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On New Types of Sets Via γ-open Sets in $(\mathfrak{a})$Topological Spaces

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ABSTRACT

In this paper, we introduced the notion of γ-semi-open sets and γ-P-semi-open sets in $(\mathfrak{a})$topological spaces which is a set equipped with countable number of topologies. Several properties of these notions are discussed.

RESUMEN

En este artículo, introducimos la noción de conjuntos γ-semi-abiertos y conjuntos γ-P-semi-abiertos en espacios $(\mathfrak{a})$topológicos, el cual es un conjunto dotado con una cantidad numerable de topologías. Discutimos diversas propiedades de estas nociones.

Keywords and Phrases: $(\mathfrak{a})$topological spaces, $(\mathfrak{a})$-γ-semi-open sets, $(\mathfrak{a})$-γ-P-semi-open sets.

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1 Introduction

The notion of bitopological space (X, τ_1, τ_2) (a non empty set X endowed with two topologies τ_1 and τ_2) is introduced by Kelly [5]. Kovár [7, 8] also studied the properties of a non empty set equipped with three topologies. Many authors studied a countable number of topologies in (ω)topological spaces and $(\aleph_0)$topological spaces in [1, 2, 3, 4]. Ogata [9] defined an operation γ on a topological space (X, τ) as a mapping from τ into the power set $P(X)$ of X such that $U \subseteq \gamma(U)$ for each $U \in \tau$, where $\gamma(U)$ denotes the value of γ at U. A susbet A of X is said to be γ-open if for each $x \in A$, there exists an open set U containing x such that $\gamma(U) \subseteq A$. In topological spaces, γ-P-open set are defined by Khalaf and Ibrahim [6]. The main purpose of this paper is to introduce the concept of γ-P-semi-open sets and γ-semi-open sets in (a)topological spaces. We give some properties related to these sets and introduce some separation axioms in (a)topological spaces. Further we define new types of functions in (a)topological spaces, namely (a)-γ-semi-continuous and (a)-γ-P-semi-continuous. An operation γ on (a)topological space $(X, \{\tau_n\})$ is a mapping $\gamma \colon \bigcup \tau_n \to P(X)$ such that $U \subseteq \gamma(U)$ for each $U \in \bigcup \tau_n$.
Throughout the paper, $\mathbb{N}$ denotes the set of natural numbers. The elements of $\mathbb{N}$ are denoted by i, m, n etc. μ stands for the discrete topology. The (τ_n)-closure (resp. (τ_n)-interior) of a set A is denoted by τ_n-cl(A) (resp. τ_n-Int(A)). By $\tau_{m\gamma}$-Int(A) and $\tau_{m\gamma}$-cl(A), we denote the $\tau_{m\gamma}$-interior of A and $\tau_{m\gamma}$-closure of A in $(X, \{\tau_n\})$, respectively. If there is no scope of confusion, we denote the (a)topological space $(X, \{\tau_n\})$ by X.

2 (a)topological spaces

Definition 2.1. *[10] If $\{\tau_n\}$ is a sequence of topologies on a set X, then the pair $(X, \{\tau_n\})$ is called an (a)topological space.*

Definition 2.2. *[9] A susbet A of X is said to be γ-open if for each $x \in A$, there exists an open set U containing x such that $\gamma(U) \subseteq A$.*

Definition 2.3. *Let X be an (a)topological space. A subset S of X is said to be*:

(i). (m, n)-*semi-open if* $S \subseteq \tau_m\text{-cl}(\tau_n\text{-Int}(S))$.

(ii). (m, n)-γ-*semi-open if* $S \subseteq \tau_{m\gamma}\text{-cl}(\tau_{n\gamma}\text{-Int}(S))$.

(iii). (m, n)-γ-P-*semi-open if* $S \subseteq \tau_m\text{-cl}(\tau_{n\gamma}\text{-Int}(S))$.

The complements of (m, n)-semi-open set, (m, n)-γ-semi-open set and (m, n)-γ-P-semi-open set are (m, n)-semi-closed, (m, n)-γ-semi-closed and (m, n)-γ-P-semi-closed, respectively.

Definition 2.4. *Let X be an (a)topological space. A subset S of X is said to be*:

(i). (a)-*semi-open if S is (m, n)-semi-open for all $m \neq n$.*

(ii). (a)-γ-*semi-open if* S *is* (m, n)-γ-*semi-open for all* $m \neq n$.

(iii). (a)-γ-P-*semi-open if* S *is* (m, n)-γ-P-*semi-open for all* $m \neq n$.

The complements of (a)-semi-open set, (a)-γ-semi-open set and (a)-γ-P-semi-open set are (a)-semi-closed, (a)-γ-semi-closed and (a)-γ-P-semi-closed, respectively.

By $SO(X)$, $\gamma SO(X)$ and $\gamma PSO(X)$, we denote the family of all (a)-semi-open sets, (a)-γ-semi-open sets and (a)-γ-P-semi-open sets in X, respectively.

Theorem 2.1. *Every* (a)-γ-P-*semi-open set is* (a)-γ-*semi-open.*

Proof. Let S be an (a)-γ-P-semi-open set. Then S is (m, n)-γ-P-semi-open for all $m \neq n$. So $S \subseteq \tau_m\text{-}cl(\tau_{n\gamma}\text{-}Int(S)) \subseteq \tau_{m\gamma}\text{-}cl(\tau_{n\gamma}\text{-}Int(S))$ for all $m \neq n$. This implies that S is (m, n)-γ-semi-open for all $m \neq n$. Thus, S is (a)-γ-semi-open. □ □

The following example shows that the converse of the above theorem is not true generally.

Example 2.5. *Consider* $X = \{a, b, c, d\}$ *with topologies* $\tau_1 = \{X, \emptyset, \{b\}, \{d\}, \{b, d\}, \{a, b, c\}\}$, $\tau_2 = \{X, \emptyset, \{a\}, \{d\}, \{a, d\}, \{a, b\}, \{a, b, d\}\}$ *and* $\tau_i = \mu$ *for* $i \neq 1, 2$. *Let* γ *be an operation on* $\bigcup \tau_n$ *defined as follows* :

$$\gamma(U) = \begin{cases} U, & \text{if } U = \{d\} \\ X, & \text{if } U \neq \{d\} \end{cases}$$

Then $\{b, c, d\}$ *is* (a)-γ-*semi-open but it is not* (a)-γ-P-*semi-open.*

Theorem 2.2. *Every* (a)-γ-P-*semi-open set is* (a)-*semi-open.*

Proof. Let S be an (a)-γ-P-semi-open set. Then S is (m, n)-γ-P-semi-open for all $m \neq n$. So $S \subseteq \tau_m\text{-}cl(\tau_{n\gamma}\text{-}Int(S)) \subseteq \tau_m\text{-}cl(\tau_n\text{-}Int(S))$ for all $m \neq n$. This implies that S is (m, n)-semi-open for all $m \neq n$. Thus, S is (a)-semi-open. □ □

The following example shows that the converse of the above theorem is not true generally.

Example 2.6. *Let* X, τ_1 *and* γ *be as in Example* 2.6. *and let* $\tau_i = \tau_2$ *for all* $i \neq 1$.
Then $\{a, b, c\}$ *is* (a)-*semi-open but not* (a)-γ-P-*semi-open.*

Following example shows that there is no relation between (a)-semi-open sets and (a)-γ-semi-open sets.

Example 2.7. *Let* $(X, \{\tau_n\})$ *and* γ *be as in Example* 2.8.
Then $\{a, b, c\}$ *is* (a)-*semi-open but not* (a)-γ-*semi-open and* $\{b, d\}$ *is* (a)-γ-*semi-open but not* (a)-*semi-open.*

Following example shows that (a)-γ-P-semi-open set need not be τ_i-open set.

Example 2.8. *Consider* $X = \{a, b, c, d\}$ *with topologies* $\tau_1 = \{X, \emptyset, \{a\}, \{d\}, \{a, d\}, \{a, b\}, \{a, b, d\}\}$, $\tau_i = \{X, \emptyset, \{b\}, \{d\}, \{b, d\}\}$ *for all* $i \neq 1$. *Let* γ *be an operation on* $\bigcup \tau_n$ *defined as follows*:

$$\gamma(U) = \begin{cases} U, & \text{if } U = \{d\} \\ X, & \text{if } U \neq \{d\} \end{cases}$$

Then $\{c, d\}$ *is* (a)-γ-P-*semi-open but is not* τ_i-*open.*

Following example shows that (a)-γ-P-semi-open set need not be γ_i-open set.

Example 2.9. *Let* $(X, \{\tau_n\})$ *and* γ *be as in Example* 2.10.
Then $\{c, d\}$ *is* (a)-γ-P-*semi-open but not* γ_i-*open* .

Theorem 2.3. *Let* $\{S_\alpha \colon \alpha \in \Lambda\}$ *be a class of* (a)-γ-P-*semi-open sets. Then* $\bigcup_{\alpha\in\Lambda} S_\alpha$ *is also an* (a)-γ-P-*semi-open set.*

Proof. Since each S_α is an (a)-γ-P-semi-open set, S_α is (m, n)-γ-P-semi-open for all $\alpha \in \Lambda$ and for all $m \neq n$. We have $S_\alpha \subseteq \tau_m\text{-cl}(\tau_{n\gamma}\text{-Int}(S_\alpha))$ for all $\alpha \in \Lambda$ and for all $m \neq n$. Hence, it is obtained

$$\begin{aligned} \bigcup_{\alpha\in\Lambda} S_\alpha \subseteq \bigcup_{\alpha\in\Lambda} \tau_m\text{-cl}(\tau_{n\gamma}\text{-Int}(S_\alpha)) \\ \subseteq \tau_m\text{-cl}(\bigcup_{\alpha\in\Lambda} \tau_{n\gamma}\text{-Int}(S_\alpha)) \\ \subseteq \tau_m\text{-cl}(\tau_{n\gamma}\text{-Int}(\bigcup_{\alpha\in\Lambda} S_\alpha)). \end{aligned}$$

Therefore, $\bigcup_{\alpha\in\Lambda} S_\alpha$ is also an (a)-γ-P-semi-open set. □ □

Following example shows that the intersection of two (a)-γ-P-semi-open sets need not be again (a)-γ-P-semi-open.

Example 2.10. *Consider* $X = \{a, b, c, d\}$ *with topologies* $\tau_1 = \{X, \emptyset, \{c\}, \{d\}, \{c, d\}\}$, $\tau_i = \{X, \emptyset, \{c\}, \{d\}, \{c, d\}, \{b, c, d\}\}$ *for all* $i \neq 1$. *Let* γ *be an operation on* $\bigcup \tau_n$ *defined as follows* :

$$\gamma(U) = \begin{cases} U, & \text{if } U \in \{\{c\}, \{d\}\} \\ X, & \text{if } U \notin \{\{c\}, \{d\}\} \end{cases}$$

Then $\{b, c\}$ *and* $\{b, d\}$ *are* (a)-γ-P-*semi-open but their intersection* $\{b\}$ *is not* (a)-γ-P-*semi-open.*

Theorem 2.4. *A subset* F *is* (a)-γ-P-*semi-closed in* (a)*topological space* $(X, \{\tau_n\})$ *if and only if* $\tau_m\text{-Int}(\tau_{n\gamma}\text{-cl}(F)) \subseteq F$ *for all* $m \neq n$.

Proof. Let F be an (a)-γ-P-semi-closed set in X. Then $X \backslash F$ is (a)-γ-P-semi-open, so $X \backslash F \subseteq \tau_m$-$\text{cl}(\tau_{n\gamma}\text{-Int}(X \backslash F))$ for all $m \neq n$.

It follows that

$$\begin{aligned} F &\supseteq X\backslash\tau_m\text{-}cl(\tau_{n\gamma}\text{-}Int(X\backslash F)) \\ &= \tau_m\text{-}Int(X\backslash\tau_{n\gamma}\text{-}Int(X\backslash F)) \\ &= \tau_m\text{-}Int(\tau_{n\gamma}\text{-}cl(F)). \end{aligned}$$

Conversely, for all $m \neq n$, we obtain

$$\begin{aligned} X\backslash F &\subseteq X\backslash\tau_m\text{-}Int(\tau_{n\gamma}\text{-}cl(F)) \\ &= \tau_m\text{-}cl(X\backslash\tau_{n\gamma}\text{-}cl(F)) \\ &= \tau_m\text{-}cl(\tau_{n\gamma}\text{-}Int(X\backslash F)). \end{aligned}$$

which completes the proof. □ □

Theorem 2.5. *Let* $\{F_\alpha\colon \alpha \in \Lambda\}$ *be a class of* (a)-γ-P-*semi-closed sets. Then* $\bigcap_{\alpha\in\Lambda} F_\alpha$ *is also an* (a)-γ-P-*semi-closed.*

Proof. For each $\alpha \in \Lambda$, F_α is an (a)-γ-P-semi-closed set. This implies that $X\backslash F_\alpha$ is an (a)-γ-P-semi open set. By Theorem 2.12., $\bigcup_{\alpha\in\Lambda} X\backslash F_\alpha$ is an (a)-γ-P-semi open set. By De Morgan's Law, $X\backslash \bigcap_{\alpha\in\Lambda} F_\alpha$ is an (a)-γ-P-semi open set. Thus, $\bigcap_{\alpha\in\Lambda} F_\alpha$ is an (a)-γ-P-semi-closed set. □ □

Following example shows that the union of two (a)-γ-P-semi-closed sets need not be (a)-γ-P-semi-closed.

Example 2.11. *Let* $(X, \{\tau_n\})$ *and* γ *be as in Example* 2.13.
Then $\{a, c\}$ *and* $\{a, d\}$ *are* (a)-γ-P-*semi-closed but their union* $\{a, c, d\}$ *is not* (a)-γ-P-*semi-closed.*

Definition 2.12. *In an* (a)*topological space* X, *a point* x *of* X *is said to be* (a)-γ-P-*semi interior* ((a)-γ-*semi interior*) *point of* S *if there exists an* (a)-γ-P-*semi-open* ((a)-γ-*semi-open*) *set* V *such that* $x \in V \subseteq S$.

By (a)-γ-PS-Int(A) (resp.(a)-γ-S-Int(A)), *we denote the* (a)-γ-PS-*interior* (resp.(a)-γ-S-*interior*) *of* A *consisting of all* (a)-γ-P-*semi interior* ((a)-γ-*semi interior*) *points of* A.

Theorem 2.6. *The following properties hold for any subset* A *of* (a)*topological space* X :

(i). (a)-γ-PS-Int(A) *is the union of all* (a)-γ-P-*semi-open sets (the largest* (a)-γ-P-*semi-open set) contained in* A.

(ii). (a)-γ-PS-Int(A) *is an* (a)-γ-P-*semi-open set.*

(iii). A *is* (a)-γ-P-*semi-open if and only if* $A = (a)$-γ-PS-Int(A).

Proof. The proof follows from definitions. □ □

Theorem 2.7. *The following properties hold for any subsets* A_1, A_2 *and any class of subsets* $\{A_\alpha\colon \alpha \in \Lambda\}$ *of* (a)*topological space* X :

(i). *If* $A_1 \subseteq A_2$, *then* (a)-γ-PS-$Int(A_1) \subseteq (a)$-γ-PS-$Int(A_2)$.

(ii). $\bigcup_{\alpha\in\Lambda}(a)$-$\gamma$-$PS$-$Int(A_\alpha) \subseteq (a)$-$\gamma$-$PS$-$Int(\bigcup_{\alpha\in\Lambda} A_\alpha)$.

(iii). (a)-γ-PS-$Int(\bigcap_{\alpha\in\Lambda} A_\alpha) \subseteq \bigcap_{\alpha\in\Lambda}(a)$-$\gamma$-$PS$-$Int(A_\alpha)$.

Proof. (i). Since $A_1 \subseteq A_2$, (a)-γ-PS-$Int(A_1)$ is an (a)-γ-P-semi-open set contained in A_2. But (a)-γ-PS-$Int(A_2)$ is the largest (a)-γ-P-semi-open set contained in A_2. So (a)-γ-PS-$Int(A_1) \subseteq (a)$-γ-PS-$Int(A_2)$.

(ii). From (i), we have (a)-γ-PS-$Int(A_\alpha) \subseteq (a)$-$\gamma$-$PS$-$Int(\bigcup_{\alpha\in\Lambda} A_\alpha)$ for all $\alpha \in \Lambda$. Hence, $\bigcup_{\alpha\in\Lambda}(a)$-$\gamma$-$PS$-$Int(A_\alpha) \subseteq (a)$-$\gamma$-$PS$-$Int(\bigcup_{\alpha\in\Lambda} A_\alpha)$.

(iii). From (i), (a)-γ-PS-$Int(\bigcap_{\alpha\in\Lambda} A_\alpha) \subseteq (a)$-$\gamma$-$PS$-$Int(A_\alpha)$ for all $\alpha \in \Lambda$. Hence, (a)-γ-PS-$Int(\bigcap_{\alpha\in\Lambda} A_\alpha) \subseteq \bigcap_{\alpha\in\Lambda}(a)$-$\gamma$-$PS$-$Int(A_\alpha)$. □

□

The reverse inclusion in (ii) and (iii) of Theorem 2.19. may not be applicable as shown in the following examples.

Example 2.13. *Consider* $X = \{a, b, c\}$ *with topologies* $\tau_1 = \{X, \emptyset, \{a\}, \{b, c\}\}$, $\tau_i = \{X, \emptyset, \{b\}\}$ *for all* $i \neq 1$. *Let* γ *be an operation on* $\bigcup \tau_n$ *defined as follows* :

$$\gamma(U) = \begin{cases} U, & \textit{if } U = \{c\} \\ X, & \textit{if } U \neq \{c\} \end{cases}$$

$\{a, b, c\} = (a)$-γ-PS-$Int\{a, b, c\} \nsubseteq (a)$-$\gamma$-$PS$-$Int\{a\} \cup (a)$-$\gamma$-$PS$-$Int\{b, c\} = \emptyset$.

Example 2.14. *Let* $(X, \{\tau_n\})$ *and* γ *be as in Example* 2.13.
$\{b\} = (a)$-γ-PS-$Int\{b, c\} \cap (a)$-γ-PS-$Int\{b, d\} \nsubseteq (a)$-$\gamma$-$PS$-$Int\{b\} = \emptyset$.

Definition 2.15. *In an* (a)*topological space* X, *a point* x *of* X *is said to be* (a)-γ-P-*semi cluster* ((a)-γ-*semi cluster*) *point of a subset* $A \subset X$ *if* $A \cap V \neq \emptyset$ *for every* (a)-γ-P-*semi-open* ((a)-γ-*semi-open set*) *containing* x.

By (a)-γ-PS-$cl(A)$ (resp.(a)-γ-S-$cl(A)$), *we denote the* (a)-γ-PS-*closure* (resp.(a)-γ-S-*closure*) *of* A *consisting of all* (a)-γ-P-*semi cluster* ((a)-γ-*semi cluster*) *points of* A.

Theorem 2.8. *The following properties hold for any subset* A *of an* (a)*topological space* X :

(i). (a)-γ-PS-$cl(A)$ *is the intersection of all* (a)-γ-P-*semi-closed sets (the smallest* (a)-γ-P-*semi-closed set) containing* A.

(ii). (a)-γ-PS-$cl(A)$ *is an* (a)-γ-P-*semi-closed set.*

(iii). A *is* (a)-γ-P-*semi-closed if and only if* $A = (a)$-γ-PS-$cl(A)$.

Proof. The proof follows from definitions. □ □

Theorem 2.9. *The following properties hold for any subsets* A_1, A_2 *and any class of subsets* $\{A_\alpha : \alpha \in \Lambda\}$ *of an* (a)*topological space* X:

(i). *If* $A_1 \subseteq A_2$, *then* (a)-γ-PS-$cl(A_1) \subseteq (a)$-γ-PS-$cl(A_2)$.

(ii). $\bigcup_{\alpha\in\Lambda}(a)$-$\gamma$-PS-$cl(A_\alpha) \subseteq (a)$-$\gamma$-PS-$cl(\bigcup_{\alpha\in\Lambda} A_\alpha)$.

(iii). (a)-γ-PS-$cl(\bigcap_{\alpha\in\Lambda} A_\alpha) \subseteq \bigcap_{\alpha\in\Lambda}(a)$-$\gamma$-PS-$cl(A_\alpha)$.

Proof. (i). Since $A_1 \subseteq A_2$, (a)-γ-PS-$cl(A_2)$ is an (a)-γ-P-semi-closed set containing A_1. But (a)-γ-PS-$cl(A_1)$ is the smallest (a)-γ-P-semi-closed set containing A_1. so (a)-γ-PS-$cl(A_1) \subseteq (a)$-γ-PS-$cl(A_2)$.

(ii). From (i), (a)-γ-PS-$cl(A_\alpha) \subseteq (a)$-$\gamma$-PS-$cl(\bigcup_{\alpha\in\Lambda} A_\alpha)$ for all $\alpha \in \Lambda$. Hence, $\bigcup_{\alpha\in\Lambda}(a)$-$\gamma$-PS-$cl(A_\alpha) \subseteq (a)$-$\gamma$-PS-$cl(\bigcup_{\alpha\in\Lambda} A_\alpha)$.

(iii). From (i), (a)-γ-PS-$cl(\bigcap_{\alpha\in\Lambda} A_\alpha) \subseteq (a)$-$\gamma$-PS-$cl(A_\alpha)$ for all $\alpha \in \Lambda$. Hence, (a)-γ-PS-$cl(\bigcap_{\alpha\in\Lambda} A_\alpha) \subseteq \bigcap_{\alpha\in\Lambda}(a)$-$\gamma$-PS-$cl(A_\alpha)$. □

□

The reverse inclusion in (ii) and (iii) of Theorem 2.24 may not be applicable as shown in the following examples.

Example 2.16. *Let* $(X, \{\tau_n\})$ *and* γ *be as in Example* 2.13.
$\{a, b, c, d\} = (a)$-γ-PS-$cl\{a, c, d\} \nsubseteq (a)$-$\gamma$-PS-$cl\{a, c\} \cup (a)$-$\gamma$-PS-$cl\{a, d\} = \{a\}$.

Example 2.17. *Consider* $X = \{a, b, c\}$ *with topologies* $\tau_1 = \{X, \emptyset, \{a\}, \{a, b\}\}$ *and* $\tau_i = \{X, \emptyset, \{b\}, \{a, b\}\}$ *for all* $i \neq 1$. *Let* γ *be an operation on* $\bigcup \tau_n$ *defined as follows* :

$$\gamma(U) = \begin{cases} U, & \text{if } U = \{a, b\} \\ X, & \text{if } U \neq \{a, b\} \end{cases}$$

$\{a, b, c\} = (a)$-γ-PS-$cl\{a, c\} \cap (a)$-γ-PS-$cl\{b, c\} \nsubseteq (a)$-$\gamma$-PS-$cl\{c\} = \{c\}$.

Theorem 2.10. *The following properties hold for a subset* A *of an* (a)*topological space* X:

(i). (a)-γ-PS-$Int(X\backslash A) = X\backslash(a)$-$\gamma$-PS-$cl(A)$.

(ii). (a)-γ-PS-$cl(X\backslash A) = X\backslash(a)$-$\gamma$-PS-$Int(A)$.

Proof. 1. By part (i). of Theorem 2.18., we have

$$\begin{aligned}(a)\text{-}\gamma\text{-PS-Int}(X\backslash A) &= \bigcup\{S \subset X\colon S \text{ is } (a)\text{-}\gamma\text{-P-semi-open and } S \subset X\backslash A\} \\ &= \bigcup\{X\backslash(X\backslash S) \subset X\colon X\backslash S \text{ is } (a)\text{-}\gamma\text{-P-semi-closed and } A \subset X\backslash S\} \\ &= X\backslash \bigcap\{X\backslash S \subset X\colon X\backslash S \text{ is } (a)\text{-}\gamma\text{-P-semi-closed and } A \subset X\backslash S\} \\ &= X\backslash \bigcap\{F \subset X\colon F \text{ is } (a)\text{-}\gamma\text{-P-semi-closed and } A \subset F\} \\ &= X\backslash(a)\text{-}\gamma\text{-PS-cl}(A).\end{aligned}$$

2. By part (i). of Theorem 2.23., we have

$$\begin{aligned}(a)\text{-}\gamma\text{-PS-cl}(X\backslash A) &= \bigcap\{S \subset X\colon S \text{ is } (a)\text{-}\gamma\text{-P-semi-closed and } X\backslash A \subset S\} \\ &= \bigcap\{X\backslash(X\backslash S) \subset X\colon X\backslash S \text{ is } (a)\text{-}\gamma\text{-P-semi-open and } X\backslash S \subset A\} \\ &= X\backslash \bigcup\{X\backslash S \subset X\colon X\backslash S \text{ is } (a)\text{-}\gamma\text{-P-semi-open and } X\backslash S \subset A\} \\ &= X\backslash \bigcup\{F \subset X\colon X\backslash F \text{ is } (a)\text{-}\gamma\text{-P-semi-open and } F \subset A\} \\ &= X\backslash(a)\text{-}\gamma\text{-PS-Int}(A).\end{aligned}$$

□

Definition 2.18. *A set* A *is said to be* (a)-γ-P-*semi neighborhood of a point* x *in an* (a)*topological space* X *if there exists an* (a)-γ-P-*semi-open set* U *such that* $x \in U \subseteq A$.

Theorem 2.11. *A subset of an* (a)*topological space* X *is* (a)-γ-P-*semi-open if and only if it is* (a)-γ-P-*semi neighborhood of each of its points.*

Proof. The proof follows from definition 2.28. □ □

Definition 2.19. *An* (a)*topological space* X *is said to be* (a)-γ-PS-T_0 *if for every distinct points* x *and* y *of* X, *there exists an* (a)-γ-P-*semi-open set* U *such that* $x \in U$ *but* $y \notin U$ *or vice versa.*

Theorem 2.12. *An* (a)*topological space* X *is* (a)-γ-PS-T_0 *if and only if for each distinct points* x *and* y *of* X (a)-γ-PS-$cl\{x\} \neq (a)$-γ-PS-$cl\{y\}$.

Proof. Let x and y be any two distinct points of X. Then there exists an (a)-γ-P-semi-open set U such that $x \in U$ but $y \notin U$ or vice versa. Without loss of generality, assume that U containing x but not y. Then we have $\{y\} \cap U = \emptyset$ which implies $x \notin (a)$-γ-PS-$cl\{y\}$. Hence, (a)-γ-PS-$cl\{x\} \neq (a)$-γ-PS-$cl\{y\}$.

Conversely, let x and y be any two distinct points of X. Then we have (a)-γ-PS-$cl\{x\} \neq (a)$-γ-PS-$cl\{y\}$. Without loss of generality let $z \in (a)$-γ-PS-$cl\{y\}$ but $z \notin (a)$-γ-PS-$cl\{x\}$. Then $\{y\} \cap U \neq \emptyset$ for every (a)-γ-P-semi-open set U containing z and $\{x\} \cap U = \emptyset$ for atleast one (a)-γ-P-semi-open set U containing z. Thus, $y \in U$ and $x \notin U$. Hence, X is (a)-γ-PS-T_0. □ □

Definition 2.20. *An* $(\mathfrak{a})$*topological space* $(X, \{\tau_n\})$ *is said to be* $(\mathfrak{a})$-γ-PS-T_1 *if for every distinct points* x *and* y *of* X*, there exist two* $(\mathfrak{a})$-γ-P-*semi-open sets which one of them contains* x *but not* y *and the other one contains* y *but not* x*.*

Theorem 2.13. *An* $(\mathfrak{a})$*topological space* X *is* $(\mathfrak{a})$-γ-PS-T_1 *if and only if for each point* x *of* X $(\mathfrak{a})$-γ-PS-$\mathsf{cl}\{x\} = \{x\}$.

Proof. Since $\{x\} \subseteq (\mathfrak{a})$-$\gamma$-$\mathsf{PS}$-$\mathsf{cl}\{x\}$, Let $y \in (\mathfrak{a})$-γ-PS-$\mathsf{cl}\{x\}$ be arbitrary. On contrary suppose that $y \notin \{x\}$. Then there exists an $(\mathfrak{a})$-γ-P-semi-open set U such that $y \in U$ but $x \notin U$. Then we have $\{x\} \cap U = \emptyset$ which implies $y \notin (\mathfrak{a})$-γ-PS-$\mathsf{cl}\{x\}$. Hence, contradiction.
Conversely, let $x \neq y$ for $x, y \in X$. Since $x \notin (\mathfrak{a})$-γ-PS-$\mathsf{cl}\{y\}$ and $y \notin (\mathfrak{a})$-γ-PS-$\mathsf{cl}\{x\}$, there exist $(\mathfrak{a})$-γ-P-semi-open sets U and V containing x and y, respectively such that $\{y\} \cap U = \emptyset$ and $\{x\} \cap V = \emptyset$. Thus, we have $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$. Hence, X is $(\mathfrak{a})$-γ-PS-T_1. □ □

Definition 2.21. *An* $(\mathfrak{a})$*topological space* X *is said to be* $(\mathfrak{a})$-γ-PS-T_2 *if for every distinct points* x *and* y *of* X*, there exist two disjoint* $(\mathfrak{a})$-γ-P-*semi-open sets* U *and* V *containing* x *and* y*, respectively.*

Theorem 2.14. *An* $(\mathfrak{a})$*topological space* X *is* $(\mathfrak{a})$-γ-PS-T_2 *if and only if for each distinct points* x *and* y *of* X *there exists an* $(\mathfrak{a})$-γ-P-*semi-open set* U *containing* x *such that* $y \notin (\mathfrak{a})$-γ-PS-$\mathsf{cl}(U)$.

Proof. Let X be an $(\mathfrak{a})$-γ-PS-T_2 space. On contrary suppose that $y \in (\mathfrak{a})$-γ-PS-$\mathsf{cl}(U)$ for all $(\mathfrak{a})$-γ-P-semi-open set U containing x. Then $U \cap V \neq \emptyset$ for every $(\mathfrak{a})$-γ-P-semi-open set V containing y and $(\mathfrak{a})$-γ-P-semi-open set U containing x. Thus, contradiction.
Conversely, let x and y be any two distinct point of X. Then there exist two disjoint $(\mathfrak{a})$-γ-P-semi-open sets U and V containing x and y, respectively. This implies that $\{y\} \cap U = \emptyset$. Hence, $y \notin (\mathfrak{a})$-γ-PS-$\mathsf{cl}(U)$. □ □

Theorem 2.15. *An* $(\mathfrak{a})$*topological space* X *is* $(\mathfrak{a})$-γ-PS-T_2 *if and only if the intersection of all* $(\mathfrak{a})$-γ-PS-*closed neighborhood of each point of* X *consists of only that point.*

Proof. Let $x \in X$ be arbitrary and $y \in X$ such that $y \neq x$. Then there exist disjoint $(\mathfrak{a})$-γ-P-semi-open sets U_y and V_y containing x and y, respectively. Since $U_y \subseteq X \backslash V_y$, $X \backslash V_y$ is an $(\mathfrak{a})$-γ-PS-closed neighborhood of x which does not contain y. Hence, $\cap\{X \backslash V_y : y \in X, y \neq x\} = \{x\}$.
Conversely, let x and y be any two distinct points of X. Since $\{x\} = \cap\{S \subset X \colon S$ is $(\mathfrak{a})$-γ-PS-closed neighborhood of $x\}$. This implies that there exists an $(\mathfrak{a})$-γ-PS-closed neighborhood U of x not containing y. Then, $y \in X \backslash U$ and $X \backslash U$ is $(\mathfrak{a})$-γ-P-semi-open. Since, U is an $(\mathfrak{a})$-γ-PS-neighborhood of x, then there exists an $(\mathfrak{a})$-γ-P-semi-open set V containing x such that $V \subseteq U$. Clearly, V and $X \backslash U$ are disjoint. Hence, $(X, \{\tau_n\})$ is $(\mathfrak{a})$-γ-PS-T_2. □ □

Remark 2.22. (i). *Every* $(\mathfrak{a})$-γ-PS-T_2 $(\mathfrak{a})$*topological space is* $(\mathfrak{a})$-γ-PS-T_1*.*

(ii). *Every* $(\mathfrak{a})$-γ-PS-T_1 $(\mathfrak{a})$*topological space is* $(\mathfrak{a})$-γ-PS-T_0*.*

Following examples shows that converse of above remark need not be true.

Example 2.23. *Let* $X = \{a, b, c, d\}$ *with topologies* $\tau_1 = \{\emptyset, X, \{c\}, \{d\}, \{c, d\}\{a, b, c\}\}$ *and* $\tau_i = \{\emptyset, X, \{c\}, \{d\}, \{c, d\}\}$ *for all* $i \neq 1$.

$$\gamma(U) = \begin{cases} U, & \textit{if } U \in \{\{c\}, \{d\}, \{a, b, c\}\} \\ X, & \textit{if } U \notin \{\{c\}, \{d\}, \{a, b, c\}\} \end{cases}$$

Then $\tau_{n\gamma} = \tau_n$ *for all* $n \in \mathbb{N}$ *and* (a)-γ-$PSO = \{\emptyset, X, \{c\}, \{d\}, \{c, d\}\{a, b, c\}, \{a, c\}, \{b, c\}\}$.
Clearly, $(X, \{\tau_n\})$ *is* (a)-γ-PS-T_0 *but not* (a)-γ-PS-T_1.

Example 2.24. *Let* $X = \{a, b, c\}$ *with topologies* $\tau_n = \mu$ *for all* n.

$$\gamma(U) = \begin{cases} U, & \textit{if } U \in \{\{a, b\}, \{a, c\}, \{b, c\}\} \\ X, & \textit{if } U \notin \{\{a, b\}, \{a, c\}, \{b, c\}\} \end{cases}$$

Then $\tau_{n\gamma} = \{\emptyset, X, \{a, b\}, \{a, c\}, \{b, c\}\}$ *for all* $n \in \mathbb{N}$ *and* (a)-γ-$PSO = \{\emptyset, X, \{a, b\}, \{a, c\}, \{b, c\}\}$.
Clearly $(X, \{\tau_n\})$ *is* (a)-γ-PS-T_1 *but not* (a)-γ-PS-T_2.

Example 2.25. *Let* $X = \{a, b, c\}$ *with topologies* $\tau_n = \mu$ *for all* n.

$$\gamma(U) = \begin{cases} U, & \textit{if } U \in \{\{a\}, \{b\}, \{c\}\} \\ X, & \textit{if } U \notin \{\{a\}, \{b\}, \{c\}\} \end{cases}$$

Then $\tau_{n\gamma} = \mu$ *for all* $n \in \mathbb{N}$ *and* (a)-γ-$PSO = \mu$
Clearly X *is* (a)-γ-PS-T_2 *space.*

Definition 2.26. *Let* $f\colon (X, \{\tau_n\}) \to (Y, \{\zeta_n\})$ *be a function and* x *be any point of* X. f *is said to be* (a)-γ-P-*semi continuous* (*resp.*(a)-γ-*semi continuous*) *at* x *if for every* ζ_n *open subset* O *of* Y *containing* $f(x)$ *there exists an* (a)-γ-P-*semi-open* (*resp.* (a)-γ-*semi-open*) *set* G *of* X *containing* x *such that* $f(G) \subseteq O$.

Theorem 2.16. *For a function* $f\colon (X, \{\tau_n\}) \to (Y, \{\zeta_n\})$, *the followings statements are equivalent* :

(i). f *is* (a)-γ-P-*semi continuous* (*resp.*(a)-γ-*semi continuous*).

(ii). *For every* ζ_n *open subset* O *of* Y, $f^{-1}(O)$ *is an* (a)-γ-P-*semi-open* (*resp.*(a)-γ-*semi-open*) *set in* X.

(iii). *For every* ζ_n *closed subset* F *of* Y, $f^{-1}(F)$ *is an* (a)-γ-P-*semi-closed* (*resp.*(a)-γ-*semi-closed*) *set in* X.

(iv). *For every subset* T *of* X, $f((a)\text{-}\gamma\text{-PS-}cl(T)) \subseteq \zeta_n\text{-}cl(f(T))$ (*resp.* $f((a)\text{-}\gamma\text{-S-}cl(T)) \subseteq \zeta_n\text{-}cl(f(T))$).

(v). *For every subset* F *of* Y, $(a)\text{-}\gamma\text{-PS-}cl(f^{-1}F) \subseteq f^{-1}(\zeta_n\text{-}cl(F))$(*resp.* $(a)\text{-}\gamma\text{-S-}cl(f^{-1}F) \subseteq f^{-1}(\zeta_n\text{-}cl(F))$).

Proof. $(1). \Longrightarrow (ii)$. Let O be ζ_n open in Y and $x \in f^{-1}(O)$ be arbitrary. Since f is (a)-γ-P-semi continuous on X, there exists an (a)-γ-P-semi-open set G of X containing x such that $f(G) \subseteq O$. Thus, we have $G \subseteq f^{-1}(O)$. Hence, $f^{-1}(O)$ is an (a)-γ-P-semi-open set in X.
$(ii). \Longrightarrow (i)$. Let x be any point of X and H be a ζ_n open set containing $f(x)$. We get $f^{-1}(H)$ is (a)-γ-P-semi-open and $x \in f^{-1}(H)$. Take $G = f^{-1}(H)$, we have $f(G) \subseteq H$. Hence, f is (a)-γ-P-semi continuous.
$(ii) \Longleftrightarrow (iii)$. Obviously.
$(i). \Longrightarrow (iv)$. Let T be a subset of X and $f(x) \in f((a)\text{-}\gamma\text{-}PS\text{-cl}(T))$, for $x \in (a)\text{-}\gamma\text{-}PS\text{-cl}(T)$. Let H be any ζ_n open set of Y containing $f(x)$. By hypothesis there exists an (a)-γ-P-semi-open set G of X containing x such that $f(G) \subseteq H$. Since $G \cap T \neq \emptyset$, $H \cap f(T) \neq \emptyset$. This implies that $f(x) \in \zeta_n\text{-cl}(f(T))$. Hence, $(a)\text{-}\gamma\text{-}PS\text{-cl}(f^{-1}F) \subseteq f^{-1}(\zeta_n\text{-cl}(F))$.
$(iv). \Longrightarrow (v)$. Let F be a subset of Y. By hypothesis, we have $f((a)\text{-}\gamma\text{-}PS\text{-cl}(F)) \subseteq \zeta_n\text{-cl}(f(F))$. Taking the pre-image on both sides, we get $(a)\text{-}\gamma\text{-}PS\text{-cl}(f^{-1}F) \subseteq f^{-1}(\zeta_n\text{-cl}(F))$.
$(v). \Longrightarrow (iii)$. Let F be ζ_n-closed in Y. By hypothesis, we have $(a)\text{-}\gamma\text{-}PS\text{-cl}(f^{-1}F) \subseteq f^{-1}(F)$. Hence, $f^{-1}(F)$ is (a)-γ-P-semi-closed in X. □ □

Corolary 1. (i). *Every* (a)-γ-P-*semi continuous function is* (a)-γ-*semi continuous.*

(ii). *Every* (a)-γ-P-*semi continuous function is* (a)-*semi continuous.*

Following example shows that (a)-γ-semi continuous function need not be (a)-γ-P-semi continuous.

Example 2.27. *Consider* $X = \{a, b, c, d\}$ *with topologies* $\tau_1 = \{X, \emptyset, \{b\}, \{d\}, \{b, d\}\}$, $\tau_i = \{X, \emptyset, \{a\}, \{d\}, \{a, d\}, \{a, b\}, \{a, b, d\}\}$ *for all* $i \neq 1$. *Let* γ *be an operation on* $\bigcup \tau_n$ *defined as follows* :

$$\gamma(U) = \begin{cases} U, & \text{if } U = \{d\} \\ X, & \text{if } U \neq \{d\} \end{cases}$$

Define $f\colon (X, \{\tau_n\}) \to (X, \{\tau_n\})$ *as* $f\{a, b, d\} = d$, $f(c) = c$. *Then* f *is* (a)-γ-*semi continuous function but not* (a)-γ-*P-semi continuous as* $\{a, b, d\}$ *is not* (a)-γ-*P-semi-open.*

Example 2.28. *Consider* $X = \{a, b, c, d\}$ *with topologies* $\tau_1 = \{X, \emptyset, \{b\}, \{d\}, \{b, d\}, \{a, b, c\}\}$, $\tau_i = \{X, \emptyset, \{a\}, \{d\}, \{a, d\}, \{a, b\}, \{a, b, d\}\}$ *for all* $i \neq 1$. *Let* γ *be an operation on* $\bigcup \tau_n$ *defined as follows* :

$$\gamma(U) = \begin{cases} U, & \text{if } U = \{a\}, \{b\} \\ X, & \text{if } U \neq \{a\}, \{b\} \end{cases}$$

Define $f\colon (X, \{\tau_n\}) \to (X, \{\tau_n\})$ *as* $f\{a, b, c\} = d$, $f(d) = c$. *Then* f *is* (a)-*semi continuous function but not* (a)-γ-*P-semi continuous as* $\{d\}$ *is not* (a)-γ-*P-semi-open.*

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