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Z_k-Magic Labeling of Path Union of Graphs

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ABSTRACT

For any non-trivial Abelian group A under addition a graph G is said to be A-*magic* if there exists a labeling $f : E(G) \to A - \{0\}$ such that, the vertex labeling f^+ defined as $f^+(v) = \sum f(uv)$ taken over all edges uv incident at v is a constant. An A-*magic* graph G is said to be Z_k-magic graph if the group A is Z_k, the group of integers modulo k and these graphs are referred as k-*magic* graphs. In this paper we prove that the graphs such as path union of cycle, generalized Petersen graph, shell, wheel, closed helm, double wheel, flower, cylinder, total graph of a path, lotus inside a circle and n-pan graph are Z_k-magic graphs.

RESUMEN

Para cualquier grupo Abeliano no-trivial A bajo adición, un grafo G se dice A-*mágico* si existe un etiquetado $f\colon E(G) \to A - \{0\}$ tal que el etiquetado de un vértice f^+ definido como $f^+(v) = \sum f(uv)$, tomado sobre todos los ejes uv incidentes en v, es constante. Un grafo A-*mágico* G se dice Z_k-mágico si el grupo A es Z_k, el grupo de enteros módulo k y estos se llaman grafos k-*mágicos.* En este paper demostramos que los grafos tales como la unión por caminos de ciclos, grafos de Petersen generalizados, concha, rueda, casco cerrado, rueda doble, flor, cilindro, el grafo total de un camino, lotos dentro de un círculo y n-sartenes son todos grafos Z_k-mágicos.

Keywords and Phrases: A-magic labeling, Z_k-magic labeling, Z_k-magic graph, generalized Petersen graph, shell, wheel, closed helm, double wheel, flower, cylinder, total graph of a path, lotus inside a circle, n-pan graph.

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1 Introduction

Graph labeling is currently an emerging area in the research of graph theory. A graph labeling is an assignment of integers to vertices or edges or both subject to certain conditions. A detailed survey was done by Gallian in [1]. If the labels of edges are distinct positive integers and for each vertex v the sum of the labels of all edges incident with v is the same for every vertex v in the given graph then the labeling is called a magic labeling. Sedláček [10] introduced the concept of A-magic graphs. A graph with real-valued edge labeling such that distinct edges have distinct non-negative labels and the sum of the labels of the edges incident to a particular vertex is same for all vertices. Low and Lee [9] examined the A-magic property of the resulting graph obtained from the product of two A-magic graphs. Shiu, Lam and Sun [12] proved that the product and composition of A-magic graphs were also A-magic.

For any non-trivial Abelian group A under addition a graph G is said to be A-*magic* if there exists a labeling $f : E(G) \to A - \{0\}$ such that, the vertex labeling f^+ defined as $f^+(v) = \sum f(uv)$ taken over all edges uv incident at v is a constant. An A-*magic* graph G is said to be Z_k-magic graph if the group A is Z_k, the group of integers modulo k. These Z_k-magic graphs are referred to as k-*magic graphs*. Shiu and Low [13] determined all positive integers k for which fans and wheels have a Z_k-magic labeling with a magic constant 0. Kavitha and Thirusangu [8] obtained a Z_k-magic labeling of two cycles with a common vertex. Motivated by the concept of A-magic graph in [10] and the results in [9, 12, 13] Jeyanthi and Jeya Daisy [2, 3, 4, 5, 6, 7] proved that some standard graphs admit Z_k-magic labeling. We use the following definitions in the subsequent section.

Definition 1.1. *Let $G_1, G_2, \dots, G_n$, $n \geq 2$, be copies of a graph G. Let $v_i \in V(G_i)$, $i = 1, 2, \dots, n$, be the vertex corresponding to the vertex $v \in V(G)$ in the i^{th} copy of G_i. We denoted by $P(n.G^v)$ the graph obtained by adding the edge v_iv_{i+1}, to G_i and G_{i+1}, $1 \leq i \leq n-1$, and we call $P(n.G^v)$ the path union of n copies of the graph G.*

Note, that up to isomorphism, we obtain $|V(G)|$ graphs $P(n.G^v)$. This operation was defined in [11].

Definition 1.2. *A generalized Petersen graph $P(n, m)$, $n \geq 3$, $1 \leq m < \frac{n}{2}$ is a 3-regular graph with the vertex set $\{u_i, v_i : i = 1, 2, \dots, n\}$ and the edge set $\{u_iv_i, u_iu_{i+1}, v_iv_{i+m} : i = 1, 2, \dots, n\}$, where the indices are taken over modulo n.*

Definition 1.3. *A shell graph S_n, $n \geq 4$, is obtained by taking $n-3$ concurrent chords in a cycle C_n. The vertex at which all the chords are concurrent is called an apex.*

Definition 1.4. *A wheel graph W_n, $n \geq 3$, is obtained by joining the vertices of a cycle C_n to an extra vertex called the centre. The vertices of degree three are called rim vertices.*

Definition 1.5. *A helm graph H_n, $n \geq 3$, is obtained from a wheel W_n by adjoining a pendant edge at each vertex of the wheel except the center.*

Definition 1.6. *A closed helm graph CH_n, $n \geq 3$, is obtained from a helm H_n by joining each pendent vertex to form a cycle.*

Definition 1.7. *A double wheel graph DW_n, $n \geq 3$, is obtained by joining the vertices of two cycles C_n to an extra vertex called the hub.*

Definition 1.8. *A flower graph Fl_n, $n \geq 3$, is obtained from a helm H_n by joining each pendent vertex to the central vertex of the helm.*

Definition 1.9. *A Cartesian product of a cycle C_n, $n \geq 3$, and a path on two vertices is called a cylinder graph $C_n \square P_2$.*

Definition 1.10. *A total graph $T(G)$ is a graph with the vertex set $V(G) \cup E(G)$ in which two vertices are adjacent whenever they are either adjacent or incident in G.*

Definition 1.11. *A lotus inside a circle LC_n, $n \geq 3$, is a graph obtained from a wheel W_n by subdividing every edge forming the outer cycle and joining these new vertices to form a cycle.*

Definition 1.12. *An n-pan graph, $n \geq 3$, is obtained by attaching a pendent edge to a vertex of a cycle C_n.*

2 Z_k-Magic Labeling of Path Union of Graphs

In this section we prove that the graphs such as path union of cycle, generalized Petersen graph, shell, wheel, closed helm, double wheel, flower, cylinder, total graph of a path, lotus inside a circle and n-pan graph are Z_k-magic graphs.

Let v be a vertex of a cycle C_r, $r \geq 3$. According to the symmetry all $P(n.C_r^v)$ are isomorphic. Thus we use the notation $P(n.C_r)$.

Theorem 2.1. *Let $r \geq 3$ and $n \geq 2$ be integers. The path union of a cycle $P(n.C_r)$ is Z_k-magic for $k \geq 3$ when r is odd.*

Proof. Let the vertex set and the edge set of $P(n.C_r)$ be $V(P(n.C_r)) = \{v_i^j : 1 \leq i \leq r, 1 \leq j \leq n\}$ and $E(P(n.C_r)) = \{v_i^j v_{i+1}^j : 1 \leq i \leq r, 1 \leq j \leq n\} \cup \{v_1^j v_1^{j+1} : \ 1 \leq j \leq n-1\}$, where the index i is taken over modulo r.

Let a, k be positive integers, $k > 2a$. Thus $k \geq 3$.

For r is odd, we define an edge labeling $f : E(P(n.C_r)) \to Z_k - \{0\}$ as follows:

$$f(v_i^1 v_{i+1}^1) = f(v_i^n v_{i+1}^n) = \begin{cases} k-a, & \text{for } i = 1,3,\dots,r, \\ a, & \text{for } i = 2,4,\dots,r-1, \end{cases}$$

$$f(v_i^j v_{i+1}^j) = \begin{cases} k-2a, & \text{for } i = 1,3,\dots,r,\ j = 2,3,\dots,n-1, \\ 2a, & \text{for } i = 2,4,\dots,r-1,\ j = 2,3,\dots,n-1, \end{cases}$$

$$f(v_1^j v_1^{j+1}) = 2a, \quad \text{for } j = 1,2,\dots,n-1.$$

Then the induced vertex labeling $f^+ : V(P(n.C_r)) \to Z_k$ is $f^+(v) \equiv 0 \pmod{k}$ for every vertex v in $V(P(n.C_r))$. □

An example of a Z_{10}-magic labeling of $P(4.C_5)$ is shown in Figure 1.

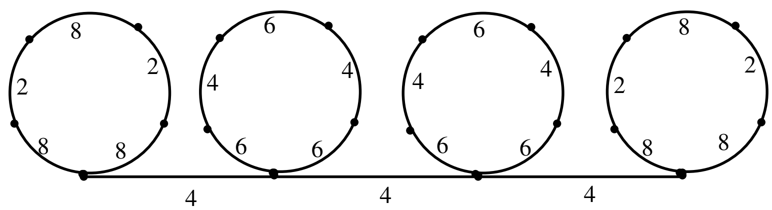

Figure 1: A Z_{10}-magic labeling of $P(4.C_5)$.

Up to isomorphism there are two graphs obtained by attaching n copies of a generalized Petersen graph $P(r,m)$, $r \geq 3$, $1 \leq m \leq \frac{r-1}{2}$ to a path P_n to get a graph $P(n.P(r,m)^v)$. We deal with the case when v is a vertex in the outer polygon of $P(r,m)$.

Theorem 2.2. *Let $r \geq 3$, $m \leq \frac{r-1}{2}$ and $n \geq 2$ be positive integers. The path union of a generalized Petersen graph $P(n.P(r,m)^v)$, where v is a vertex in the outer polygon of $P(r,m)$, is Z_k-magic for $k \geq 5$ when r is odd.*

Proof. Let the vertex set and the edge set of $P(n.P(r,m)^v)$ be $V(P(n.P(r,m)^v)) = \{u_i^j, v_i^j : 1 \leq i \leq r, 1 \leq j \leq n\}$ and $E(P(n.P(r,m)^v)) = \{u_i^j v_i^j : 1 \leq i \leq r, 1 \leq j \leq n\} \cup \{u_i^j u_{i+1}^j : 1 \leq i \leq r, 1 \leq j \leq n\} \cup \{u_1^j u_1^{j+1} : 1 \leq j \leq n-1\} \cup \{v_i^j v_{i+m}^j : 1 \leq i \leq r, 1 \leq j \leq n\}$, where the index i is taken over modulo r.

Let a, k be positive integers, $k > 4a$. Thus $k \geq 5$.

Define an edge labeling $f : E(P(n.P(r,m)^v)) \to Z_k - \{0\}$ as follows:

$$
\begin{aligned}
f(v_i^j v_{i+m}^j) &= a, \quad \text{for } i = 1,2,\dots,r,\ j = 1,2,\dots,n-1,\\
f(u_i^j v_i^j) &= k-2a, \quad \text{for } i = 1,2,\dots,r,\ j = 1,2,\dots,n-1,\\
f(u_i^1 u_{i+1}^1) &= \begin{cases} k-a, & \text{for } i = 1,3,\dots,r,\\ 3a, & \text{for } i = 2,4,\dots,r-1,\end{cases}\\
f(u_i^j u_{i+1}^j) &= a, \quad \text{for } i = 1,2,\dots,r,\ j = 2,3,\dots,n-1,\\
f(v_i^n v_{i+m}^n) &= \begin{cases} k-a, & \text{for } n \text{ is odd},\\ a, & \text{for } n \text{ is even},\end{cases}\\
f(u_i^n v_i^n) &= \begin{cases} 2a, & \text{for } n \text{ is odd},\\ k-2a, & \text{for } n \text{ is even},\end{cases}\\
f(u_i^n u_{i+1}^n) &= \begin{cases} a, & \text{for } i = 1,3,\dots,r \text{ and } n \text{ is odd},\\ k-3a, & \text{for } i = 2,4,\dots,r-1 \text{ and } n \text{ is odd},\\ k-a, & \text{for } i = 1,3,\dots,r \text{ and } n \text{ is even},\\ 3a, & \text{for } i = 2,4,\dots,r-1 \text{ and } n \text{ is even},\end{cases}\\
f(u_1^j u_1^{j+1}) &= \begin{cases} 4a, & \text{for } j = 1,3,\dots \text{ and } j \le n-1,\\ k-4a, & \text{for } j = 2,4,\dots \text{ and } j \le n-1.\end{cases}
\end{aligned}
$$

Then the induced vertex labeling $f^+ : V(P(n.P(r,m)^v)) \to Z_k$ is $f^+(u) \equiv 0 \pmod{k}$ for all $u \in V(P(n.P(r,m)^v))$. Thus $V(P(n.P(r,m)^v))$ is a Z_k-magic graph. □

An example of a Z_{15}-magic labeling of $P(5.P(5,2)^v)$ is shown in Figure 2.

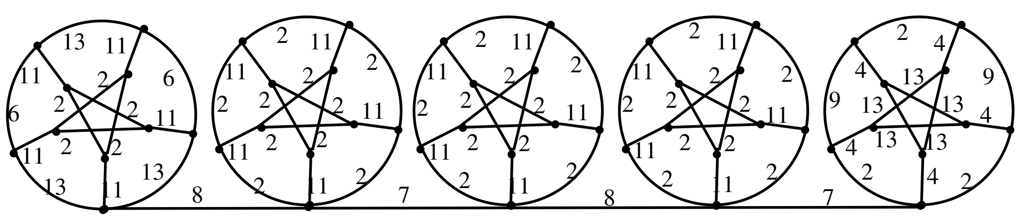

Figure 2: A Z_{15}-magic labeling of $P(5.P(5,2)^v)$.

Theorem 2.3. *Let $r \ge 4$ and $n \ge 2$ be positive integers. The path union of a shell graph $P(n.S_r^v)$, where $v \in V(S_r)$ is the vertex of degree $r-1$, is Z_k-magic for $k \ge 2r-3$ when r is odd and for $k \ge r-1$ when k is even.*

Proof. Let the vertex set and the edge set of $P(n.S_r^v)$ be $V(P(n.S_r^v)) = \{v_i^j : 1 \le i \le r, 1 \le j \le n\}$ and $E(P(n.S_r^v)) = \{v_i^j v_{i+1}^j : 1 \le i \le r, 1 \le j \le n\} \cup \{v_1^j v_i^j : 3 \le i \le r, 1 \le j \le n\} \cup \{v_1^j v_1^{j+1} : 1 \le j \le n-1\}$ with the index i taken over modulo r.

We consider the following two cases according to the parity of r.

Case (i): when r is odd.

Let a, k be positive integers, $k > 2(r-2)a$. Thus $k \geq 2r-3$.

Define an edge labeling $f : E(P(n.S_r^v)) \to Z_k - \{0\}$ as follows:

$$
\begin{aligned}
f(v_1^1 v_i^1) &= 2a, \quad \text{for } i = 3, 4, \ldots, r-1,\\
f(v_1^1 v_2^1) &= f(v_r^1 v_1^1) = a,\\
f(v_i^1 v_{i+1}^1) &= k-a, \quad \text{for } i = 2, 3, \ldots, r-1,\\
f(v_1^j v_1^{j+1}) &= \begin{cases} k-2a(r-2), & \text{for } j = 1, 3, \ldots \text{ and } j \leq n-1,\\ 2a(r-2), & \text{for } j = 2, 4, \ldots \text{ and } j \leq n-1, \end{cases}\\
f(v_1^j v_i^j) &= a, \quad \text{for } i = 3, 4, \ldots, r-1,\ j = 2, 3, \ldots, n-1,\\
f(v_i^j v_{i+1}^j) &= \begin{cases} \frac{(r-3)a}{2}, & \text{for } i = 2, 4, \ldots, r-1,\ j = 2, 3, \ldots, n-1,\\ k - \frac{(r-1)a}{2}, & \text{for } i = 3, 5, \ldots, r-2,\ j = 2, 3, \ldots, n-1, \end{cases}\\
f(v_1^j v_2^j) &= f(v_r^j v_1^j) = k - \frac{(r-3)a}{2}, \quad \text{for } j = 2, 3, \ldots, n-1,\\
f(v_1^n v_i^n) &= \begin{cases} k-2a, & \text{for } i = 3, 4, \ldots, r-1 \text{ and } n \text{ is odd},\\ 2a, & \text{for } i = 3, 4, \ldots, r-1 \text{ and } n \text{ is even}, \end{cases}\\
f(v_1^n v_2^n) = f(v_r^n v_1^n) &= \begin{cases} k-a, & \text{for } n \text{ is odd},\\ a, & \text{for } n \text{ is even}, \end{cases}\\
f(v_i^n v_{i+1}^n) &= \begin{cases} a, & \text{for } i = 2, 3, \ldots, r-1 \text{ and } n \text{ is odd},\\ k-a, & \text{for } i = 2, 3, \ldots, r-1 \text{ and } n \text{ is even}. \end{cases}
\end{aligned}
$$

Then the induced vertex labeling $f^+ : V(p(n.S_r^v)) \to Z_k$ is $f^+(u) \equiv 0 \pmod{k}$ for all $u \in V(P(n.S_r^v))$.

Case (ii): when r is even.

Let a, k be positive integers, $k > (r-2)a$. Thus $k \geq r-1$.

Define an edge labeling $f : E(P(n.S_r^v)) \to Z_k - \{0\}$ in the following way.

$$f(v_1^1 v_i^1) = a, \quad \text{for } i = 3, 4, \dots, r-1,$$
$$f(v_1^1 v_2^1) = k - a,$$
$$f(v_r^1 v_1^1) = 2a,$$
$$f(v_i^1 v_{i+1}^1) = \begin{cases} a, & \text{for } i = 2, 4, \dots, r-2, \\ k-2a, & \text{for } i = 3, 5, \dots, r-1, \end{cases}$$
$$f(v_1^j v_1^{j+1}) = \begin{cases} k - a(r-2), & \text{for } j = 1, 3, \dots \text{ and } j \le n-1, \\ a(r-2), & \text{for } j = 2, 4, \dots \text{ and } j \le n-1, \end{cases}$$
$$f(v_1^j v_i^j) = \tfrac{k}{2}, \quad \text{for } i = 3, 4, \dots, r-1,\ j = 2, 3, \dots, n-1,$$
$$f(v_i^j v_{i+1}^j) = \begin{cases} \frac{3k}{4}, & \text{for } i = 2, 3, \dots, r-1,\ j = 2, 3, \dots, n-1 \text{ and } k \equiv 0 \pmod 4, \\ \frac{3k+2}{4}, & \text{for } i = 2, 4, \dots, r-2,\ j = 2, 3, \dots, n-1 \text{ and } k \equiv 2 \pmod 4, \\ \frac{3k-2}{4}, & \text{for } i = 3, 5, \dots, r-1,\ j = 2, 3, \dots, n-1 \text{ and } k \equiv 2 \pmod 4, \end{cases}$$
$$f(v_1^j v_2^j) = \begin{cases} \frac{k}{4}, & \text{for } j = 2, 3, \dots, n-1 \text{ and } k \equiv 0 \pmod 4, \\ \frac{k-2}{4}, & \text{for } j = 2, 3, \dots, n-1 \text{ and } k \equiv 2 \pmod 4, \end{cases}$$
$$f(v_r^j v_1^j) = \begin{cases} \frac{k}{4}, & \text{for } j = 2, 3, \dots, n-1 \text{ and } k \equiv 0 \pmod 4, \\ \frac{k+2}{4}, & \text{for } j = 2, 3, \dots, n-1 \text{ and } k \equiv 2 \pmod 4, \end{cases}$$
$$f(v_1^n v_i^n) = \begin{cases} k-a, & \text{for } i = 3, 4, \dots, r-1 \text{ and } n \text{ is odd}, \\ a, & \text{for } i = 3, 4, \dots, r-1 \text{ and } n \text{ is even}, \end{cases}$$
$$f(v_1^n v_2^n) = \begin{cases} a, & \text{for } n \text{ is odd}, \\ k-a, & \text{for } n \text{ is even}, \end{cases}$$
$$f(v_r^n v_1^n) = \begin{cases} k-2a, & \text{for } n \text{ is odd}, \\ 2a, & \text{for } n \text{ is even}, \end{cases}$$
$$f(v_i^n v_{i+1}^n) = \begin{cases} k-a, & \text{for } i = 2, 4, \dots, r-2 \text{ and } n \text{ is odd} \\ 2a, & \text{for } i = 3, 5, \dots, r-1 \text{ and } n \text{ is odd}, \\ a, & \text{for } i = 2, 4, \dots, r-2 \text{ and } n \text{ is even}, \\ k-2a, & \text{for } i = 3, 5, \dots, r-1 \text{ and } n \text{ is even}. \end{cases}$$

Then the induced vertex labeling $f^+ : V(P(n.S_r^v)) \to Z_k$ is $f^+(u) \equiv 0 \pmod k$ for all $u \in V(P(n.S_r^v))$. Thus $P(n.S_r^v)$ is a Z_k-magic graph for r is even. □

An example of a Z_{11}-magic labeling of $P(3.S_7^v)$ is shown in Figure 3.

According to the symmetry of wheels there exist two non isomorphic graphs $P(n.W_r^v)$. We deal with the case when v is a rim vertex, that is a vertex of degree three in W_r.

Theorem 2.4. *Let $r \ge 4$ and $n \ge 2$ be integers. The path union of a wheel graph $P(n.W_r^v)$, where $v \in V(W_r)$ is a vertex of degree 3, is Z_k-magic for $k \ge r$ when r is odd and for $k \ge 2r-1$ when r*

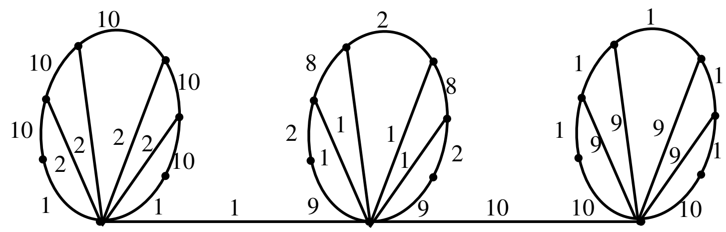

Figure 3: A Z_{11}-magic labeling of $P(3.S_7^v)$.

is even.

Proof. Let the vertex set and the edge set of $P(n.W_r^v)$ be $V(P(n.W_r^v)) = \{w_j, v_i^j : 1 \leq i \leq r, 1 \leq j \leq n\}$ and $E(P(n.W_r^v)) = \{v_i^j v_{i+1}^j : 1 \leq i \leq r, 1 \leq j \leq n\} \cup \{w_j v_i^j : 1 \leq i \leq r, 1 \leq j \leq n\} \cup \{u_1^j u_1^{j+1} : 1 \leq j \leq n-1\}$, where the index i is taken over modulo r.

We consider the following two cases according to the parity of r.

Case (i): when r is odd.

Let a, k be positive integers, $k > (r-1)a$. This implies $k \geq r$.

Define an edge labeling $f : E(P(n.W_r^v)) \to Z_k - \{0\}$ as follows:

$$f(w_j v_i^j) = a, \quad \text{for } i = 2, 3, \ldots, r, \ j = 1, 2, \ldots, n-1,$$
$$f(w_j v_1^j) = k - (r-1)a, \quad \text{for } j = 1, 2, \ldots, n-1,$$
$$f(v_i^1 v_{i+1}^1) = \begin{cases} a, & \text{for } i = 1, 3, \ldots, r, \\ k - 2a, & \text{for } i = 2, 4, \ldots, r-1, \end{cases}$$
$$f(v_i^j v_{i+1}^j) = \begin{cases} \frac{(r-1)a}{2}, & \text{for } i = 1, 3, \ldots, r, \ j = 2, 3, \ldots, n-1, \\ k - \frac{(r+1)a}{2}, & \text{for } i = 2, 4, \ldots, r-1, \ j = 2, 3, \ldots, n-1, \end{cases}$$
$$f(w_n v_1^n) = \begin{cases} (r-1)a, & \text{for } n \text{ is odd}, \\ k - (r-1)a, & \text{for } n \text{ is even}, \end{cases}$$
$$f(w_n v_i^n) = \begin{cases} k - a, & \text{for } i = 2, 3, \ldots, r \text{ and } n \text{ is odd}, \\ a, & \text{for } i = 2, 3, \ldots, r \text{ and } n \text{ is even}, \end{cases}$$
$$f(v_i^n v_{i+1}^n) = \begin{cases} k - a, & \text{for } i = 1, 3, \ldots, r \text{ and } n \text{ is odd}, \\ 2a, & \text{for } i = 2, 4, \ldots, r-1 \text{ and } n \text{ is odd}, \\ a, & \text{for } i = 1, 3, \ldots, r \text{ and } n \text{ is even}, \\ k - 2a, & \text{for } i = 2, 4, \ldots, r-1 \text{ and } n \text{ is even}, \end{cases}$$
$$f(v_1^j v_1^{j+1}) = \begin{cases} a(r-3), & \text{for } j = 1, 3, \ldots \text{ and } j \leq n-1, \\ k - a(r-3), & \text{for } j = 2, 4, \ldots \text{ and } j \leq n-1. \end{cases}$$

This means that for the induced vertex labeling $f^+ : V(P(n.W_r^v)) \to Z_k$ is $f^+(u) \equiv 0 \pmod{k}$ for all $u \in V(P(n.W_r^v))$.

Case (ii): when r is even.

Let a, k be positive integers, $k > 2(r-1)a$.

Define an edge labeling $f : E(P(n.W_r^v)) \to Z_k - \{0\}$ in the following way.

$$f(w_1 v_1^j) = f(w_n v_1^n) = k - (r-1)a,$$
$$f(w_1 v_i^1) = f(w_n v_i^n) = a, \quad \text{for } i = 2, 3, \dots, r,$$
$$f(v_i^1 v_{i+1}^1) = f(v_i^n v_{i+1}^n) = \begin{cases} a, & \text{for } i = 1, 3, \dots, r-1, \\ k - 2a, & \text{for } i = 2, 4, \dots, r, \end{cases}$$
$$f(w_j v_1^j) = k - 2(r-1)a, \quad \text{for } j = 2, 3, \dots, n-1,$$
$$f(w_j v_i^j) = 2a, \quad \text{for } i = 2, 3, \dots, r,\ j = 2, 3, \dots, n-1,$$
$$f(v_i^j v_{i+1}^j) = k - a, \quad \text{for } i = 1, 2, \dots, r,\ j = 2, 3, \dots, n-1,$$
$$f(v_1^j v_1^{j+1}) = ra, \quad \text{for } j = 1, 2, \dots, n-1.$$

Then the induced vertex labeling $f^+ : V(P(n.W_r^v)) \to Z_k$ is $f^+(u) \equiv 0 \pmod{k}$ for all $u \in V(P(n.W_r^v))$. Hence f^+ is constant that means $P(n.W_r^v)$ admits a Z_k-magic labeling. □

An example of a Z_{12}-magic labeling of $P(3.W_6^v)$ is shown in Figure 4.

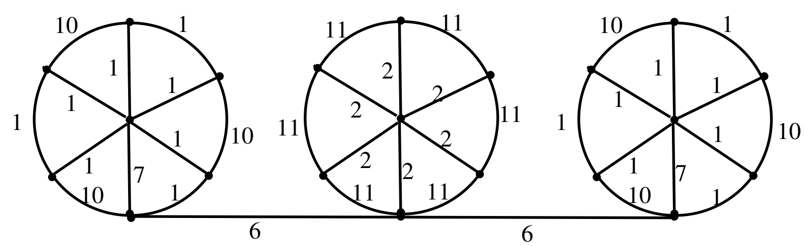

Figure 4: A Z_{12}-magic labeling of $P(3.W_6^v)$.

In the next theorem we deal with the path union of a closed helm graph $P(n.CH_r^v)$, where v is a vertex of degree three in CH_r.

Theorem 2.5. *Let $r \geq 4$ and $n \geq 2$ be integers. The path union of a closed helm graph $P(n.CH_r^v)$, where v is a vertex of degree 3 in CH_r, is Z_k-magic for $k \geq r$ when r is odd and for even $k \geq r$ when r is even.*

Proof. Let the vertex set and the edge set of $P(n.CH_r^v)$ be $V(P(n.CH_r^v)) = \{w_j, v_i^j, u_i^j : 1 \leq i \leq r, 1 \leq j \leq n\}$ and $E(P(n.CH_r^v)) = \{v_i^j v_{i+1}^j : 1 \leq i \leq r, 1 \leq j \leq n\} \cup \{u_i^j u_{i+1}^j : 1 \leq i \leq r, 1 \leq j \leq n\} \cup \{w_j v_i^j : 1 \leq i \leq r, 1 \leq j \leq n\} \cup \{v_i^j u_i^j : 1 \leq i \leq r, 1 \leq j \leq n\} \cup \{u_1^j u_1^{j+1} : 1 \leq j \leq n-1\}$, where the index i is taken over modulo r.

Case (i): when r is odd.

Let a, k be positive integers, $k > (r-1)a$. Thus $k \geq r$.

Define an edge labeling $f : E(P(n.CH_r^v)) \to Z_k - \{0\}$ as follows:

$$f(w_j v_1^j) = k-(r-1)a, \quad \text{for } j = 1,2,\dots,n-1,$$
$$f(w_j v_i^j) = a, \quad \text{for } i = 2,3,\dots,r,\ j = 1,2,\dots,n-1,$$
$$f(v_i^j v_{i+1}^j) = \begin{cases} (r-1)a, & \text{for } i = 1,3,\dots,r,\ j = 1,2,\dots,n-1, \\ k-(r-1)a, & \text{for } i = 2,4,\dots,r-1,\ j = 1,2,\dots,n-1, \end{cases}$$
$$f(u_i^1 u_{i+1}^1) = \begin{cases} (r-1)a, & \text{for } i = 1,3,\dots,r, \\ k-(r-2)a, & \text{for } i = 2,4,\dots,r-1, \end{cases}$$
$$f(v_i^j u_i^j) = k-a, \quad \text{for } i = 2,3,\dots,r,\ j = 1,2,\dots,n-1,$$
$$f(v_1^j u_1^j) = k-(r-1)a, \quad \text{for } j = 1,2,\dots,n-1,$$
$$f(u_i^j u_{i+1}^j) = \begin{cases} \frac{(r-1)a}{2}, & \text{for } i = 1,3,\dots,r,\ j = 2,3,\dots,n-1, \\ k-\frac{(r-3)a}{2}, & \text{for } i = 2,4,\dots,r-1,\ j = 2,3,\dots,n-1, \end{cases}$$
$$f(w_n v_1^n) = \begin{cases} (r-1)a, & \text{for } n \text{ is odd}, \\ k-(r-1)a, & \text{for } n \text{ is even}, \end{cases}$$
$$f(w_n v_i^n) =, \begin{cases} k-a, & \text{for } i = 2,3,\dots,r \text{ and } n \text{ is odd}, \\ a, & \text{for } i = 2,3,\dots,r \text{ and } n \text{ is even}, \end{cases}$$
$$f(v_1^n u_1^n) = \begin{cases} (r-1), & \text{for } n \text{ is odd}, \\ k-(r-1)a, & \text{for } n \text{ is even}, \end{cases}$$
$$f(v_i^n u_i^n) = \begin{cases} a, & \text{for } i = 2,3,\dots,r \text{ and } n \text{ is odd}, \\ k-a, & \text{for } i = 2,3,\dots,r \text{ and } n \text{ is even}, \end{cases}$$
$$f(v_i^n v_{i+1}^n) = \begin{cases} k-(r-1)a, & \text{for } i = 1,3,\dots,r \text{ and } n \text{ is odd}, \\ (r-1)a, & \text{for } i = 2,4,\dots,r-1 \text{ and } n \text{ is odd}, \\ (r-1)a, & \text{for } i = 1,3,\dots,r \text{ and } n \text{ is even}, \\ k-(r-1)a, & \text{for } i = 2,4,\dots,r-1 \text{ and } n \text{ is even}, \end{cases}$$
$$f(u_i^n u_{i+1}^n) = \begin{cases} k-(r-1)a, & \text{for } i = 1,3,\dots,r \text{ and } n \text{ is odd}, \\ (r-2)a, & \text{for } i = 2,4,\dots,r-1 \text{ and } n \text{ is odd}, \\ (r-1)a, & \text{for } i = 1,3,\dots,r \text{ and } n \text{ is even}, \\ k-(r-2)a, & \text{for } i = 2,4,\dots,r-1 \text{ and } n \text{ is even}, \end{cases}$$
$$f(u_1^j u_1^{j+1}) = \begin{cases} k-(r-1)a, & \text{for } j = 1,3,\dots \text{ and } j \leq n-1, \\ (r-1)a, & \text{for } j = 2,4,\dots \text{ and } j \leq n-1. \end{cases}$$

Then the induced vertex labeling $f^+ : V(P(n.CH_r^v)) \to Z_k$ is $f^+(u) \equiv 0 \pmod{k}$ for all $u \in V(P(n.CH_r^v))$.

Case (ii): when r is even.

Let a be a positive integer and $k > (r-2)a$ be an even integer. Thus $k \geq r$.

Define an edge labeling $f : E(P(n.CH_r^v)) \to Z_k - \{0\}$ such that

$$f(w_1v_1^1) = k-(r-1)a,$$
$$f(w_1v_i^1) = a, \quad \text{for } i = 2,3,\dots,r,$$
$$f(v_i^1v_{i+1}^1) = \begin{cases} (r-1)a, & \text{for } i = 1,3,\dots,r-1, \\ k-(r-1)a, & \text{for } i = 2,4,\dots,r, \end{cases}$$
$$f(u_i^1u_{i+1}^1) = \begin{cases} (r-1)a, & \text{for } i = 1,3,\dots,r-1, \\ k-(r-2)a, & \text{for } i = 2,4,\dots,r-1, \end{cases}$$
$$f(v_1^1u_1^1) = (r-1)a,$$
$$f(v_i^1u_i^1) = k-a, \quad \text{for } i = 2,3,\dots,r,$$
$$f(w_jv_i^j) = f(v_i^ju_i^j) = f(v_i^jv_{i+1}^j) = \tfrac{k}{2}, \quad \text{for } i = 1,2,\dots,r,\ j = 2,3,\dots,n-1,$$
$$f(u_i^ju_{i+1}^j) = \begin{cases} \frac{k}{4}, & \text{for } i = 1,2,\dots,r,\ j = 2,3,\dots,n-1 \text{ and } k \equiv 0 \pmod 4, \\ \frac{k-2}{4}, & \text{for } i = 1,3,\dots,r-1,\ j = 2,3,\dots,n-1 \text{ and } k \equiv 2 \pmod 4, \\ \frac{k+2}{4}, & \text{for } i = 2,4,\dots,r,\ j = 2,3,\dots,n-1 \text{ and } k \equiv 2 \pmod 4, \end{cases}$$
$$f(w_nv_1^n) = \begin{cases} (r-1)a, & \text{for } n \text{ is odd}, \\ k-(r-1)a, & \text{for } n \text{ is even}, \end{cases}$$
$$f(w_nv_i^n) = \begin{cases} k-a, & \text{for } i = 2,3,\dots,r \text{ and } n \text{ is odd}, \\ a, & \text{for } i = 2,3,\dots,r \text{ and } n \text{ is even}, \end{cases}$$
$$f(v_1^nu_1^n) = \begin{cases} k-(r-1)a, & \text{for } n \text{ is odd}, \\ (r-1)a, & \text{for } n \text{ is even}, \end{cases}$$
$$f(v_i^nu_i^n) = \begin{cases} a, & \text{for } i = 2,3,\dots,r \text{ and } n \text{ is odd}, \\ k-a, & \text{for } i = 2,3,\dots,r \text{ and } n \text{ is even}, \end{cases}$$
$$f(v_i^nv_{i+1}^n) = \begin{cases} k-(r-1)a, & \text{for } i = 1,3,\dots,r-1 \text{ and } n \text{ is odd}, \\ (r-1)a, & \text{for } i = 2,4,\dots,r \text{ and } n \text{ is odd}, \\ (r-1)a, & \text{for } i = 1,3,\dots,r-1 \text{ and } n \text{ is even}, \\ k-(r-1)a, & \text{for } i = 2,4,\dots,r \text{ and } n \text{ is even}, \end{cases}$$
$$f(u_i^nu_{i+1}^n) = \begin{cases} k-(r-1)a, & \text{for } i = 1,3,\dots,r-1 \text{ and } n \text{ is odd}, \\ (r-2)a, & \text{for } i = 2,4,\dots,r \text{ and } n \text{ is odd}, \\ (r-1)a, & \text{for } i = 1,3,\dots,r-1 \text{ and } n \text{ is even}, \\ k-(r-2)a, & \text{for } i = 2,4,\dots,r \text{ and } n \text{ is even}, \end{cases}$$
$$f(u_1^ju_1^{j+1}) = \begin{cases} k-ra, & \text{for } j = 1,3,\dots \text{ and } j \leq n-1, \\ ra, & \text{for } j = 2,4,\dots \text{ and } j \leq n-1. \end{cases}$$

Then the induced vertex labeling $f^+ : V(P(n.CH_r^v)) \to Z_k$ is $f^+(u) \equiv 0 \pmod k$ for all $u \in$

$V(P(n.CH_r^v))$. Hence f^+ is constant equal to 0 (mod k). Therefore $P(n.CH_r^v)$ is a Z_k-magic graph. $\square$

An example of a Z_6-magic labeling of $P(3.CH_6^v)$ is shown in Figure 5.

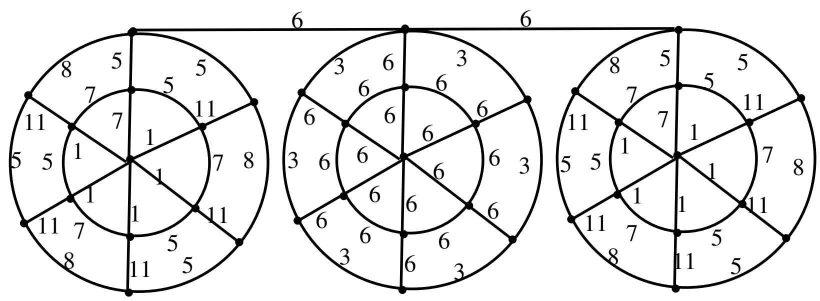

Figure 5: A Z_{12}-magic labeling of $P(3.CH_6^v)$.

Theorem 2.6. *Let $r \geq 3$ and $n \geq 2$ be integers. The path union of a double wheel graph $P(n.DW_r^v)$, where $v \in V(DW_r)$ is a vertex of degree 3, is Z_k-magic for $k \geq 5$ when r is odd.*

Proof. Let the vertex set and the edge set of $C(n.DW_r^v)$ be $V(P(n.DW_r^v)) = \{v_j, v_i^j, u_i^j : 1 \leq i \leq r, 1 \leq j \leq n\}$ and $E(P(n.DW_r^v)) = \{v_j v_i^j, v_j u_i^j : 1 \leq i \leq r, 1 \leq j \leq n\} \cup \{v_i^j v_{i+1}^j : 1 \leq i \leq r, 1 \leq j \leq n\} \cup \{u_i^j u_{i+1}^j : 1 \leq i \leq r, 1 \leq j \leq n\} \cup \{u_1^j u_1^{j+1} : 1 \leq j \leq n-1\}$ with index i taken over modulo r.

Let a, k be positive integers, $k > 4a$. Thus $k \geq 5$.

For r is odd we define an edge labeling $f : E(P(n.DW_r^v)) \to Z_k - \{0\}$ as follows:

$$f(v_j v_i^j) = 2a, \quad \text{for } i = 1, 2, \dots, r,\ j = 1, 2, \dots, n-1,$$

$$f(v_j u_i^j) = k - 2a, \quad \text{for } i = 1, 2, \dots, r,\ j = 1, 2, \dots, n-1,$$

$$f(v_i^j v_{i+1}^j) = k - a, \quad \text{for } i = 1, 2, \dots, r,\ j = 1, 2, \dots, n-1,$$

$$f(u_i^1 u_{i+1}^1) = \begin{cases} k - a, & \text{for } i = 1, 3, \dots, r, \\ 3a, & \text{for } i = 2, 4, \dots, r-1, \end{cases}$$

$$f(u_i^j u_{i+1}^j) = a, \quad \text{for } i = 1, 2, \dots, r,\ j = 2, 3, \dots, n-1,$$

$$f(v_n v_i^n) = \begin{cases} k - 2a, & \text{for } i = 1, 2, \dots, r \text{ and } n \text{ is odd}, \\ 2a, & \text{for } i = 1, 2, \dots, r \text{ and } n \text{ is even}, \end{cases}$$

$$f(v_n u_i^n) = \begin{cases} 2a, & \text{for } i = 1, 2, \dots, r \text{ and } n \text{ is odd}, \\ k - 2a, & \text{for } i = 1, 2, \dots, r \text{ and } n \text{ is even}, \end{cases}$$

$$f(v_i^n v_{i+1}^n) = \begin{cases} a, & \text{for } i = 1, 2, \dots, r-1 \text{ and } n \text{ is odd}, \\ k - a, & \text{for } i = 1, 2, \dots, r-1 \text{ and } n \text{ is even}, \end{cases}$$

$$f(v_r^n v_1^n) = \begin{cases} a, & \text{for } n \text{ is odd}, \\ k - a, & \text{for } n \text{ is even}, \end{cases}$$

$$f(u_i^n u_{i+1}^n) = \begin{cases} a, & \text{for } i = 1, 3, \dots, r \text{ and } n \text{ is odd}, \\ k - 3a, & \text{for } i = 2, 4, \dots, r-1 \text{ and } n \text{ is odd}, \\ k - a, & \text{for } i = 1, 3, \dots, r \text{ and } n \text{ is even}, \\ 3a, & \text{for } i = 2, 4, \dots, r-1 \text{ and } n \text{ is even}, \end{cases}$$

$$f(u_1^j u_1^{j+1}) = \begin{cases} 4a, & \text{for } j = 1, 3, \dots \text{ and } j \le n-1, \\ k - 4a, & \text{for } j = 2, 4, \dots \text{ and } j \le n-1. \end{cases}$$

Then the induced vertex labeling $f^+ : V(P(n.DW_r^v)) \to Z_k$ is $f^+(u) \equiv 0 \pmod{k}$ for all $u \in V(P(n.DW_r^v))$. □

An example of a Z_7-magic labeling of $P(3.DW_7^v)$ is shown in Figure 6.

Theorem 2.7. *Let $r \ge 3$ and $n \ge 2$ be positive integers. The path union of a flower graph $P(n.Fl_r^v)$, where $v \in V(Fl_r)$ is the vertex of degree 4, is Z_k-magic for $k \ge 5$ when r is odd and for $k \ge 3$ when k is even.*

Proof. Let the vertex set and the edge set of $P(n.Fl_r^v)$ be $V(P(n.Fl_r^v)) = \{w_j, v_i^j, u_i^j : 1 \le i \le r, 1 \le j \le n\}$ and $E(P(n.Fl_r^v)) = \{w_j v_i^j : 1 \le i \le r, 1 \le j \le n\} \cup \{v_i^j u_i^j : 1 \le i \le r, 1 \le j \le n\} \cup \{w_j u_i^j : 1 \le i \le r, 1 \le j \le n\} \cup \{v_i^j v_{i+1}^j : 1 \le i \le r, 1 \le j \le n\} \cup \{v_1^j v_1^{j+1} : 1 \le j \le n-1\}$, with index i taken over modulo r.

Case (i): when r is odd.

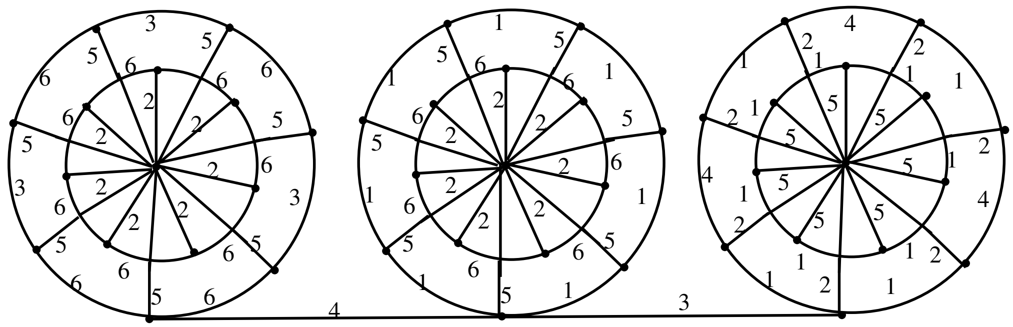

Figure 6: A Z_7-magic labeling of $P(3.DW_7^v)$.

Let a, k be positive integers, $k > 4a$. This means $k \geq 5$.

Define an edge labeling $f : E(P(n.Fl_r^v)) \to Z_k - \{0\}$ as follows:

$$f(w_j v_i^j) = f(v_i^j u_i^j) = a, \quad \text{for } i = 1, 2, \ldots, r,\ j = 1, 2, \ldots, n-1,$$
$$f(u_i^j w_j) = k - a, \quad \text{for } i = 1, 2, \ldots, r,\ j = 1, 2, \ldots, n-1,$$
$$f(v_i^1 v_{i+1}^1) = \begin{cases} a, & \text{for } i = 1, 3, \ldots, r, \\ k - 3a, & \text{for } i = 2, 4, \ldots, r-1, \end{cases}$$
$$f(v_i^j v_{i+1}^j) = k - a, \quad \text{for } i = 1, 2, \ldots, r,\ j = 2, 3, \ldots, n-1,$$
$$f(w_n v_i^n) = f(v_i^n u_i^n) = \begin{cases} k - a, & \text{for } i = 1, 2, \ldots, r \text{ and } n \text{ is odd}, \\ a, & \text{for } i = 1, 2, \ldots, r \text{ and } n \text{ is even}, \end{cases}$$
$$f(u_i^n w_n) = \begin{cases} a, & \text{for } i = 1, 2, \ldots, r \text{ and } n \text{ is odd}, \\ k - a, & \text{for } i = 1, 2, \ldots, r \text{ and } n \text{ is even}, \end{cases}$$
$$f(v_i^n v_{i+1}^n) = \begin{cases} k - a, & \text{for } i = 1, 3, \ldots, r \text{ and } n \text{ is odd}, \\ 3a, & \text{for } i = 2, 4, \ldots, r-1 \text{ and } n \text{ is odd}, \\ a, & \text{for } i = 1, 3, \ldots, r \text{ and } n \text{ is even}, \\ k - 3a, & \text{for } i = 2, 4, \ldots, r-1 \text{ and } n \text{ is even}, \end{cases}$$
$$f(v_1^j v_1^{j+1}) = \begin{cases} k - 4a, & \text{for } j = 1, 3, \ldots \text{ and } j \leq n-1, \\ 4a, & \text{for } j = 2, 4, \ldots \text{ and } j \leq n-1. \end{cases}$$

Then the induced vertex labeling $f^+ : V(P(n.Fl_r^v)) \to Z_k$ is $f^+(u) \equiv 0 \pmod{k}$ for all $u \in V(P(n.Fl_r^v))$.

Case (ii): when r is even.

Let a, k be positive integers, $k > 2a$. Thus $k \geq 3$.

Define an edge labeling $f : E(P(n.Fl_r^v)) \to Z_k - \{0\}$ as follows:

$$
\begin{aligned}
f(w_1v_1^1) &= f(v_1^1u_1^1) = 2a,\\
f(u_1^1w_1) &= k-2a,\\
f(v_i^jv_{i+1}^j) &= k-a, \quad \text{for } i=1,2,\dots,r,\ j=2,3,\dots,n-1,\\
f(w_jv_i^j) &= f(v_i^ju_i^j) = a, \quad \text{for } i=2,3,\dots,r,\ j=1,2,\dots,n-1,\\
f(w_ju_i^j) &= k-a, \quad \text{for } i=2,3,\dots,r,\ j=1,2,\dots,n-1,\\
f(w_nv_1^n) &= f(v_1^nu_1^n) = \begin{cases} k-2a, & \text{for } n \text{ is odd},\\ 2a, & \text{for } n \text{ is even},\end{cases}\\
f(w_nu_1^n) &= \begin{cases} 2a, & \text{for } n \text{ is odd},\\ k-2a, & \text{for } n \text{ is even},\end{cases}\\
f(w_nv_i^n) &= f(v_i^nu_i^n) = \begin{cases} k-a, & \text{for } i=2,3,\dots,r \text{ and } n \text{ is odd},\\ a, & \text{for } i=2,3,\dots,r \text{ and } n \text{ is even},\end{cases}\\
f(w_nu_i^n) &= \begin{cases} a, & \text{for } i=2,3,\dots,r \text{ and } n \text{ is odd},\\ k-a, & \text{for } i=2,3,\dots,r \text{ and } n \text{ is even},\end{cases}\\
f(v_i^nv_{i+1}^n) &= \begin{cases} a, & \text{for } i=1,2,\dots,r \text{ and } n \text{ is odd},\\ k-a, & \text{for } i=1,2,\dots,r \text{ and } n \text{ is even},\end{cases}\\
f(v_1^jv_1^{j+1}) &= \begin{cases} k-2a, & \text{for } j=1,3,\dots \text{ and } j\le n-1,\\ 2a, & \text{for } j=2,4,\dots \text{ and } j\le n-1.\end{cases}
\end{aligned}
$$

The induced vertex labeling $f^+ : V(P(n.Fl_r^v)) \to Z_k$ is $f^+(u) \equiv 0 \pmod k$ for all $u \in V(P(n.Fl_r^v))$.

□

An example of a Z_{10}-magic labeling of $P(4.Fl_3^v)$ is shown in Figure 7.

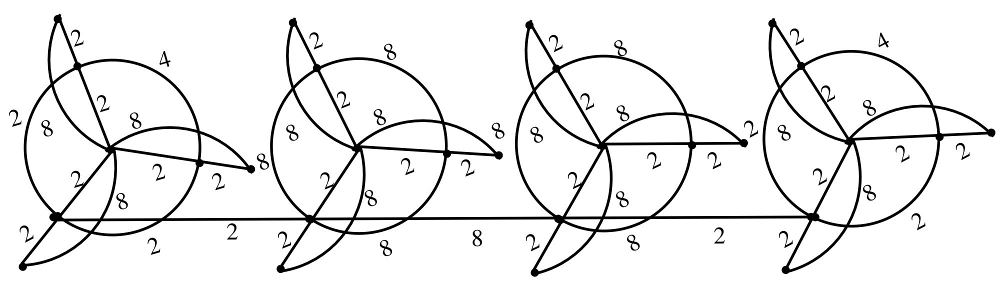

Figure 7: A Z_{10}-magic labeling of $P(4.Fl_3^v)$.

Let v be a vertex of a cylinder graph $C_r\square P_2$, $r \ge 3$. According to the symmetry all $P(n.(C_r\square P_2)^v)$ are isomorphic. Thus we use the notation $P(n.(C_r\square P_2))$.

Theorem 2.8. *Let $r \ge 3$, $n \ge 2$ be integers. The path union of a cylinder graph $P(n.(C_r\square P_2))$ is Z_k-magic for $k \ge 5$ when r is odd.*

Proof. Let the vertex set and the edge set of $P(n.(C_r\square P_2))$ be $V(P(n.(C_r\square P_2))) = \{v_i^j, u_i^j : 1 \leq i \leq r, 1 \leq j \leq n\}$ and $E(P(n.(C_r\square P_2))) = \{u_i^j v_i^j : 1 \leq i \leq r, 1 \leq j \leq n\} \cup \{u_i^j u_{i+1}^j : 1 \leq i \leq r, 1 \leq j \leq n\} \cup \{v_i^j v_{i+1}^j : 1 \leq i \leq r, 1 \leq j \leq n\} \cup \{u_1^j u_1^{j+1} : 1 \leq j \leq n-1\}$, with index i taken over modulo r.

Let a, k be positive integers, $k > 4a$. Thus $k \geq 5$.

For r odd we define an edge labeling $f : E(P(n.(C_r\square P_2))) \to Z_k - \{0\}$ as follows:

$$f(v_i^j u_i^j) = k - 2a, \quad \text{for } i = 1, 2, \ldots, r,\ j = 1, 2, \ldots, n-1,$$
$$f(v_i^j v_{i+1}^j) = a, \quad \text{for } i = 1, 2, \ldots, r,\ j = 1, 2, \ldots, n-1,$$
$$f(u_i^1 u_{i+1}^1) = \begin{cases} k-a, & \text{for } i = 1, 3, \ldots, r, \\ 3a, & \text{for } i = 2, 4, \ldots, r-1, \end{cases}$$
$$f(u_i^j u_{i+1}^j) = a, \quad \text{for } i = 1, 2, \ldots, r,\ j = 2, 3, \ldots, n-1,$$
$$f(v_i^n v_{i+1}^n) = \begin{cases} k-a, & \text{for } n \text{ is odd}, \\ a, & \text{for } n \text{ is even}, \end{cases}$$
$$f(v_i^n u_i^n) = \begin{cases} 2a, & \text{for } n \text{ is odd}, \\ k-2a, & \text{for } n \text{ is even}, \end{cases}$$
$$f(u_i^n u_{i+1}^n) = \begin{cases} a, & \text{for } i = 1, 3, \ldots, r \text{ and } n \text{ is odd}, \\ k-3a, & \text{for } i = 2, 4, \ldots, r-1 \text{ and } n \text{ is odd}, \\ k-a, & \text{for } i = 1, 3, \ldots, r \text{ and } n \text{ is even}, \\ 3a, & \text{for } i = 2, 4, \ldots, r-1 \text{ and } n \text{ is even}, \end{cases}$$
$$f(u_1^j u_1^{j+1}) = \begin{cases} 4a, & \text{for } j = 1, 3, \ldots,\ j \leq n-1, \\ k-4a, & \text{for } j = 2, 4, \ldots,\ j \leq n-1. \end{cases}$$

Then the induced vertex labeling $f^+ : V(P(n.(C_r\square P_2))) \to Z_k$ is $f^+(v) \equiv 0 \equiv k$ for all $v \in V(P(n.(C_r\square P_2)))$. Hence f^+ is constant and is equal to $0 \equiv k$. $\square$

An example of a Z_9-magic labeling of $P(3.(C_7\square P_2))$ is shown in Figure 8.

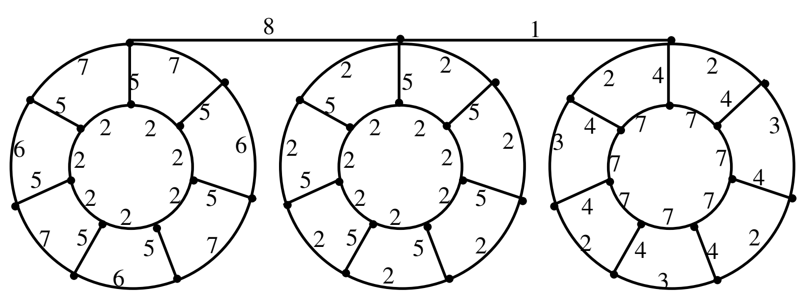

Figure 8: A Z_9-magic labeling of $P(3.(C_7\square P_2)^v)$.

Theorem 2.9. *Let $r \geq 5$ and $n \geq 2$ be positive integers. The path union of a total graph of a path $P(n.T(P_r)^v)$, where $v \in V(T(P_r))$ is a vertex of degree two, is Z_k-magic for $k \geq 3$.*

Proof. Let the vertex set and the edge set of $P(n.T(P_r)^v)$ be $V(P(n.T(P_r)^v)) = \{u_i^j : 1 \leq i \leq r, 1 \leq j \leq n\} \cup \{v_i^j : 1 \leq i \leq r-1, 1 \leq j \leq n\}$ and $E(P(n.T(P_r)^v)) = \{u_i^j u_{i+1}^j : 1 \leq i \leq r-1, 1 \leq j \leq n\} \cup \{v_i^j v_{i+1}^j : 1 \leq i \leq r-2, 1 \leq j \leq n\} \cup \{u_{i+1}^j v_i^j : 1 \leq i \leq r-1, 1 \leq j \leq n\} \cup \{u_i^j v_i^j : 1 \leq i \leq r-1, 1 \leq j \leq n\} \cup \{u_1^j u_1^{j+1} : 1 \leq j \leq n-1\}$.

We consider the following two cases according to the parity of r.

Case (i): when r is odd.

Let a, k be positive integers, $k > 2a$. Thus $k \geq 3$.

Define an edge labeling $f : E(P(n.T(P_r)^v)) \rightarrow Z_k - \{0\}$ as follows:

$$
\begin{aligned}
f(u_i^1 u_{i+1}^1) &= \begin{cases} a, & \text{for } i = 1, 3, \ldots, r, \\ 2a, & \text{for } i = 2, 4, \ldots, r-3, \end{cases} \\
f(u_{r-1}^1 u_r^1) &= f(v_1^1 v_2^1) = a, \\
f(v_i^1 v_{i+1}^1) &= \begin{cases} 2a, & \text{for } i = 3, 5, \ldots, r, \\ a, & \text{for } i = 2, 4, \ldots, r-1, \end{cases} \\
f(u_1^1 v_1^1) &= a, \\
f(u_2^1 v_2^1) &= k - a, \\
f(u_i^1 v_i^1) &= k - 2a, \quad \text{for } i = 3, 4, \ldots, r-2, \\
f(u_{r-1}^1 v_{r-1}^1) &= k - a, \\
f(v_1^1 u_2^1) &= k - 2a, \\
f(v_i^1 u_{i+1}^1) &= k - a, \quad \text{for } i = 2, 3, \ldots, r-1, \\
f(u_1^j v_1^j) &= f(u_2^j v_1^j) = a, \quad \text{for } j = 2, 3, \ldots, n-1, \\
f(u_1^j u_2^j) &= f(u_{r-1}^j u_r^j) = k - a, \quad \text{for } j = 2, 3, \ldots, n-1, \\
f(u_i^j u_{i+1}^j) &= k - 2a, \quad \text{for } i = 2, 3, \ldots, r-2,\ j = 2, 3, \ldots, n-1, \\
f(v_i^j v_{i+1}^j) &= k - 2a, \quad \text{for } i = 1, 2, \ldots, r-2,\ j = 2, 3, \ldots, n-1, \\
f(u_i^j v_i^j) &= f(u_{i+1}^j v_i^j) = 2a, \quad \text{for } i = 2, 3, \ldots, r-2,\ j = 2, 3, \ldots, n-1, \\
f(u_r^j v_{r-1}^j) &= f(u_{r-1}^j v_{r-1}^j) = a, \quad \text{for } j = 2, 3, \ldots, n-1,
\end{aligned}
$$

$$f(u^n_{r-1}u^n_r) = \begin{cases} k-a, & \text{for } n \text{ is odd},\\ a, & \text{for } n \text{ is even},\end{cases}$$

$$f(u^n_i u^n_{i+1}) = \begin{cases} k-a, & \text{for } i = 1,3,\dots,r \text{ and } n \text{ is odd},\\ k-2a, & \text{for } i = 2,4,\dots,r-3 \text{ and } n \text{ is odd},\\ a, & \text{for } i = 1,3,\dots,r \text{ and } n \text{ is even},\\ 2a, & \text{for } i = 2,4,\dots,r-3 \text{ and } n \text{ is odd},\end{cases}$$

$$f(v^n_1 v^n_2) = \begin{cases} k-a, & \text{for } n \text{ is odd},\\ a, & \text{for } n \text{ is even},\end{cases}$$

$$f(v^n_i v^n_{i+1}) = \begin{cases} k-2a, & \text{for } i = 3,5,\dots,r \text{ and } n \text{ is odd},\\ k-a, & \text{for } i = 2,4,\dots,r-1 \text{ and } n \text{ is odd},\\ 2a, & \text{for } i = 3,5,\dots,r \text{ and } n \text{ is even},\\ a, & \text{for } i = 2,4,\dots,r-1 \text{ and } n \text{ is even},\end{cases}$$

$$f(u^n_1 v^n_1) = \begin{cases} k-a, & \text{for } n \text{ is odd},\\ a, & \text{for } n \text{ is even},\end{cases}$$

$$f(u^n_2 v^n_2) = \begin{cases} a, & \text{for } n \text{ is odd},\\ k-a, & \text{for } n \text{ is even},\end{cases}$$

$$f(u^n_i v^n_i) = \begin{cases} 2a, & \text{for } i = 3,4,\dots,r-2 \text{ and } n \text{ is odd},\\ k-2a, & \text{for } i = 3,4,\dots,r-2 \text{ and } n \text{ is even},\end{cases}$$

$$f(u^n_{r-1} v^n_{r-1}) = \begin{cases} a, & \text{for } n \text{ is odd},\\ k-a, & \text{for } n \text{ is even},\end{cases}$$

$$f(v^n_1 u^n_2) = \begin{cases} 2a, & \text{for } n \text{ is odd},\\ k-2a, & \text{for } n \text{ is even},\end{cases}$$

$$f(v^n_i u^n_{i+1}) = \begin{cases} a, & \text{for } i = 2,3,\dots,r-1 \text{ and } n \text{ is odd},\\ k-a, & \text{for } i = 2,3,\dots,r-1 \text{ and } n \text{ is even},\end{cases}$$

$$f(u^j_1 u^{j+1}_1) = \begin{cases} k-2a, & \text{for } j = 1,3,\dots \text{ and } j \le n-1,\\ 2a, & \text{for } j = 2,4,\dots \text{ and } j \le n-1.\end{cases}$$

Then the induced vertex labeling $f^+ : V(P(n.T(P_r)^v)) \to Z_k$ is $f^+(u) \equiv 0 \pmod{k}$ for all $u \in V(P(n.T(P_r)^v))$.

Case (ii): when r is even.

Let a, k be positive integers, $k > 2a$. Thus $k \geq 3$.

Define an edge labeling $f : E(P(n.T(P_r)^v)) \to Z_k - \{0\}$ as follows:

$$f(u_i^1u_{i+1}^1) = f(v_i^1v_{i+1}^1) = \begin{cases} k-a, & \text{for } i = 1,3,\dots,r-1, \\ k-2a, & \text{for } i = 2,4,\dots,r, \end{cases}$$

$$f(v_1^1u_1^1) = k-a,$$

$$f(v_i^1u_i^1) = a, \quad \text{for } i = 2,3,\dots,r-1,$$

$$f(v_i^1u_{i+1}^1) = 2a, \quad \text{for } i = 1,2,\dots,r-2,$$

$$f(v_{r-1}^1u_r^1) = a,$$

$$f(u_1^jv_1^j) = f(u_2^jv_1^j) = a, \quad \text{for } j = 2,3,\dots,n-1,$$

$$f(u_1^ju_2^j) = f(u_{r-1}^ju_r^j) = k-a, \quad \text{for } j = 2,3,\dots,n-1,$$

$$f(u_i^ju_{i+1}^j) = k-2a, \quad \text{for } i = 2,3,\dots,r-2,\ j = 2,3,\dots,n-1,$$

$$f(v_i^jv_{i+1}^j) = k-2a, \quad \text{for } i = 1,2,\dots,r-2,\ j = 2,3,\dots,n-1,$$

$$f(u_i^jv_i^j) = f(u_{i+1}^jv_i^j) = 2a, \quad \text{for } i = 2,3,\dots,r-2,\ j = 2,3,\dots,n-1,$$

$$f(u_r^jv_{r-1}^j) = f(u_{r-1}^jv_{r-1}^j) = a, \quad \text{for } j = 2,3,\dots,n-1,$$

$$f(u_i^nu_{i+1}^n) = f(v_i^nv_{i+1}^n) = \begin{cases} a, & \text{for } i = 1,3,\dots,r-1 \text{ and } n \text{ is odd}, \\ 2a, & \text{for } i = 2,4,\dots,r \text{ and } n \text{ is odd}, \\ k-a, & \text{for } i = 1,3,\dots,r-1 \text{ and } n \text{ is even}, \\ k-2a, & \text{for } i = 2,4,\dots,r \text{ and } n \text{ is even}, \end{cases}$$

$$f(u_1^nv_1^n) = \begin{cases} a, & \text{for } n \text{ is odd}, \\ k-a, & \text{for } n \text{ is even}, \end{cases}$$

$$f(u_i^nv_i^n) = \begin{cases} k-a, & \text{for } i = 2,3,\dots,r-1 \text{ and } n \text{ is odd}, \\ a, & \text{for } i = 2,3,\dots,r-1 \text{ and } n \text{ is even}, \end{cases}$$

$$f(v_i^nu_{i+1}^n) = \begin{cases} k-2a, & \text{for } i = 1,2,\dots,r-2 \text{ and } n \text{ is odd}, \\ 2a, & \text{for } i = 1,2,\dots,r-2 \text{ and } n \text{ is even}, \end{cases}$$

$$f(v_{r-1}^nu_r^n) = \begin{cases} k-a, & \text{for } n \text{ is odd}, \\ a, & \text{for } n \text{ is even}, \end{cases}$$

$$f(u_1^ju_1^{j+1}) = \begin{cases} 2a, & \text{for } j = 1,3,\dots \text{ and } j \leq n-1, \\ k-2a, & \text{for } j = 2,4,\dots \text{ and } j \leq n-1. \end{cases}$$

Then the induced vertex labeling $f^+ : V(P(n.T(P_r)^v)) \to Z_k$ is $f^+(u) \equiv 0 \pmod{k}$ for all $u \in V(P(n.T(P_r)^v))$. Hence $P(n.T(P_r)^v)$ is a Z_k-magic graph. □

An example of a Z_5-magic labeling of $P(5.T(P_6)^v)$ is shown in Figure 9.

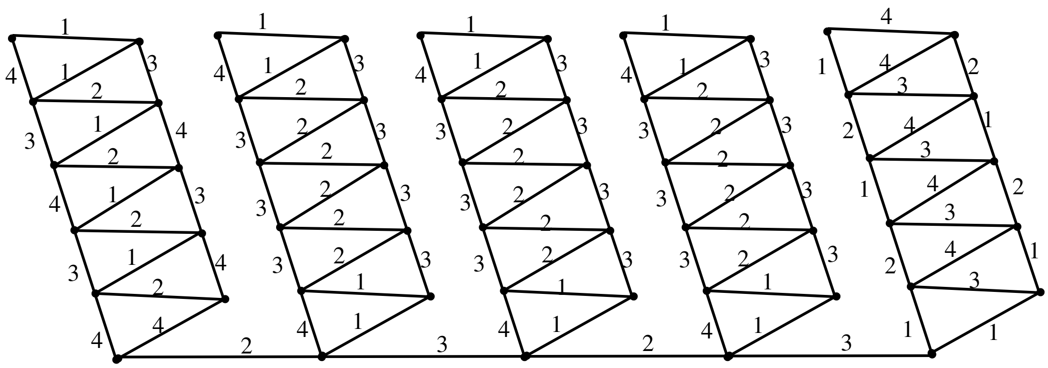

Figure 9: A Z_5-magic labeling of $P(5.T(P_6)^v)$.

Theorem 2.10. *Let $r \geq 3$ and $n \geq 2$ be integers. Let v is a vertex of degree 2 in LC_r. The path union of a lotus inside a circle graph $P(n.LC_r^v)$, is Z_k-magic for $k \geq r$.*

Proof. Let the vertex set and the edge set of $P(n.LC_r^v)$ be $V(P(n.LC_r^v)) = \{w_j, v_i^j, u_i^j : 1 \leq i \leq r, 1 \leq j \leq n\}$ and $E(P(n.LC_r^v)) = \{w_j v_i^j : 1 \leq i \leq r, 1 \leq j \leq n\} \cup \{v_i^j u_i^j : 1 \leq i \leq r, 1 \leq j \leq n\} \cup \{v_i^j u_{i+1}^j : 1 \leq i \leq r, 1 \leq j \leq n\} \cup \{u_i^j u_{i+1}^j : 1 \leq i \leq r, 1 \leq j \leq n\} \cup \{u_1^j u_1^{j+1} : 1 \leq j \leq n-1\}$, where the index i is taken over modulo r.

We consider the following two cases according to the parity of r.

Case (i): when r is odd.

Let a, k be positive integers, $k > (r-1)a$. Thus $k \geq r$.

Define an edge labeling $f : E(P(n.LC_r^v)) \to Z_k - \{0\}$ in the following way.

$$
\begin{aligned}
f(w_j v_1^j) &= k-(r-1)a, \quad \text{for } j=1,2,\dots,n-1,\\
f(w_j v_i^j) &= a, \quad \text{for } i=2,3,\dots,r,\ j=1,2,\dots,n-1,\\
f(v_1^j u_1^j) &= (r-2)a, \quad \text{for } j=1,2,\dots,n-1,\\
f(v_i^j u_i^j) &= k-2a, \quad \text{for } i=2,3,\dots,r,\ j=1,2,\dots,n-1,\\
f(v_i^j u_{i+1}^j) &= a, \quad \text{for } i=1,2,\dots,r,\ j=1,2,\dots,n-1,\\
f(u_i^1 u_{i+1}^1) &= \begin{cases} k-a, & \text{for } i=1,3,\dots,r,\\ 2a, & \text{for } i=2,4,\dots,r-1,\end{cases}\\
f(u_i^j u_{i+1}^j) &= \begin{cases} k-\frac{(r-1)a}{2}, & \text{for } i=1,3,\dots,r,\ j=2,3,\dots,n-1,\\ \frac{(r+1)a}{2}, & \text{for } i=2,4,\dots,r-1,\ j=2,3,\dots,n-1,\end{cases}\\
f(u_1^j u_1^{j+1}) &= \begin{cases} k-(r-3)a, & \text{for } j=1,3,\dots,\ j\le n-1,\\ (r-3)a, & \text{for } j=2,4,\dots,\ j\le n-1,\end{cases}\\
f(w_n v_1^n) &= \begin{cases} (r-1)a, & \text{for } n \text{ is odd},\\ k-(r-1)a, & \text{for } n \text{ is even},\end{cases}\\
f(w_n v_i^n) &= \begin{cases} k-a, & \text{for } i=2,3,\dots,r \text{ and } n \text{ is odd},\\ a, & \text{for } i=2,3,\dots,r \text{ and } n \text{ is even},\end{cases}\\
f(v_1^n u_1^n) &= \begin{cases} k-(r-2)a, & \text{for } n \text{ is odd},\\ (r-2)a, & \text{for } n \text{ is even},\end{cases}\\
f(v_i^n u_i^n) &= \begin{cases} 2a, & \text{for } i=2,3,\dots,r \text{ and } n \text{ is odd},\\ k-2a, & \text{for } i=2,3,\dots,r \text{ and } n \text{ is even},\end{cases}\\
f(v_i^n u_{i+1}^n) &= \begin{cases} k-a, & \text{for } i=1,2,\dots,r \text{ and } n \text{ is odd},\\ a, & \text{for } i=1,2,\dots,r \text{ and } n \text{ is even},\end{cases}\\
f(u_i^n u_{i+1}^n) &= \begin{cases} a, & \text{for } i=1,3,\dots,r \text{ and } n \text{ is odd},\\ k-2a, & \text{for } i=2,4,\dots,r-1 \text{ and } n \text{ is odd},\\ k-a, & \text{for } i=1,3,\dots,r \text{ and } n \text{ is even},\\ 2a, & \text{for } i=2,4,\dots,r-1 \text{ and } n \text{ is even}.\end{cases}
\end{aligned}
$$

Then the induced vertex labeling $f^+ : V(P(n.LC_r^v)) \to Z_k$ is $f^+(u) \equiv 0 \pmod{k}$ for all $u \in V(P(n.LC_r^v))$.

Case (ii): when r is even.

Let a, k be positive integers, $k > (r-1)a$. Thus $k \geq r$.

Define an edge labeling $f : E(P(n.LC_r)) \to Z_k - \{0\}$ as follows:

$$
\begin{aligned}
f(w_1v_1^1) &= k-(r-1)a,\\
f(w_1v_i^1) &= a, \quad \text{for } i=2,3,\dots,r,\\
f(v_1^1u_1^1) &= (r-2)a,\\
f(v_i^1u_i^1) &= k-2, \quad \text{for } i=2,3,\dots,r,\\
f(v_i^1u_{i+1}^1) &= a, \quad \text{for } i=1,2,\dots,r,\\
f(u_i^1u_{i+1}^1) &= \begin{cases} k-a, & \text{for } i=1,3,\dots,r-1,\\ 2a, & \text{for } i=2,4,\dots,r, \end{cases}\\
f(w_jv_i^j) &= \begin{cases} a, & \text{for } i=1,3,\dots,r-1,\ j=2,3,\dots,n-1,\\ k-a, & \text{for } i=2,4,\dots,r,\ j=2,3,\dots,n-1, \end{cases}\\
f(v_i^ju_i^j) &= \begin{cases} k-2a, & \text{for } i=1,3,\dots,r-1,\ j=2,3,\dots,n-1,\\ k-a, & \text{for } i=2,4,\dots,r,\ j=2,3,\dots,n-1, \end{cases}\\
f(v_i^ju_{i+1}^j) &= \begin{cases} a, & \text{for } i=1,3,\dots,r-1,\ j=1,2,\dots,n-1,\\ 2a, & \text{for } i=2,4,\dots,r,\ j=1,2,\dots,n-1, \end{cases}\\
f(u_i^ju_{i+1}^j) &= \begin{cases} k-a, & \text{for } i=1,3,\dots,r-1,\ j=2,3,\dots,n-1,\\ a, & \text{for } i=2,4,\dots,r,\ j=2,3,\dots,n-1, \end{cases}\\
f(w_nv_1^n) &= \begin{cases} (r-1)a, & \text{for } n \text{ is odd},\\ k-(r-1)a, & \text{for } n \text{ is even}, \end{cases}\\
f(w_nv_i^n) &= \begin{cases} k-a, & \text{for } i=2,3,\dots,r \text{ and } n \text{ is odd},\\ a, & \text{for } i=2,3,\dots,r \text{ and } n \text{ is even}, \end{cases}\\
f(v_1^nu_1^n) &= \begin{cases} k-(r-2)a, & \text{for } n \text{ is odd},\\ (r-2)a, & \text{for } n \text{ is even}, \end{cases}\\
f(v_i^nu_i^n) &= \begin{cases} 2a, & \text{for } i=2,3,\dots,r \text{ and } n \text{ is odd},\\ k-2a, & \text{for } i=2,3,\dots,r \text{ and } n \text{ is even}, \end{cases}\\
f(v_i^nu_{i+1}^n) &= \begin{cases} k-a, & \text{for } i=1,2,\dots,r \text{ and } n \text{ is odd},\\ a, & \text{for } i=1,2,\dots,r \text{ and } n \text{ is even}, \end{cases}\\
f(u_i^nu_{i+1}^n) &= \begin{cases} a, & \text{for } i=1,3,\dots,r-1 \text{ and } n \text{ is odd},\\ k-2a, & \text{for } i=2,4,\dots,r \text{ and } n \text{ is odd},\\ k-a, & \text{for } i=1,3,\dots,r-1 \text{ and } n \text{ is even},\\ 2a, & \text{for } i=2,4,\dots,r \text{ and } n \text{ is even}, \end{cases}\\
f(u_1^ju_1^{j+1}) &= \begin{cases} k-ra, & \text{for } j=1,3,\dots,\ j\le n-1,\\ ra, & \text{for } j=2,4,\dots,\ j\le n-1. \end{cases}
\end{aligned}
$$

Then the induced vertex labeling $f^+ : V(P(n.LC_r^v)) \to Z_k$ is $f^+(u) \equiv 0 \pmod{k}$ for all $u \in$

$V(P(n.LC_r^v))$. Hence f^+ is constant and is equal to $\equiv 0 \pmod{k}$. $\square$

An example of a Z_{10}-magic labeling of $P(3.LC_6^v)$ is shown in Figure 10.

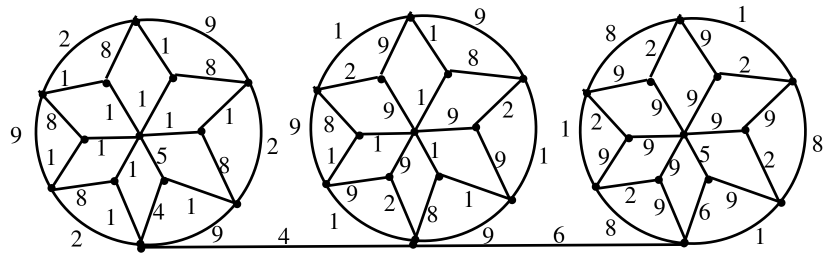

Figure 10: A Z_{10}-magic labeling of $P(3.LC_6^v)$.

In the last theorem we deal with the path union of an r-pan graph $P(n.(r\text{-pan})^v)$, where v is a vertex of degree two in an r-pan graph.

Theorem 2.11. *Let $r \geq 3$, $n \geq 2$ be integers. The path union of an r-pan graph $P(n.(r\text{-pan})^v)$, where v is a vertex of degree two in an r-pan graph, is Z_k-magic for $k \geq 5$ when r is odd.*

Proof. Let v be a vertex of degree two in an r-pan graph. Let the vertex set and the edge set of $P(n.(r\text{-pan})^v)$ be $V(P(n.(r\text{-pan})^v)) = \{w_j, v_i^j : 1 \leq i \leq r, 1 \leq j \leq n\}$ and $E(P(n.(r\text{-pan})^v)) = \{v_i^j v_{i+1}^j : 1 \leq i \leq r, 1 \leq j \leq n\} \cup \{v_1^j w_j : 1 \leq j \leq n\} \cup \{w_1^j w_1^{j+1} : 1 \leq j \leq n-1\}$, where the index i is taken over modulo r.

Let a, k be positive integers, $k > 2a$. Thus $k \geq 5$.

For r odd we define an edge labeling $f : E(P(n.(r\text{-pan})^v)) \to Z_k - \{0\}$ as follows:

$$f(v_i^1 v_{i+1}^1) = f(v_i^n v_{i+1}^n) = \begin{cases} k-a, & \text{for } i = 1, 3, \dots, r, \\ a, & \text{for } i = 2, 4, \dots, r-1, \end{cases}$$

$$f(v_i^j v_{i+1}^j) = \begin{cases} k-2a, & \text{for } i = 1, 3, \dots, r,\ j = 2, 3, \dots, n-1, \\ 2a, & \text{for } i = 2, 4, \dots, r-1,\ j = 2, 3, \dots, n-1, \end{cases}$$

$$f(v_1^1 w_1) = f(v_1^n w_n) = 2a,$$

$$f(v_1^j w_j) = 4a, \quad \text{for } j = 2, 3, \dots, n-1,$$

$$f(w_1^j w_1^{j+1}) = k-2a, \quad \text{for } j = 1, 2, \dots, n-1.$$

Then the induced vertex labeling $f^+ : V(P(n.(r\text{-pan})^v)) \to Z_k$ is $f^+(u) \equiv 0 \pmod{k}$ for all $u \in V(P(n.(r\text{-pan})^v))$. This means that $P(n.(r\text{-pan})^v)$ is a Z_k-magic graph. $\square$

An example of a Z_9-magic labeling of $P(4.(5\text{-pan})^v)$ is illustrated in Figure 11.

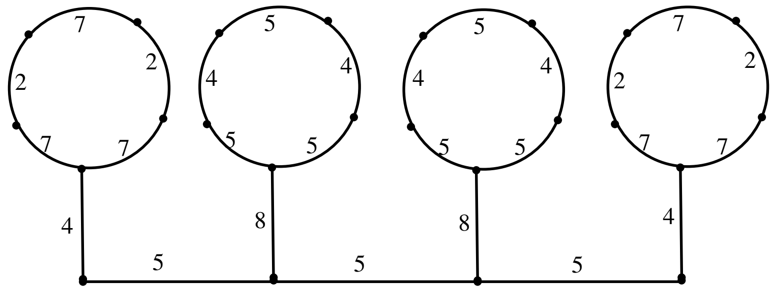

Figure 11: A Z_9-magic labeling of $P(4.(5\text{-pan})^v)$.

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