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New approach to prove the existence of classical solutions for a class of nonlinear parabolic equations

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ABSTRACT

In this article, we consider a class of nonlinear parabolic equations. We use an integral representation combined with a sort of fixed point theorem to prove the existence of classical solutions for the initial value problem (1.1), (1.2). We also obtain a result on continuous dependence on the initial data. We propose a new approach for investigation for existence of classical solutions of some classes nonlinear parabolic equations.

RESUMEN

En este artículo, consideramos una clase de ecuaciones parabólicas nolineales. Usamos una representación integral combinada con una especie de teorema de punto fijo para probar la existencia de soluciones clásicas para el problema de valor inicial (1.1), (1.2). También obtenemos un resultado sobre la dependencia continua de la data inicial. Proponemos una estrategia nueva para la investigación de la existencia de soluciones clásicas de algunas clases de ecuaciones parabólicas nolineales.

Keywords and Phrases: parabolic equation, existence, differentiability with respect to the initial data

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1 Introduction

Here, we consider the Cauchy problem

$$u_t - u_{xx} = f(t, x, u, u_x) \qquad \text{in} \qquad (0, \infty) \times \mathbb{R}, \tag{1.1}$$

$$u(0, x) = \phi(x) \qquad \text{in} \qquad \mathbb{R}, \tag{1.2}$$

where $\phi \in \mathcal{C}^2(\mathbb{R})$, $f : [0, \infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longmapsto \mathbb{C}$ is a given continuous function, $u : [0, \infty) \times \mathbb{R} \longmapsto \mathbb{C}$ is the main unknown.

Our main results are as follows.

Theorem 1.1. *Let $f \in \mathcal{C}([0, \infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})$, $\phi \in \mathcal{C}^2(\mathbb{R})$. Then there exists $m \in (0, 1)$ such that the problem (1.1), (1.2) has a solution $u \in \mathcal{C}^1([0, m], \mathcal{C}^2([0, 1]))$.*

Theorem 1.2. *Let $f \in \mathcal{C}([0, \infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})$, $\phi \in \mathcal{C}^2(\mathbb{R})$. Then there exists $m \in (0, 1)$ such that the problem (1.1), (1.2) has a solution $u \in \mathcal{C}^1([0, m], \mathcal{C}^2(\mathbb{R}))$.*

For $O_1, O_2 \subset \mathbb{R}$ with $\mathcal{C}^1(O_1, \mathcal{C}^2(O_2))$ we denote the space of all continuous functions u on $O_1 \times O_2$ such that u_t, u_x and u_{xx} exist and are continuous on $O_1 \times O_2$.

Example 1.3. *Let $p > 1$ and $a \in \mathbb{C}$ be chosen so that $a^{p-1} = -\dfrac{1}{p-1}$. Consider the Cauchy problem*

$$u_t - u_{xx} = u^p \quad \textit{in} \quad (0, \infty) \times \mathbb{R}$$

$$u(0, x) = a \quad \textit{in} \quad \mathbb{R}.$$

Then $u(t, x) = a(t + 1)^{-\frac{1}{p-1}}$ is its solution. Actually,

$$u_t(t, x) \quad = \quad -\frac{a}{p-1}(t + 1)^{-\frac{p}{p-1}},$$

and

$$u_{xx}(t, x) \quad = \quad 0,$$

and

$$(u(t, x))^p = -\frac{a}{p-1}(t + 1)^{-\frac{p}{p-1}}.$$

Therefore

$$u_t(t, x) - u_{xx}(t, x) = (u(t, x))^p \quad \textit{in} \quad (0, \infty) \times \mathbb{R}$$

and

$$u(0, x) = a \quad \textit{in} \quad \mathbb{R}.$$

To prove our main result we propose new integral representation of the solutions of the initial value problem (1.1), (1.2). Many works have been devoted to the investigation of initial value problems for parabolic equations and systems (see, for example, [13]-[16] and the references therein). We note that in the references the IVP (1.1), (1.2) is connected with the dimension n, Fujita exponent, Sobolev critical exponents, bounded and unbounded domain. In this article we propose new idea which tell us that the local existence of classical solutions of the IVP is connected with the integral representation of the solutions, it is not connected with the dimension n and if the domain is bounded or not.

As an application of our new integral representation we deduce some results connected with the continuous dependence on the initial data and parameters of the problem (1.1), (1.2).

Theorem 1.4. *Let* $f \in \mathcal{C}([0,\infty)\times\mathbb{R}\times\mathbb{R}\times\mathbb{R})$, $\frac{\partial f}{\partial u}$, $\frac{\partial f}{\partial u_x}$ *exist and are continuous in* $[0,\infty)\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$, $\phi \in \mathcal{C}^2(\mathbb{R})$. *Let also,* $u(t,x,\phi) \in \mathcal{C}^1([0,m],\mathcal{C}^2([c,d]))$ *be a solution to the problem (1.1), (1.2) for some* $m \in (0,1)$ *and for some* $[c,d] \subset \mathbb{R}$. *Then* $u(t,x,\phi)$ *is differentiable with respect to* ϕ *and* $v(t,x) = \frac{\partial u}{\partial \phi}(t,x,\phi)$ *satisfies the following initial value problem*

$$\begin{aligned} v_t - v_{xx} &= \tfrac{\partial f}{\partial u}(t,x,u(t,x,\phi),u_x(t,x,\phi))v \\ &\quad + \tfrac{\partial f}{\partial u_x}(t,x,u(t,x,\phi),u_x(t,x,\phi))v_x \qquad \text{in} \qquad [0,m]\times[c,d], \end{aligned} \tag{1.3}$$

$$v(0,x) = 1 \qquad \text{in} \qquad [c,d]. \tag{1.4}$$

2 Auxiliary results

We will start with the following useful lemma.

Lemma 2.1. *Let* $f \in \mathcal{C}([a,b]\times[c,d]\times\mathbb{R}\times\mathbb{R})$, $g \in \mathcal{C}^2([c,d])$. *Then the function* $u \in \mathcal{C}^1([a,b],\mathcal{C}^2([c,d]))$ *is a solution to the problem*

$$u_t - u_{xx} = f(t,x,u,u_x) \qquad \text{in} \qquad (a,b]\times[c,d], \tag{2.1}$$

$$u(a,x) = g(x) \qquad \text{in} \qquad [c,d], \tag{2.2}$$

if and only if it is a solution to the integral equation

$$\begin{aligned} &\textstyle\int_c^x\int_c^y (u(t,z)-g(z))\,dzdy - \int_a^t (u(\tau,x)-u(\tau,c)-(x-c)u_x(\tau,c))\,d\tau \\ &= \textstyle\int_a^t\int_c^x\int_c^y f(\tau,z,u(\tau,z),u_x(\tau,z))dzdyd\tau, \qquad x\in[c,d], \quad t\in[a,b]. \end{aligned} \tag{2.3}$$

Proof. (1) Let $u \in \mathcal{C}^1([a,b],\mathcal{C}^2([c,d]))$ is a solution to the problem (2.1), (2.2).

We integrate the equation (2.1) with respect to x and we get

$$\begin{aligned} &\textstyle\int_c^x u_t(t,z)dz - \int_c^x u_{xx}(t,z)dz \\ &= \textstyle\int_c^x f(t,z,u(t,z),u_x(t,z))dz, \qquad x\in[c,d], \quad t\in[a,b], \end{aligned}$$

or

$$\int_c^x u_t(t,z)dz - u_x(t,x) + u_x(t,c)$$

$$= \int_c^x f(t,z,u(t,z),u_x(t,z))dz, \qquad x \in [c,d], \quad t \in [a,b].$$

Now we integrate the last equation with respect to x and we find

$$\int_c^x \int_c^y u_t(t,z)dzdy - \int_c^x (u_x(t,z) - u_x(t,c))\, dz$$

$$= \int_c^x \int_c^y f(t,z,u(t,z),u_x(t,z))dzdy, \qquad x \in [c,d], \quad t \in [a,b],$$

or

$$\int_c^x \int_c^y u_t(t,z)dzdy - u(t,x) + u(t,c) + (x-c)u_x(t,c)$$

$$= \int_c^x \int_c^y f(t,z,u(t,z),u_x(t,z))dzdy, \qquad x \in [c,d], \quad t \in [a,b].$$

We integrate the last equality with respect to t and we obtain

$$\int_a^t \int_c^x \int_c^y u_t(s,z)dzdyds - \int_a^t (u(s,x) - u(s,c) - (x-c)u_x(s,c))\, ds$$

$$= \int_a^t \int_c^x \int_c^y f(s,z,u(s,z),u_x(s,z))dzdyds, \qquad x \in [c,d], \quad t \in [a,b],$$

or

$$\int_c^x \int_c^y (u(t,z) - g(z))\, dzdy - \int_a^t (u(s,x) - u(s,c) - (x-c)u_x(s,c))\, ds$$

$$= \int_a^t \int_c^x \int_c^y f(s,z,u(s,z),u_x(s,z))dzdyds, \qquad x \in [c,d], \quad t \in [a,b],$$

i.e., u satisfies the equation (2.3).

(2) Let $u \in \mathcal{C}^1([a,b],\mathcal{C}^2([c,d]))$ be a solution to the integral equation (2.3).
We differentiate the equation (2.3) with respect to x and we get

$$\int_c^x (u(t,z) - g(z))\, dz - \int_a^t (u_x(s,x) - u_x(s,c))\, ds$$

$$= \int_a^t \int_c^x f(s,z,u(s,z),u_x(s,z))dzds, \qquad x \in [c,d], \quad t \in [a,b].$$

Again we differentiate with respect to x and we find

$$\begin{aligned} &u(t,x) - g(x) - \int_a^t u_{xx}(s,x)ds \\ &= \int_a^t f(s,x,u(s,x),u_x(s,x))ds, \qquad x \in [c,d], \quad t \in [a,b]. \end{aligned} \tag{2.4}$$

Now we put $t = a$ in the last equation and we find

$$u(a,x) = g(x), \qquad x \in [c,d],$$

i.e., the function u satisfies (2.2).

Now we differentiate the equation (2.4) with respect to t and we find

$$u_t(t,x) - u_{xx}(t,x) = f(t,x,u(t,x),u_x(t,x)), \qquad x \in [c,d], t \in [a,b].$$

□

The proof of the existence results are based on the following theorem.

Theorem 2.2 ([14]). *Let X be a nonempty closed convex subset of a Banach space Y. Suppose that T and S map X into Y such that*

(1) *S is continuous and $S(X)$ contained in a compact subset of Y.*

(2) *$T : X \longmapsto Y$ is expansive and onto.*

Then there exists a point $x^* \in X$ such that

$$Sx^* + Tx^* = x^*.$$

Definition 2.3. *Let (X,d) be a metric space and M be a subset of X. The mapping $T : M \longmapsto X$ is said to be expansive if there exists a constant $h > 1$ such that*

$$d(Tx,Ty) \geq hd(x,y)$$

for any $x, y \in M$.

3 Proof of Theorem 1.1

Let $B > \|\phi\|_{\mathcal{C}^2([0,1])}$ be arbitrarily chosen. Since $\phi \in \mathcal{C}([0,1])$, $f \in \mathcal{C}([0,1]\times[0,1]\times[-B,B]\times[-B,B])$ we have that there exists a constant $M_{11} > 0$ such that

$$\begin{aligned} |\phi(x)| &\leq M_{11} \quad \text{in} \quad [0,1], \\ |f(t,x,y,z)| &\leq M_{11} \quad \text{in} \quad [0,1]\times[0,1]\times[-B,B]\times[-B,B]. \end{aligned}$$

We take $l, m \in (0,1)$ so that

$$\begin{aligned} &lB + l(B + M_{11}) + 3lBm + lM_{11}m \leq B \\ &l(5B + 2M_{11}) \leq B. \end{aligned} \tag{3.1}$$

Let $E_{11} = \mathcal{C}^1([0,m],\mathcal{C}^2([0,1]))$ be endowed with the norm

$$\begin{aligned} \|u\| = \max\Big\{ &\max_{(t,x)\in[0,m]\times[0,1]} |u(t,x)|, \quad \max_{(t,x)\in[0,m]\times[0,1]} |u_t(t,x)|, \\ &\max_{(t,x)\in[0,m]\times[0,1]} |u_x(t,x)|, \quad \max_{(t,x)\in[0,m]\times[0,1]} |u_{xx}(t,x)| \Big\}. \end{aligned}$$

By $\tilde{K}_{11}$ we denote the set of all equi-continuous families in E_{11}, i.e., for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that

$$|u(t_1, x_1) - u(t_2, x_2)| < \epsilon, \qquad |u_t(t_1, x_1) - u_t(t_2, x_2)| < \epsilon,$$

$$|u_x(t_1, x_1) - u_x(t_2, x_2)| < \epsilon, \qquad |u_{xx}(t_1, x_1) - u_{xx}(t_2, x_2)| < \epsilon$$

whenever $|t_1 - t_2| < \delta$, $|x_1 - x_2| < \delta$. Let also,

$$K'_{11} = \overline{\tilde{K}_{11}}, \quad K_{11} = \{u \in K'_{11} : \|u\| \leq B\}$$

and

$$L_{11} = \{u \in K'_{11} : \|u\| \leq (1 + l)B\}.$$

We note that K_{11} is a closed convex subset of L_{11}.

For $u \in L_{11}$ we define the operators

$$\begin{aligned} T_{11}(u)(t, x) &= (1 + l)u(t, x), \\ S_{11}(u)(t, x) &= -lu(t, x) + l\int_0^x \int_0^y (u(t, z) - \phi(z))dzdy \\ &\quad -l\int_0^t (u(\tau, x) - u(\tau, 0) - xu_x(\tau, 0))d\tau \\ &\quad -l\int_0^t \int_0^x \int_0^y f(\tau, z, u(\tau, z), u_x(\tau, z))dzdyd\tau. \end{aligned}$$

We will prove that the problem

$$u_t - u_{xx} = f(t, x, u_x) \quad \text{in} \quad [0, m] \times [0, 1], \tag{3.2}$$

$$u(0, x) = \phi(x) \quad \text{in} \quad [0, 1], \tag{3.3}$$

has a solution $u \in \mathcal{C}^1([0, m], \mathcal{C}^2([0, 1]))$.

a) $S_{11} : K_{11} \longmapsto K_{11}$**.** Let $u \in K_{11}$. Then $S_{11}(u) \in \mathcal{C}^1([0, m], \mathcal{C}^2([0, 1]))$ and for $(t, x) \in [0, m] \times$

$[0,1]$, using the first inequality of (3.1), we get

$$\begin{aligned}
|S_{11}(u)(t,x)| &= \Big| -lu(t,x) + l\int_0^x\int_0^y (u(t,z)-\phi(z))dzdy \\
&\quad -l\int_0^t (u(\tau,x)-u(\tau,0)-xu_x(\tau,0))d\tau \\
&\quad -l\int_0^t\int_0^x\int_0^y f(\tau,z,u(\tau,z),u_x(\tau,z))dzdyd\tau\Big| \\
&\leq l|u(t,x)| + l\int_0^x\int_0^y (|u(t,z)|+|\phi(z)|)\,dzdy \\
&\quad +l\int_0^t (|u(\tau,x)|+|u(\tau,0)|+x|u_x(\tau,0)|)\,d\tau \\
&\quad +l\int_0^t\int_0^x\int_0^y |f(\tau,z,u(\tau,z),u_x(\tau,z))|dzdyd\tau \\
&\leq lB + l(B+M_{11}) + 3lBm + lM_{11}m \\
&\leq B.
\end{aligned}$$

Note that

$$\begin{aligned}
S_{11}(u)_t(t,x) &= -lu_t(t,x) + l\int_0^x\int_0^y u_t(t,z)dzdy \\
&\quad -l(u(t,x)-u(t,0)-xu_x(t,0)) \\
&\quad -l\int_0^x\int_0^y f(t,z,u(t,z),u_x(t,z))dzdy, \\
&\quad (t,x)\in[0,m]\times[0,1].
\end{aligned}$$

Then, using the second inequality of (3.1), we obtain

$$
\begin{aligned}
|S_{11}(u)_t(t,x)| &= \Big| -lu_t(t,x) + l\int_0^x\int_0^y u_t(t,z)dzdy \\
&\quad -l(u(t,x)-u(t,0)-xu_x(t,0)) \\
&\quad -l\int_0^x\int_0^y f(t,z,u(t,z),u_x(t,z))dzdy\Big| \\
&\leq l|u_t(t,x)| + l\int_0^x\int_0^y |u_t(t,z)|dzdy \\
&\quad +l\left(|u(t,x)|+|u(t,0)|+x|u_x(t,0)|\right) \\
&\quad +l\int_0^x\int_0^y |f(t,z,u(t,z),u_x(t,z))|dzdy \\
&\leq lB + lB + 3lB + lM_{11} \\
&= l(5B+M_{11}) \\
&\leq B, \quad (t,x)\in[0,m]\times[0,1].
\end{aligned}
$$

Also,

$$
\begin{aligned}
S_{11}(u)_x(t,x) &= -lu_x(t,x) + l\int_0^x (u(t,z)-\phi(z))dz \\
&\quad -l\int_0^t (u_x(\tau,x)-u_x(\tau,0))d\tau \\
&\quad -l\int_0^t\int_0^x f(\tau,z,u(\tau,z),u_x(\tau,z))dzd\tau, \\
&\quad (t,x)\in[0,m]\times[0,1].
\end{aligned}
$$

Hence, using the first inequality of (3.1),

$$
\begin{aligned}
|S_{11}(u)_x(t,x)| &= \Big| -lu_x(t,x) + l\int_0^x (u(t,z)-\phi(z))dz \\
&\quad -l\int_0^t (u_x(\tau,x)-u_x(\tau,0))d\tau \\
&\quad -l\int_0^t\int_0^x f(\tau,z,u(\tau,z),u_x(\tau,z))dzd\tau\Big| \\
&\leq l|u_x(t,x)| + l\int_0^x (|u(t,z)|+|\phi(z)|)\,dz \\
&\quad +l\int_0^t (|u_x(\tau,x)|+|u_x(\tau,0)|)\,d\tau \\
&\quad +l\int_0^t\int_0^x |f(\tau,z,u(\tau,z),u_x(\tau,z))|dzd\tau \\
&\leq lB + l(B+M_{11}) + 2lBm + lM_{11}m \\
&\leq B, \quad (t,x)\in[0,m]\times[0,1].
\end{aligned}
$$

For $(t,x)\in[0,m]\times[0,1]$ we have

$$
\begin{aligned}
S_{11}(u)_{xx}(t,x) &= -lu_{xx}(t,x) + l(u(t,x)-\phi(x)) \\
&\quad -l\int_0^t u_{xx}(\tau,x)d\tau \\
&\quad -l\int_0^t f(\tau,x,u(\tau,x),u_x(\tau,x))d\tau,
\end{aligned}
$$

from where, using the first inequality of (3.1),

$$\begin{aligned}
|S_{11}(u)_{xx}(t,x)| &= \Big| -lu_{xx}(t,x) + l(u(t,x) - \phi(x)) \\
&\quad -l\int_0^t u_{xx}(\tau,x)d\tau \\
&\quad -l\int_0^t f(\tau,x,u(\tau,x),u_x(\tau,x))d\tau \Big| \\
&\leq l|u_{xx}(t,x)| + l\left(|u(t,x)| + |\phi(x)|\right) \\
&\quad +l\int_0^t |u_{xx}(\tau,x)|d\tau \\
&\quad +l\int_0^t |f(\tau,x,u(\tau,x),u_x(\tau,x))|d\tau \\
&\leq lB + l(B + M_{11}) + lBm + lM_{11}m \\
&\leq B.
\end{aligned}$$

We note that $\{S_{11}(u) : u \in K_{11}\}$ is an equi-continuous family in E_{11}. Consequently $S_{11} : K_{11} \longmapsto K_{11}$. Also, $S_{11}(K_{11}) \subset K_{11} \subset L_{11}$, i.e., $S_{11}(K_{11})$ resides in a compact subset of L_{11}.

b) $S_{11} : K_{11} \longmapsto K_{11}$ **is a continuous operator.** We note that if $\{u_n\}_{n=1}^{\infty}$ be a sequence of elements of K_{11} such that $u_n \longrightarrow u$ in K_{11} as $n \longrightarrow \infty$, then $S_{11}(u_n) \longrightarrow S_{11}(u)$ in K_{11} as $n \longrightarrow \infty$. Therefore $S_{11} : K_{11} \longmapsto K_{11}$ is a continuous operator.

c) $T_{11} : K_{11} \longmapsto L_{11}$ **is an expansive operator and onto.** For $u, v \in K_{11}$ we have that

$$||T_{11}(u) - T_{11}(v)|| = (1 + l)||u - v||,$$

i.e., $T_{11} : K_{11} \longmapsto L_{11}$ is an expansive operator with constant $1 + l$.

Let $v \in L_{11}$. Then $\frac{v}{1+l} \in K_{11}$ and

$$T_{11}\left(\frac{v}{1+l}\right) = v,$$

i.e., $T_{11} : K_{11} \longmapsto L_{11}$ is onto.

From a), b), c) and from Theorem 2.2, it follows that there is $u_{11} \in K_{11}$ such that

$$T_{11}u_{11} + S_{11}u_{11} = u_{11}$$

or

$$\begin{aligned}
&(1+l)u_{11}(t,x) - lu_{11}(t,x) + l\int_0^x\int_0^y (u_{11}(t,z)-\phi(z))dzdy \\
&\quad -l\int_0^t (u_{11}(\tau,x) - u_{11}(\tau,0) - xu_{11x}(\tau,0))d\tau \\
&\quad -l\int_0^t\int_0^x\int_0^y f(\tau,z,u_{11}(\tau,z),u_{11x}(\tau,z))dzdyd\tau \\
&= u_{11}(t,x),
\end{aligned}$$

or

$$\begin{aligned}
&\int_0^x\int_0^y (u_{11}(t,z)-\phi(z))dzdy - \int_0^t (u_{11}(\tau,x) - u_{11}(\tau,0) - xu_{11x}(\tau,0))d\tau \\
&\quad -\int_0^t\int_0^x\int_0^y f(\tau,z,u_{11}(\tau,z),u_{11x}(\tau,z))dzdyd\tau \\
&= 0, \quad (t,x)\in[0,m]\times[0,1],
\end{aligned}$$

whereupon, using Lemma 2.1, we conclude that $u_{11} \in \mathcal{C}^1([0,1],\mathcal{C}^2([0,1]))$ is a solution to the problem (3.2), (3.3).

4 Proof of Theorem 1.2

Now we consider the problem

$$u_t - u_{xx} = f(t,x,u(t,x),u_x(t,x)) \qquad \text{in} \qquad (0,m]\times[1,2], \tag{4.1}$$

$$u(0,x) = \phi(x) \qquad \text{in} \qquad [1,2]. \tag{4.2}$$

Let $E_{12} = \mathcal{C}^1([0,m],\mathcal{C}^2([1,2]))$ be endowed with the norm

$$\begin{aligned}
\|u\| &= \max\Big\{\max_{(t,x)\in[0,m]\times[1,2]}|u(t,x)|, \quad \max_{(t,x)\in[0,m]\times[1,2]}|u_t(t,x)|, \\
&\quad \max_{(t,x)\in[0,m]\times[1,2]}|u_x(t,x)|, \quad \max_{(t,x)\in[0,m]\times[1,2]}|u_{xx}(t,x)|\Big\}.
\end{aligned}$$

By $\tilde{K}_{12}$ we denote the set of all equi-continuous families in E_{12}.

Let $K'_{12} = \overline{\tilde{K}_{12}}$,

$$K_{12} = \{u \in K'_{12} : \|u\| \le B\}.$$

Since $\phi \in \mathcal{C}([1,2])$, $f \in \mathcal{C}([0,m] \times [1,2] \times [-B,B] \times [-B,B])$ we have that there exists a constant $M_{12} > 0$ such that

$$\begin{aligned} |\phi(x)| &\leq M_{12} \quad \text{in} \quad [1,2], \\ |f(t,x,y,z)| &\leq M_{12} \quad \text{in} \quad [0,m] \times [1,2] \times [-B,B] \times [-B,B]. \end{aligned}$$

Let $l_1 > 0$ be chosen so that

$$l_1(5B + 2M_{12}) \leq B$$

$$l_1 B + l_1(B + M_{12}) + 3l_1 Bm + l_1 M_{12} m \leq B$$

Let also,

$$L_{12} = \{u \in K'_{12} : \|u\| \leq (1 + l_1)B\}.$$

We note that K_{12} is a closed convex subset of L_{12}.

For $u \in L_{12}$ we define the operators

$$\begin{aligned} T_{12}(u)(t,x) &= (1 + l_1)u(t,x), \\ S_{12}(u)(t,x) &= -l_1 u(t,x) + l_1 \int_1^x \int_1^y (u(t,z) - \phi(z))dzdy \\ &\quad -l_1 \int_0^t (u(\tau,x) - u_{11}(\tau,1) - (x-1)u_{11x}(\tau,1))d\tau \\ &\quad -l_1 \int_0^t \int_1^x \int_1^y f(\tau,z,u(\tau,z),u_x(\tau,z))dzdyd\tau. \end{aligned}$$

As in the previous section one can prove that there is $u_{12} \in \mathcal{C}^1([0,1], \mathcal{C}^2([1,2]))$ which is a solution to the problem (4.1), (4.2). This solution u_{12} satisfies the integral equation

$$\begin{aligned} &\int_1^x \int_1^y (u_{12}(t,z) - \phi(z))dzdy \\ &\quad - \textstyle\int_0^t (u_{12}(\tau,x) - u_{11}(\tau,1) - (x-1)u_{11x}(\tau,1))d\tau \\ &\quad - \textstyle\int_0^t \int_1^x \int_1^y f(\tau,z,u_{12}(\tau,z),u_{12x}(\tau,z))dzdyd\tau \\ &= 0, \quad (t,x) \in [0,m] \times [1,2]. \end{aligned} \tag{4.3}$$

Now we put $x = 1$ in (4.3) and we find

$$\int_0^t (u_{12}(\tau,1) - u_{11}(\tau,1))d\tau = 0,$$

which we differentiate with respect to t and we get

$$u_{12}(t,1) = u_{11}(t,1) \qquad \text{in} \qquad [0,m]. \tag{4.4}$$

Now we differentiate (4.3) with respect to x and we find

$$\int_1^x (u_{12}(t,z) - \phi(z))dz - \int_0^t (u_{12x}(\tau,x) - u_{11x}(\tau,1))d\tau$$
$$- \int_0^t \int_1^x f(\tau,z,u_{12}(\tau,z),u_{12x}(\tau,z))dzd\tau = 0, \quad (t,x) \in [0,m] \times [1,2].$$

In the last equation we put $x = 1$ and we become

$$\int_0^t (u_{12x}(\tau,x) - u_{11x}(\tau,1))d\tau = 0, \quad (t,x) \in [0,m] \times [1,2],$$

which we differentiate with respect to t and we find

$$u_{12x}(t,1) = u_{11x}(t,1) \qquad \text{in} \qquad [0,m]. \tag{4.5}$$

Now we differentiate (4.4) with respect to t and we get

$$u_{12t}(t,1) = u_{11t}(t,1) \qquad \text{in} \qquad [0,m].$$

Hence, (4.4), (4.5) and

$$f(t,1,u_{11}(t,1),u_{11x}(t,1)) = f(t,1,u_{12}(t,1),u_{12x}(t,1)),$$

we find

$$\begin{aligned} u_{12xx}(t,1) &= u_{12t}(t,1) - f(t,1,u_{12}(t,1),u_{12x}(t,1)) \\ &= u_{11t}(t,1) - f(t,1,u_{11}(t,1),u_{11x}(t,1)) \\ &= u_{11xx}(t,1) \qquad \text{in} \qquad [0,m]. \end{aligned}$$

Consequently the function

$$u(t,x) = \begin{cases} u_{11}(t,x) & \text{in} \quad [0,m] \times [0,1] \\ u_{12}(t,x) & \text{in} \quad [0,m] \times [1,2], \end{cases}$$

is a $\mathcal{C}^1([0,m],\mathcal{C}^2([0,2]))$-solution to the problem

$$\begin{aligned} u_t - u_{xx} &= f(t,x,u(t,x),u_x(t,x)) \qquad \text{in} \qquad (0,m] \times [0,2], \\ u(0,x) &= \phi(x) \qquad \text{in} \qquad [0,2]. \end{aligned}$$

Then we consider the problem

$$\begin{aligned} u_t - u_{xx} &= f(t,x,u(t,x),u_x(t,x)) \qquad \text{in} \qquad (0,m]\times[2,3] \\ u(0,x) &= \phi(x) \qquad \text{in} \qquad [2,3]. \end{aligned} \tag{4.6}$$

As in above there is $u_{13} \in \mathcal{C}^1([0,m],\mathcal{C}^2([2,3]))$ which is a solution to the problem (4.6) and satisfies the integral equation

$$\begin{aligned} &\int_2^x \int_2^y (u_{13}(t,z) - \phi(z))dzdy \\ &\qquad - \int_0^t (u_{13}(\tau,x) - u_{12}(\tau,2) - (x-2)u_{12x}(\tau,2))d\tau \\ &\qquad - \int_0^t \int_2^x \int_2^y f(\tau,z,u_{13}(\tau,z),u_{13x}(\tau,z))dzdyd\tau \\ &= 0, \quad t\in[0,m], \quad x\in[2,3]. \end{aligned}$$

The function

$$u(t,x) = \begin{cases} u_{11}(t,x) & \text{in} \quad [0,m]\times[0,1] \\ u_{12}(t,x) & \text{in} \quad [0,m]\times[1,2] \\ u_{13}(t,x) & \text{in} \quad [0,m]\times[2,3] \end{cases}$$

is a $\mathcal{C}^1([0,m],\mathcal{C}^2([0,3]))$-solution to the problem

$$u_t - u_{xx} = f(t,x,u(t,x),u_x(t,x)) \qquad \text{in} \qquad [0,m]\times[0,3],$$

$$u(0,x) = \phi(x) \qquad \text{in} \qquad [0,3].$$

An so on. We construct a solution $u_1 \in \mathcal{C}^1([0,m],\mathcal{C}^2(\mathbb{R}))$ which is a solution to the problem

$$u_t - u_{xx} = f(t,x,u(t,x),u_x(t,x)) \qquad \text{in} \qquad (0,m]\times\mathbb{R},$$

$$u(0,x) = \phi(x) \qquad \text{in} \qquad \mathbb{R}.$$

5 Proof of Theorem 1.4

We have that the solution $u(t,x,\phi)$ satisfies the following integral equation

$$\begin{aligned}
Q(\phi) &= \int_c^x \int_c^y (u(t,z,\phi(z)) - \phi(z))dzdy \\
&\quad - \int_0^t (u(\tau,x,\phi(x)) - u(\tau,c,\phi(c)) - (x-c)u_x(\tau,c,\phi(c)))d\tau \\
&\quad - \int_0^t \int_c^x \int_c^y f(\tau,z,u(\tau,z,\phi(z)),u_x(\tau,z,\phi(z)))dz \\
&= 0, \quad t \in [0,m], \quad x \in [c,d].
\end{aligned}$$

Then

$$\begin{aligned}
Q(\phi) - Q(\phi_1) &= \int_c^x \int_c^y (u(t,z,\phi(z)) - u(t,z,\phi_1(z)) - (\phi(z) - \phi_1(z)))dzdy \\
&\quad - \int_0^t (u(\tau,x,\phi(x)) - u(\tau,x,\phi_1(x)))d\tau \\
&\quad + \int_0^t (u(\tau,c,\phi(c)) - u(\tau,c,\phi_1(c)))d\tau \\
&\quad + \int_0^t (x-c)(u_x(\tau,c,\phi(c)) - u_x(\tau,c,\phi_1(c)))d\tau \\
&\quad - \int_0^t \int_c^x \int_c^y \Big(f(\tau,z,u(\tau,z,\phi(z)),u_x(\tau,z,\phi(z))) \\
&\quad - f(\tau,z,u(\tau,z,\phi_1(z)),u_x(\tau,z,\phi_1(z))) \Big) dzdyd\tau \\
&= \int_c^x \int_c^y \left(\frac{\partial u}{\partial \phi}(t,z,\phi(z)) - 1 \right) dzdy
\end{aligned}$$

$$\begin{aligned}
&- \int_0^t \frac{\partial u}{\partial \phi}(\tau,x,\phi(x))d\tau + \int_0^t \frac{\partial u}{\partial \phi}(\tau,c,\phi(c))d\tau + \int_0^t (x-c) \left(\frac{\partial u}{\partial \phi} \right)_x (\tau,c,\phi(c))d\tau \\
&- \int_0^t \int_c^x \int_c^y \frac{\partial f}{\partial u}(\tau,z,u(\tau,z,\phi(z)),u_x(\tau,z,\phi(z)))\frac{\partial u}{\partial \phi}(\tau,z,\phi(z))dzdyd\tau \\
&- \int_0^t \int_c^x \int_c^y \frac{\partial f}{\partial u_x}(\tau,z,u(\tau,z,\phi(z)),u_x(\tau,z,\phi(z))) \left(\frac{\partial u}{\partial \phi} \right)_x (\tau,z,\phi(z))dzdyd\tau \\
&+ \delta\{\phi,\phi_1\},
\end{aligned}$$

where $\delta\{\phi,\phi_1\}\longrightarrow 0$ as $\phi(x)\longrightarrow\phi_1(x)$ for every $x\in[c,d]$. Hence, when $\phi(x)\longrightarrow\phi_1(x)$ for every $x\in[c,d]$, we get

$$\begin{aligned} 0 &= \int_c^x\int_c^y(v(t,z)-1)dzdy-\int_0^t v(\tau,x)d\tau \\ &\quad+\int_0^t v(\tau,c)d\tau+\int_0^t xv_x(\tau,c)d\tau \\ &\quad-\int_0^t\int_c^x\int_c^y \tfrac{\partial f}{\partial u}(\tau,z,u(\tau,z,\phi(z)),u_x(\tau,z,\phi(z)))v(\tau,z)dzdyd\tau \\ &\quad-\int_0^t\int_c^x\int_c^y \tfrac{\partial f}{\partial u_x}(\tau,z,u(\tau,z,\phi(z)),u_x(\tau,z,\phi(z)))v_x(\tau,z)dzdyd\tau, \end{aligned} \tag{5.1}$$

which we differentiate twice in x and once in t and we get that v satisfies (1.3). Now we put $t=0$ in (5.1) and then we differentiate twice in x, and we find that v satisfies (1.4).

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