom{m}{K}=1$

That is, every point P of H_1 is a baricenter of $P_0,\ldots,P_m$. On the other hand, we have the concept of mixture of algebras, as given by Heuch [9], Holgate [10] or Worz-Busekros [13]: If $\mu_1,\ldots,\mu_s:A\times A\to A$ are bilinear mappings

and $\lambda_1, \ldots, \lambda_s \in R$ with $\lambda_1+\ldots+\lambda_s = 1$, then $\lambda_1\mu_1 +\ldots+ \lambda_s\mu_s$: $A \times A \blacktriangleright A$ is called the mixture of $\mu_1, \ldots, \mu_s$ with coefficients $\lambda_1, \ldots, \lambda_s$.
If, in addition, $\lambda_i \geq 0$ we call $\lambda_1 \mu_1 +\ldots+ \lambda_s\mu_s$ a proper mixture (or a convex combination)of $\mu_1, \ldots, \mu_s$.
Suppose now we are given points $Q_1, \ldots, Q_s$ of H_1. Let $Q = \lambda_1Q_1+\ldots+\lambda_sQ_s \in H_1$, with $\lambda_1+\ldots+\lambda_s = 1$, a baricenter of $Q_1, \ldots, Q_s$.
Let now $\mu, \mu_1, \ldots, \mu_s$ be the bilinear mappings (that is, algebras belonging to Ω_{m+1}) corresponding to $Q, Q_1, \ldots, Q_s$ as in Prop.2. It is easy to prove that $\mu = \lambda_1\mu_1+\ldots+\lambda_s\mu_s$. This is the content of:

PROPOSITION 3:

The correspondence between H_1 and Ω_{m+1} given above is such that to a baricenter of points there corresponds the mixture of corresponding algebras.

If we call A_i $(i=0,1,\ldots,m)$the algebras corresponding to the points $P_0, P_1, \ldots, P_m$ defined above, we have:

COROLLARY:

Every member of Ω_{m+1} is a mixture of $A_0, A_1, \ldots, A_m$.

PROPOSITION 4:

Every member of Ω_{m+1} is completely determined by its left (or right) train roots.

PROOF (left)

We have seen that the left train roots are given by

$$\alpha_{oj} = \sum_{i=0}^{m-j} \binom{m-j}{i} \alpha_{i,m-i} \quad (o \leq j \leq m)$$

This system of linear equalities can be reversed giving $\alpha_{i,m-i}$ as linear combinations of the α_{oj}:

$$\alpha_{k,m-k} = \sum_{i=0}^{k} (-1)^{i+k} \binom{k}{i} \alpha_{o,m-i} \quad (0 \leq k \leq m)$$

If we call $H_2 = \{(x_0, x_1, \ldots, x_m) \in R^{m+1} : x_0 = 1\}$, formulae above give a one-to-one correspondence between

H_1 and H_2. By composition we get a one-to-one correspondence between Ω_{m+1} and H_2. But now the H_2-coordinates of any A in Ω_{m+1} are its left train roots, hence our result.

REMARK 1:

The H_1-coordinates of $G(m+1, 2)$ are $(½, 0, \dots, 0, ½)$. Its H_2-coordinates are, of course, $(1, ½, ½, \dots, ½)$. Its is clear that the H_2-coordinates of $G(2, 2m)$ are $(1, t_1, \dots, t_m)$

where $t_k = \binom{2m}{k}\binom{m}{k}^{-1}$.

We could ask: which is the member of Ω_{m+1} whose H_1-coordinates are proportional to the sequence $\binom{m}{0}$

$\binom{m}{1}, \dots, \binom{m}{m-1}, \binom{m}{m}$?

If $(x_0, x_1, \dots, x_m)\rho\binom{m}{0}, \binom{m}{1}, \dots, \binom{m}{m}$, we must have

$$\sum_{k=0}^{m}\binom{m}{k}\rho\binom{m}{k} = \sum_{k=0}^{m}\rho\binom{m}{k}^2 = \rho\sum_{k=0}^{m}\binom{m}{k}^2 = 1$$

hence

$\rho\binom{2m}{m}^{-1}$ because $\sum_{k=0}^{m}\binom{m}{k}^2 = \binom{2m}{m}$. So we have

the coordinates $\binom{2m}{m}^{-1}(\binom{m}{0}, \binom{m}{1}, \dots, \binom{m}{m}$ and we see, with some calculations, that the answer is just $G(2, 2m)$.

REMARK 2:

It is clear that $A \in \Omega_{m+1}$ is a commutative algebra if and only if its H_1-coordinates are symmetric: $\alpha_{k, m-k} = \alpha_{m-k, k}$ for all $k = 0, 1, \dots, m$ or what is the same, A is symmetric.

REMARK 3:

$G(m+1, 2) = ½ A_0 + ½ A_m$. In fact, the matrices of A_0 and A_m are

and $\lambda_1, \ldots, \lambda_s \in R$ with $\lambda_1 + \ldots + \lambda_s = 1$, then $\lambda_1\mu_1 + \ldots + \lambda_s\mu_s$: $A \times A \rightarrow A$ is called the mixture of $\mu_1, \ldots, \mu_s$ with coefficients $\lambda_1, \ldots, \lambda_s$.
If, in addition, $\lambda_i \geq 0$ we call $\lambda_1 \mu_1 + \ldots + \lambda_s\mu_s$ a proper mixture (or a convex combination) of $\mu_1, \ldots, \mu_s$.
Suppose now we are given points $Q_1, \ldots, Q_s$ of H_1. Let $Q = \lambda_1 Q_1 + \ldots + \lambda_s Q_s \in H_1$, with $\lambda_1 + \ldots + \lambda_s = 1$, a baricenter of $Q_1, \ldots, Q_s$.
Let now $\mu, \mu_1, \ldots, \mu_s$ be the bilinear mappings (that is, algebras belonging to Ω_{m+1}) corresponding to $Q, Q_1, \ldots, Q_s$ as in Prop.2. It is easy to prove that $\mu = \lambda_1\mu_1 + \ldots + \lambda_s\mu_s$. This is the content of:

PROPOSITION 3:

The correspondence between H_1 and Ω_{m+1} given above is such that to a baricenter of points there corresponds the mixture of corresponding algebras.

If we call A_i $(i=0,1,\ldots,m)$ the algebras corresponding to the points $P_0, P_1, \ldots, P_m$ defined above, we have:

COROLLARY:

Every member of Ω_{m+1} is a mixture of $A_0, A_1, \ldots, A_m$.

PROPOSITION 4:

Every member of Ω_{m+1} is completely determined by its left (or right) train roots.

PROOF (left)

We have seen that the left train roots are given by

$$\alpha_{oj} = \sum_{i=0}^{m-j} \binom{m-j}{i} \alpha_{i,m-1} \quad (o \leq j \leq m)$$

This system of linear equalities can be reversed giving $\alpha_{i,m-i}$ as linear combinations of the α_{oj}:

$$\alpha_{K,m-K} = \sum_{i=0}^{K} (-1)^{i+k} \binom{K}{i} \alpha_{o,m-i} \quad (0 \leq K \leq m)$$

If we call $H_2 = \{(x_0, x_1, \ldots, x_m) \in R^{m+1} : x_0 = 1\}$, formulae above give a one-to-one correspondence between

H_1 and H_2. By composition we get a one-to-one correspondence between Ω_{m+1} and H_2. But now the H_2-coordinates of any A in Ω_{m+1} are its left train roots, hence our result.

REMARK 1:

The H_1-coordinates of G(m+1, 2) are $(\frac{1}{2}\ 0, \ldots, 0, \frac{1}{2})$. Its H_2-coordinates are, of course, $(1, \frac{1}{2}\ \frac{1}{2} \ldots, \frac{1}{2})$. Its is clear that the H_2-coordinates of G(2, 2m) are $(1, t_1, \ldots, t_m)$

where $t_k = \binom{2m}{k}\binom{m}{k}^{-1}$.

We could ask: which is the member of Ω_{m+1} whose H_1-coordinates are proportional to the sequence $\binom{m}{0}$

$\binom{m}{1}, \ldots, \binom{m}{m-1}, \binom{m}{m}$?

If $(x_0, x_1, \ldots, x_m) \rho \binom{m}{0}, \binom{m}{1}, \ldots, \binom{m}{m}$, we must have

$$\sum_{k=0}^{m} \binom{m}{k} \rho \binom{m}{k} = \sum_{k=0}^{m} \rho \binom{m}{k}^2 = \rho \sum_{k=0}^{m} \binom{m}{k}^2 = 1$$

hence

$\rho \binom{2m}{m}^{-1}$ because $\sum_{k=0}^{m} \binom{m}{k}^2 = \binom{2m}{m}$. So we have

the coordinates $\binom{2m}{m}^{-1} (\binom{m}{0}, \binom{m}{1}, \ldots, \binom{m}{m}$ and we see, with some calculations, that the answer is just G(2, 2m).

REMARK 2:

It is clear that $A \in \Omega_{m+1}$ is a commutative algebra if and only if its H_1-coordinates are symmetric: $\alpha_{k,m-k} = \alpha_{m-k,k}$ for all $k = 0, 1, \ldots, m$ or what is the same, A is symmetric.

REMARK 3:

$G(m+1, 2) = \frac{1}{2} A_0 + \frac{1}{2} A_m$. In fact, the matrices of A_0 and A_m are

$$\tilde{A}_0 = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ 0 & 0 & & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & & 0 \end{bmatrix} \quad \tilde{A}_m = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 1 & 0 & \ldots & 0 \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ 1 & 0 & \ldots & 0 \end{bmatrix}$$

and so the matrix of $\frac{1}{2} A_0 + \frac{1}{2} A_m$ is

$$\begin{bmatrix} 1 & \frac{1}{2} & . & . & \frac{1}{2} \\ \frac{1}{2} & 0 & . & . & 0 \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ \frac{1}{2} & 0 & . & . & 0 \end{bmatrix}$$

So $C_iC_0 = C_0C_i = \frac{1}{2}C_i$ $(1 \le i \le m)$ and $C_iC_j = 0$ $(1 \le i, j \le m)$.

We give now another characterization of $G(2, 2m)$ in Ω_{m+1}. We have seen in [1], th. 3, that the sequence $\frac{t_1}{t_0 - t_1}, \frac{t_2}{t_1 - t_2}, \ldots, \frac{t_m}{t_{m-1} - t_m}$ is the arithmetic sequence

$\frac{m}{m}, \frac{m-1}{m}, \ldots, \frac{1}{m}$.

This means that in the matrix $\tilde{A} = (\alpha_{ij})$ corresponding to $G(2, 2m)$ we have.

$$\frac{\alpha_{0,\, m-k+1}}{\alpha_{1,\, m-k}} = \frac{k}{m} \quad (k = 1, \ldots, m)$$

PROPOSITION 5:

The only member A of Ω_{m+1} whose matrix $\tilde{A} = (\alpha_{ij})$ satisfies the relations: $m\alpha_{0, m-k+1} = k\alpha_{1, m-k}$ $(1 \le k \le m)$ is the gametic algebra $G(2,2m)$.

PROOF:

By Prop. 4 it is enough to determine the left t-roots of A. We have, with $k=1$, $m\,\alpha_{0m} = \alpha_{1, m-1}$ and so $\alpha_{0, m-1} = \alpha_{0m} + \alpha_{1, m-1} = (m+1)\alpha_{0m}$. From $m\alpha_{0, m-1} = 2\,\alpha_{1, m-2}$, we obtain $\alpha_{1, m-2} = \frac{m}{2}\,\alpha_{0, m-1} = \frac{m(m+1)}{2}\alpha_{0m} = \binom{m+1}{2}\alpha_{0m}$. In a similar way, $\alpha_{0, m-2} = \binom{m+2}{2}\alpha_{0m} \ldots, \alpha_{01} = \binom{2m-1}{m-1}\alpha_{0m}$ $\alpha_{00} = \binom{2m}{m}\alpha_{0m}$.

But $\alpha_{00}=1$ so $\alpha_{0m}=\binom{2m}{m}^{-1}$ and $\alpha_{0,m-k}=\binom{2m}{m}^{-1}\binom{m+k}{k}$ which are exactly the train roots of $G(2, 2m)$.

3- ALTERNATE ALGEBRAS

In this paragraph, we describe a subclass of Ω_{m+1}, closely related to the gametic algebra $G(m+1,2)$.

Recall that every algebra A has an opposite algebra denoted A^0, where $(xy)^0=yx$ for all $x, y \in A$. It is clear that if $A \in \Omega_{m+1}$, the same holds for A^0 and ${}^t\tilde{A}=\tilde{A}^0$, where "t" means transpose.

THEOREM 1:

Let A be a member of Ω_{m+1}, with corresponding matrix $\tilde{A}=(\alpha_{ij})$, $0\leq i,j\leq m$. The following conditions are equivalent:

(i) $\alpha_{0j}+\alpha_{j0}=1$ $(1\leq j\leq m)$
(ii) $\alpha_{0m}+\alpha_{m0}=1$ and $\alpha_{k,m-k}+\alpha_{m-k,k}=0(1\leq k\leq m-1)$
(iii) the submatrix (α_{ij}), $1\leq i,j\leq m$, is skew symmetric
(iv) For all $u,v \in N=\ker \omega$, $uv+vu=0$
(v) For all $x,y\in A$, $xy+yx=\omega(x)y+\omega(y)x$
(vi) For all $x \in A$, $x^2=\omega(x)x$
(vii) $\frac{1}{2} A + \frac{1}{2} A^0=G(m+1,2)$

PROOF:

(i) $\Longrightarrow$ (ii): Follows by direct computation: write the system of equalities $\alpha_{0j}+\alpha_{j0} = 1$ $(j=1,\ldots,m)$ using (2) and (3) and reduce by elementary transformations of linear equations.

(ii) $\Longrightarrow$ (iii): follows directly from formula (1)

(iii) $\Longrightarrow$ (iv): if $u=\sum_{i=1}^{m}\lambda_i C_i$ and $v=\sum_{j=1}^{m}\mu_j C_j$ then

$$uv+vu=\sum_{i,j=1}^{m}\lambda_i\mu_j(C_iC_j+C_jC_i)=\sum_{i+j\leq m}\lambda_i\mu_j(\alpha_{ij}+\alpha_{ji})C_{i+j}=0$$

(iv) $\Longrightarrow$ (v): First of all, $C_kC_{m-k}+C_{m-k}C_k=0$ $(1\leq k\leq m-1)$ implies $\alpha_{k,m-k}+\alpha_{m-k,k}=0$. It follows by direct computation that $\alpha_{0j}+\alpha_{j0}=1(j=1,\ldots,m)$. We have now, for any $u \in N$,

$$C_0u+uC_0=C_0(\sum_{i=1}^{m}\lambda_iC_i)+(\sum_{i=1}^{m}\lambda_iC_i)C_0=\sum_{i=1}^{m}\lambda_i(C_0C_i+C_iC_0)=$$

$$=\sum_{i=1}^{m}\lambda_i(\alpha_{oi}+\alpha_{io})C_i=\sum_{i=1}^{m}\lambda_iC_i=u$$

Take now $x=\omega(x)C_0+u, y=\omega(y)C_0+v$, $u,v \in N$. Then:

$$xy=\omega(x)\omega(y)C_0+\omega(x)C_0v+\omega(y)uC_0+uv$$

$$yx=\omega(x)\omega(y)C_0+\omega(y)C_0u+\omega(x)vC_0+vu$$

$$xy+yx=2\omega(x)\omega(y)C_0+\omega(x)(C_0v+vC_0)+\omega(y)(C_0u+uC_0)=$$
$$=2\omega(x)\omega(y)C_0+\omega(x)v+\omega(y)u=\omega(x)y+\omega(y)x.$$

(v) $\Longrightarrow$(vi): Take $x=y$
(vi) $\Longrightarrow$(vii): For any $x \in A$, we have $x^2 = \omega(x)x$, equality involving the second power of x in $\frac{1}{2} A + \frac{1}{2} A^0$. But this algebra is commutative and it is well known that $G(m+1,2)$ is the only commutative baric algebra satisfying this equation.

(vii)$\Longrightarrow$(i): The matrix of $\frac{1}{2} A + \frac{1}{2} A^0$ is $\frac{1}{2} \tilde{A} + \frac{1}{2}{}^t\tilde{A}$ and so

$$\frac{1}{2}\tilde{A} + \frac{1}{2}\,{}^t\tilde{A} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \frac{1}{2} & 0 & 0 & & 0 \end{bmatrix}$$

Hence $\frac{1}{2}\alpha_{oj} + \frac{1}{2}\alpha_{oj} = \frac{1}{2}$, $\alpha_{oj}+\alpha_{jo}=1$, $1 \leq j \leq m$

DEFINITION:

Any member A of Ω_{m+1} satisfying the equivalent conditions of th.1 is called an alternate algebra.

REMARK:

In every alternate algebra A, we have multiple t-roots: $\alpha_{01} = \alpha_{02}$ and $\alpha_{10} = \alpha_{20}$, because $\alpha_{11} = 0$. It is also clear that $G(m+1, 2)$ is the only alternate commutative algebra in Ω_{m+1}.

THEOREM 2:

Let $A \in \Omega_{m+1}$. The following conditions are equivalent:

(i) A is a maximal element of Ω_{m+1}
(ii) There exist $\lambda, \mu \in R$ such that $\lambda+\mu=1$ and $A=\lambda A_0+\mu A_m$

PROOF:

(i) $\Longrightarrow$(ii): As dim Der $A=m(m+1)$, we must have Der $A=\{d:A\to A \mid \omega o d=0\}$. In fact, by ([1],th 1) every derivation d must satisfy $\omega o d=0$ and the subspace of linear mappings $d: A\to A$ such that $\omega o d= 0$ has dimension $m\,(m+1)$, as the Kernel of the linear mappings $d \to \omega o d$.

Take now any $a \in A$ with $\omega(a)=1$ and define $d_a:A\to A$ by $d_a(x)=\omega(x)a-x$ (see [1], th.2). We have $\omega o d_a=0$ and so d_a is a derivation of A. It follows that $d_a(a^2)= a-a^2= =ad(a)+d(a)a=0$ so $a=a^2$, and every element of weight 1 is an idempotent. Observe that d_a, restricted to N, is the reflexion $x\to -x$(for any a). Taking $1\leq i\leq m$ we have

$$C_0+C_i= (C_{0i}+C_i)^2= C_0+C_0C_i+C_iC_0+C_i^2=C_0+(\alpha_{0i}+\alpha_{i0})C_i+C_i^2$$

which implies $\alpha_{0i}+\alpha_{i0}=1$ so A is alternate.
Moreover, if $1\leq i,j\leq m$, we have

$$d_{C_0}(C_iC_j)=-C_iC_j=-C_iC_j-C_iC_j=-2C_iC_j$$

so $C_iC_j=0$ It follows that $\alpha_{01}=\ldots=\alpha_{0m}=\lambda$ and $\alpha_{10}=\ldots=\alpha_{m0}=\mu$ and finally $A=\lambda A_0+\mu A_m$, with $\lambda+\mu=1$.
(ii)$\Longrightarrow$(i): The matrices of A_0 and A_m are respectively

$$\tilde{A}_0 = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ 0 & 0 & \ldots & 0 \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ 0 & 0 & \ldots & 0 \end{bmatrix} \text{ and } \tilde{A}_m = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 1 & 0 & \ldots & 0 \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ 1 & 0 & \ldots & 0 \end{bmatrix}$$

so the matrix of $\lambda A_0+\mu A_m$ is

$$\begin{bmatrix} 1 & \lambda & \lambda & \dots & \lambda \\ \mu & 0 & 0 & \dots & 0 \\ \mu & 0 & 0 & \dots & 0 \\ . & . & . & & . \\ . & . & . & & . \\ . & . & . & & . \\ \mu & 0 & 0 & \dots & 0 \end{bmatrix}$$

We have: $C_0u=\lambda u$, $uC_0=\mu u$ and $uv=0$ for all u, $v \in N$. If $x=\omega(x)C_0+u$ and $y=\omega(y)C_0+v$, $u,v \in N$, then $xy=\omega(x)\omega(y)C_0+$ $+\lambda\omega(x)v+\mu\omega(y)u$. Suppose we have $B_0,B_1,\dots,B_m \in N$. Define $d:A \rightarrow A$ by $d(C_i)=B_i$ $(i=0,1,\dots,m)$. We prove d is a derivation:

$$xd(y)+d(x)y=x(\omega(y)B_0+d(v))+(\omega(x)B_0+d(u))y=\omega(y)xB_0+xd(v)+\omega(x)B_0y+d(u)y=\omega(y)(\omega(x)C_0+u)B_0+(\omega(x)C_0+u)d(v)+\omega(x)B_0(\omega(y)C_0+v)+d(u)(\omega(y)C_0+v)=\omega(y)\omega(x)\lambda B_0+\omega(x)\lambda d(v)+\omega(x)\omega(y)\mu B_0+\omega(y)\mu d(u)=\omega(x)\omega(y)B_0+\omega(x)\lambda d(v)+\omega(y)\mu d(u)=d(xy).$$

So, there is a one-to-one correspondence between derivations of $\lambda A_0 + \mu A_m$ and sequences $(B_0,B_1,\dots,B_m)$, $B_i \in N$. This means $\dim \mathrm{Der}(\lambda A_0+\mu A_m)=m(m+1)$ and A is maximal in Ω_{m+1}.

4.- MINIMAL ELEMENTS

THEOREM 3:

Suppose the H_1-coordinates $(\alpha_{om},\alpha_{1,m-1},\dots,\alpha_{m-1,1},\alpha_{mo})$ of $A \in \Omega_{m+1}$ satisfy:

(1) $\alpha_{j,m-j} \geq 0$, $0 \leq j \leq m$

(2) There exists $1 \leq k \leq m-1$ such that $\alpha_{k,m-k} > 0$

Then A is a minimal element of Ω_{m+1}.

PROOF

We remark that A and its opposite algebra A^0 have the same derivation algebra and the H_1-coordinates of A^0 are$(\alpha_{m0}, \alpha_{m-1,1}, \ldots, \alpha_{1,m-1}, \alpha_{0m})$. Hence we may suppose that $k \leq m-k$ in (2).

We denote again by k the least k such that $\alpha_{k,m-k} > 0$. We look now to the matrix $\tilde{A}$, which can be broken in four blocks $\tilde{A}_1, \ldots, \tilde{A}_4$ as follows:

$$\tilde{A} = \begin{array}{cc} \begin{array}{c} \text{column } m-k \\ \left[\begin{array}{c|c} \tilde{A}_1 & \tilde{A}_2 \\ \hline \tilde{A}_3 & \tilde{A}_4 \end{array}\right] \end{array} & \text{row } k \end{array}$$

The sizes of the blocks are: $\tilde{A}_1 : (k+1) \times (m-k+1)$; $\tilde{A}_2 : (k+1) \times k$; $\tilde{A}_3 : (m-k) \times (m-k+1)$; $\tilde{A}_4 : (m-k) \times k$. It is clear that $\tilde{A}_4 = 0$ and

$$\tilde{A}_2 = \left[\begin{array}{ccc} \alpha_{om} & \alpha_{om} \ldots & \alpha_{om} \\ 0 & 0 & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & 0 & 0 \end{array}\right]$$

by our choice of k. (This means α_{om} is a left t-root with multiplicity $\geq k$). The elements of $\tilde{A}_3$ are all non negative.

We look to $\tilde{A}_1$. Every element of $\tilde{A}_1$ is strictly positive because it depends on $\alpha_{k,m-k}$. Moreover, $\alpha_{oj} + \alpha_{jo} < 1$ for $j=2, \ldots, m$: this follows from formulae (2) and (3). Also $\alpha_{o1} > \alpha_{oj}$ $(j=2, \ldots, n)$, a consequence of the same formulae.

Suppose now d is a derivation of A. If $d(C_0)=\sum_{i=1}^{m}\alpha_iC_i$, $C^2{}_0=C_0$ implies $\sum_{i=1}^{m}\alpha_i(\alpha_{0i}+\alpha_{i0})C_i=\sum_{i=1}^{m}\alpha_iC_i$.

By equating coordinates, we get:

$$\alpha_i(\alpha_{0i}+\alpha_{i0})=\alpha_i \quad (i=1,\dots,m)$$

As $\alpha_{01}+\alpha_{10}=1$ and $\alpha_{0j}+\alpha_{j0}<1$ if $j=2,\dots,m$, we get $\alpha_2=\dots=\alpha_m=0$ and so:

$d(C_0)=\alpha C_1$, where $\alpha=\alpha_1$

Again, from $C_0C_1=\alpha_{01}C_1$ and calling $d(C_1)=\sum_{i=1}^{m}\beta_iC_i$, we obtain:

$$C_0\Big(\sum_{i=1}^{m}\beta_iC_i\Big)+\alpha C_1^2=\alpha_{01}\Big(\sum_{i=1}^{m}\beta_iC_i\Big) \text{ or}$$

$$\sum_{i=1}^{m}\beta_i\alpha_{0i}C_i+\alpha\alpha_{11}C_2=\sum_{i=1}^{m}\alpha_{01}\beta_iC_i$$

By equating coordinates:

$$\begin{cases}\beta_1\alpha_{01}=\beta_1\alpha_{01}\\ \beta_2\alpha_{02}+\alpha\alpha_{11}=\beta_2\alpha_{01}\\ \beta_i\alpha_{0i}=\beta_i\alpha_{01} \quad (3\le i\le m)\end{cases}$$

The first equation is an identity, the second gives $\beta_2=\alpha$ and the remaining ones give $\beta_i=0$, $3\le i\le m$, which means

$d(C_1)=\beta C_1+\alpha C_2$, where $\beta=\beta_1$.

Observe now the left principal powers of C_1:

$$C_1^2=\alpha_{11}C_2\neq0$$

$$C_1^3=C_1C_1^2=C_1\alpha_{11}C_2=\alpha_{11}\alpha_{12}C_3\neq0$$

$$C_1^{m-K+1}=C_1C_1^{m-K}=\alpha_{11}\alpha_{12}\cdots\alpha_{1,m-K}C_{m-K+1}\neq0$$

From these equations, we have:

$$d(C_2)=\frac{1}{\alpha_{11}}\, d(C_1^2)=\frac{1}{\alpha_{11}}[C_1(\beta C_1+\alpha C_2)+(\beta C_1+\alpha C_2)C_1]=$$

$$=\frac{1}{\alpha_{11}}[\beta_1\alpha_{11}C_2+\alpha\alpha_{12}C_3+\beta_1\alpha_{11}C_2+\alpha\alpha_{21}C_3]=2\beta C_2+\alpha C_3.$$

Similarly we obtain

$d(C_j)=j\beta C_j+\alpha C_{j+1}$ for $2\leq j\leq m-k$.

The effect of d on the remaining vectors $C_{m-k+1},\ldots,C_m$ can be obtained from the last row of $\tilde{A}_1$. In fact, $C_{m-k+1}=\alpha^{-1}{}_{k,m-2k+1}C_kC_{m-2k+1},\ldots,C_m=\alpha^{-1}{}_{k,m-k}C_kC_{m-k}$, which gives, by a direct computation,

$d(C_j)=j\beta C_j+\alpha C_{j+1}$ for $m-k+1\leq j\leq m$.

We have proved that $d=\alpha n+\beta\partial$

We don't know whether theorem 3 gives all minimal elements of Ω_{m+1}.

5.- DIAGONALIZABLE DERIVATIONS

Every member A of Ω_{m+1} has at least one diagonalizable derivation namely ∂. It may happen that A have several linearly independent diagonalizable derivations, as it happens to G(m+1,2) ([1],th 4).

THEOREM 4:

Let $a\in\Omega_{M+1}$, whose matrix(α_{ij}) satisfies:

(1) $\alpha_{oj}\neq\alpha_{o,j+1}(j=1,\ldots,m-1)$
(2) $\alpha_{om}+\alpha_{mo}\neq 1$

If $d:A\rightarrow A$ is a derivation such that $d(C_1)=\lambda C_1,\lambda\in R$, then $d=\lambda\partial$.

PROOF:

The first condition $\alpha_{oj}\neq\alpha_{o,j+1}$ means $\alpha_{1j}\neq 0$ and so the left principal powers of C_1 are all non zero:

$C_1^2=\alpha_{11}C_2,\ldots,C_1^m=\alpha_{11}\alpha_{12}\ldots\alpha_{1,m-1}C_m$

We have $d(C_1^2)=\lambda C_1$. Then $d(C_1)=d(\alpha_{11}C_2)=\alpha_{11}d(C_2)$ and

$d(C_1^2)=d(C_1)C_1+C_1d(C_1)=2\lambda C_1^2=2\lambda\alpha_{11}C_2$, that is, $d(C_2)=2\lambda C_2$.

Similarly, $d(C_3)=3\lambda C_3,\ldots,d(C_m)=m\lambda C_m$ and $C_1,\ldots,C_m$ are proper vectors of d, with proper values $\lambda,2\lambda,\ldots,m^{\lambda}$.

According to ([1],th 1),0 must be a proper value of d. Let us prove that $d(c_0)=0$ (which gives $d=\lambda\partial$). Call $d(C_0)=\beta_1C_1+\ldots+\beta_mC_m,\beta_i\in R$.
From $C_1C_0=\alpha_{10}C_1$,we get:$\lambda C_1C_0+C_1d(C_0)=\alpha_{10}\lambda C_1$ or $C_1d(C_0)=0$

Now $0=C_1(\beta_1C_1+\ldots+\beta_mC_m)=\beta_1\alpha_{11}C_2+\ldots+\beta_{m-1}\alpha_{1,m-1}C_m$ implies $\beta_1=\ldots=\beta_{m-1}=0$ and so $d(C_0)=\beta_mC_m$. But, as $C_0^2=C_0$,we have:

$C_0(\beta_mC_m)+(\beta_mC_m)C_0=\beta_mC_m$ or $\beta_m(\alpha_{om}+\alpha_{mo})=\beta_m$ and by (2), $\beta_m=0$

REMARK:

Let A be the alternate algebra of Ω_6 whose H_1-coordinates are $(½,1,0,0,-1,½)$. It is routine to verify that $d: A \to A$ given by $C_0\to 0,C_1\to C_1,C_2\to 0,C_3\to C_3,C_4\to 2C_4,C_5\to 3C_5$ is a derivation, C_1 is a proper vector of d but ∂ and d are linearly independent. We have, in this example, $\alpha_{01}=\alpha_{02}$ and $\alpha_{05}+\alpha_{50}=1$.

REFERENCES

[1] Costa, R. On the derivation algebra of gametic algebras for polyploidy with multiple alleles. Bol.Soc.Bras.Mat. 13, Nº2, 69-81 (1982).

[2] Costa, R. On the derivation algebra of zygotic algebras for polyploidy with multiple alleles.Bol.Soc.Bras.Mat. 14, Nº1, 63-80 (1983).

[3] Etherington, I.M.H.Genetic algebras.Proc.Roy.Soc. Edinburgh 59 (1939) 242-258.

[4] Etherington, I.M.H. Non-associative algebra and the symbolism of genetics.Proc.Roy.Soc.Edinburgh B 61 (1941) 24-42.

[5] Gonshor, H.Special train algebras arising in genetics. Proc.Edinburgh Math.Soc.(2) 12 (1960) 41-53.

[6] Gonshor, H.Special train algebras arising in genetics II. Proc.Edinburgh Math.Soc.(2) 14 (1965) 333-338.

[7] Heuch,I.An explicit formula for frequency changes in genetic algebras. J.Math.Biol.5 (1977) 43-53.

[8] Heuch, I. The genetic algebras for polyploidy with an arbitrary amount of double reduction. J.Math.Biol 6 (1978) 343-352.

[9] Heuch, I. Genetic algebras considered as elements in a vector space. SIAM J.Appl.Math. 35 (1978) 659-703.

[10] Holgate, P. Genetic algebras associated with polyploidy. Proc Edinburgh Math.Soc.(2) 15 (1966) 1-9. Corrigendum, ibid. 17 (1970) 120.

[11] Jacobson, N. Lie algebras. Interscience Publishers, New York, London, Sydney, 1962.

[12] Micali, A.; Campos, T.; Silva, C.; Ferreira, S.; Costa, R. Dérivations dans les algébras gametiques. Comm. in Algebra 12 (2) 239-243 (1984)

[13] Wörz-Busekros, A. Polyploidy with an arbitrary mixture of chromosome and chromatid segregation. J.Math.Biol. (1978) 353-365.

[14] Worz-Busekros, A. Algebras in genetics. Lecture Notes in Biomathematics, 36, Springer-Verlag, 1980.

Universidad de São Paulo
Instituto de Matemática e Estadística
Caixa Postal 20.570- Agência Iguatemi
São Paulo- SP.BRASIL.