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η-Ricci Solitons on 3-dimensional Trans-Sasakian Manifolds

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ABSTRACT

In this paper, we study η-Ricci solitons on 3-dimensional trans-Sasakian manifolds. Firstly we give conditions for the existence of these geometric structures and then observe that they provide examples of η-Einstein manifolds. In the case of ϕ-Ricci symmetric trans-Sasakian manifolds, the η-Ricci soliton condition turns them to Einstein manifolds. Afterward, we study the implications in this geometric context of the important tensorial conditions $R \cdot S = 0$, $S \cdot R = 0$, $W_2 \cdot S = 0$ and $S \cdot W_2 = 0$.

RESUMEN

En este artículo estudiamos solitones η-Ricci en variedades trans-Sasakianas tridimensionales. En primer lugar damos condiciones para la existencia de estas estructuras geométricas y luego observamos que ellas dan ejemplos de variedades η-Einstein. En el caso de variedades trans-Sasakianas ϕ-Ricci simétricas, la condición de solitón η-Ricci las convierte en variedades Einstein. A continuación estudiamos las implicancias en este contexto geométrico de las importantes condiciones tensoriales $R \cdot S = 0$, $S \cdot R = 0$, $W_2 \cdot S = 0$ y $S \cdot W_2 = 0$.

Keywords and Phrases: Trans-Sasakian manifold, η-Ricci solitons.

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1 Introduction

In 1982, the notion of the Ricci flow was introduced by Hamilton [10] to find a canonical metric on a smooth manifold.The Ricci flow is an evolution equation for Riemannian metric $g(t)$ on a smooth manifold M given by

$$\frac{\partial}{\partial t}g(t) = -2S.$$

A solution to this equation (or a Ricci flow) is a one-parameter family of metrics $g(t)$, parameterized by t in a non-degenerate interval I, on a smooth manifold M satisfying the Ricci flow equation. If I has an initial point t_0, then $(M, g(t_0))$ is called the initial condition of or the initial metric for the Ricci flow (or of the solution) [14].

Ricci solitons and η-Ricci solitons are natural generalizations of Einstein metrics. A Ricci soliton on a Riemannian manifold (M, g) is defined by

$$S + \frac{1}{2}\mathcal{L}_X g = \lambda g$$

where $\mathcal{L}_X g$ is the Lie derivative along the vector field X, S is the Ricci tensor of the metric and λ is a real constant. If $X = \nabla f$ for some function f on M, the Ricci soliton becomes gradient Ricci soliton. Ricci solitons appear as self-similar solutions to Hamiltons's Ricci flow and often arise as limits of dilations of singularities in the Ricci flow [11]. A soliton is called shrinking, steady and expanding according as $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$ respectively.

In 2009, the notion of η-Ricci soliton was introduced by J.C. Cho and M. Kimura [6]. J.C. Cho and M. Kimura proved that a real hypersurface admitting an η-Ricci soliton in a non-flat complex space form is a Hopf-hypersurface [6]. An η-Ricci soliton on a Riemannian manifold (M, g) is defined by the following equation

$$2S + \mathcal{L}_\xi g + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{1.1}$$

where $\mathcal{L}_\xi$ is the Lie derivative operator along the vector field ξ, S is the Ricci tensor of the metric and λ, μ are real constants. If $\mu = 0$, then η-Ricci soliton becomes Ricci soliton.

In the last few years, many authors have worked on Ricci solitons and their generalizations in different Contact metric manfolds in [1], [7], [8], [9], [12] etc. In 2014, B. Y. Chen and S. Deshmukh have established the characterizations of compact shrinking trivial Ricci solitons in [5]. Also, in [2], A. Bhattacharyya, T. Dutta, and S. Pahan studied the torqued vector field and established some applications of torqued vector field on Ricci soliton and conformal Ricci soliton. A.M. Blaga [3], D. G. Prakasha and B. S. Hadimani [17] observed η-Ricci solitons on different contact metric manifolds satisfying some certain curvature conditions.

In this paper we study the existence of η-Ricci soliton on 3-dimensional trans-Sasakian manifold. Next we show that η-Ricci soliton on 3-dimensional trans-Sasakian manifolds becomes η-Einstein Manifold under some conditions. Next we prove that ϕ-Ricci symmetric trans-Sasakian manifold (M, g) manifold satisfying an η-Ricci soliton becomes an Einstein manifold. Next we give an example of an η-Ricci soliton on 3-dimensional trans-Sasaian manifold with $\lambda = -2$ and $\mu = 6$. Later we obtain some different types of curvature tensors and their properties under certain conditions.

2 Preliminaries

The product $\bar{M} = M \times R$ has a natural almost complex structure J with the product metric G being Hermitian metric. The geometry of the almost Hermitian manifold $(\bar{M}, J, G)$ gives the geometry of the almost contact metric manifold (M, ϕ, ξ, η, g). Sixteen different types of structures on M like Sasakian manifold, Kenmotsu manifold etc are given by the almost Hermitian manifold $(\bar{M}, J, G)$. The notion of trans-Sasakian manifolds was introduced by Oubina [15] in 1985. Then J. C. Marrero [13] have studied the local structure of trans-Sasakian manifolds. In general a trans-Sasakian manifold $(M, \phi, \xi, \eta, g, \alpha, \beta)$ is called a trans-Sasakian manifold of type (α, β). An n $(= 2m + 1)$ dimensional Riemannian manifold (M, g) is called an almost contact manifold if there exists a (1,1) tensor field ϕ, a vector field ξ and a 1-form η on M such that

$$\phi^2(X) = -X + \eta(X)\xi, \tag{2.1}$$

$$\eta(\xi) = 1, \eta(\phi X) = 0, \tag{2.2}$$

$$\phi\xi = 0, \tag{2.3}$$

$$\eta(X) = g(X, \xi), \tag{2.4}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.5}$$

$$g(X, \phi Y) + g(Y, \phi X) = 0, \tag{2.6}$$

for any vector fields X, Y on M. A 3-dimensional almost contact metric manifold M is called a trans-Sasakian manifold if it satisfies the following condition

$$(\nabla_X \phi)(Y) = \alpha\{g(X, Y)\xi - \eta(Y)X\} + \beta\{g(\phi X, Y)\xi - \eta(Y)\phi X\}, \tag{2.7}$$

for some smooth functions α, β on M and we say that the trans-Sasakian structure is of type (α, β). For 3-dimensional trans-Sasakian manifold, from (2.7) we have,

$$\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi), \tag{2.8}$$

$$(\nabla_X\eta)(Y) = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{2.9}$$

In a 3-dimensional trans-Sasakian manifold, we have

$$\begin{aligned} R(X,Y)Z &= [\tfrac{r}{2} - 2(\alpha^2 - \beta^2 - \xi\beta)][g(Y,Z)X - g(X,Z)Y] \\ &- [\tfrac{r}{2} - 3(\alpha^2 - \beta^2) + \xi\beta][g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\xi \\ &+ [g(Y,Z)\eta(X) - g(X,Z)\eta(Y)][\phi \ grad \ \alpha - \ grad \ \beta] \\ &- [\tfrac{r}{2} - 3(\alpha^2 - \beta^2) + \xi\beta]\eta(Z)[\eta(Y)X - \eta(X)Y] \\ &- [Z\beta + (\phi Z)\alpha]\eta(Z)[\eta(Y)X - \eta(X)Y] \\ &- [X\beta + (\phi X)\alpha][g(Y,Z)\xi - \eta(Z)Y] \\ &- [Y\beta + (\phi Y)\alpha][g(X,Z)\xi - \eta(Z)X], \end{aligned}$$

$$\begin{aligned} S(X,Y) &= [\tfrac{r}{2} - (\alpha^2 - \beta^2 - \xi\beta)]g(X,Y) \\ &- [\tfrac{r}{2} - 3(\alpha^2 - \beta^2) + \xi\beta]\eta(X)\eta(Y) \\ &- [Y\beta + (\phi Y)\alpha]\eta(X) - [X\beta + (\phi X)\alpha]\eta(Y). \end{aligned}$$

When α and β are constants the above equations reduce to,

$$R(\xi, X)\xi = (\alpha^2 - \beta^2)(\eta(X)\xi - X), \tag{2.10}$$

$$S(X, \xi) = 2(\alpha^2 - \beta^2)\eta(X), \tag{2.11}$$

$$R(\xi, X)Y = (\alpha^2 - \beta^2)(g(X, Y)\xi - \eta(Y)X). \tag{2.12}$$

$$R(X, Y)\xi = (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y). \tag{2.13}$$

Definition 2.1. A trans-Sasakian manifold M^3 is said to be η-Einstein manifold if its Ricci tensor S is of the form

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y),$$

where a, b are smooth functions.

3 η-Ricci solitons on trans-Sasakian manifolds

To study the existence conditions of η-Ricci solitons on 3-dimensional trans-Sasakian manifolds, we prove the following theorem.

Theorem 3.1: Let $(M, g, \phi, \eta, \xi, \alpha, \beta)$ be a 3-dimensional trans-Sasakian manifold with α, β constants ($\beta \neq 0$). If the symmetric $(0, 2)$ tensor field h satisfying the condition $\beta h(X, Y) - \frac{\alpha}{2}[h(\phi X, Y) + h(X, \phi Y)] = \mathcal{L}_\xi g(X, Y) + 2S(X, Y) + 2\mu\eta(X)\eta(Y)$ is parallel with respect to the Levi-Civita connection associated to g. Then (g, ξ, μ) becomes an η-Ricci soliton.

Proof: We consider a symmetric $(0, 2)$-tensor field h which is parallel with respect to the Levi-Civita connection ($\nabla h = 0$). Then it follows that

$$h(R(X, Y)Z, W) + h(R(X, Y)Z, W) = 0, \tag{3.1}$$

for an arbitary vector field W, X, Y, Z on M. Put $X = Z = W = \xi$ we get

$$h(R(X, Y)\xi, \xi) = 0, \tag{3.2}$$

for any X, $Y \in \chi(M)$ By using the equation (2.13)

$$h(Y, \xi) = g(Y, \xi)h(\xi, \xi), \tag{3.3}$$

for any $Y \in \chi(M)$. Differentiating the equation (3.3) covariantly with respect to the vector field $X \in \chi(M)$ we have

$$h(\nabla_X Y, \xi) + h(Y, \nabla_X \xi) = g(\nabla_X Y, \xi)h(\xi, \xi) + g(Y, \nabla_X \xi)h(\xi, \xi), \tag{3.4}$$

Using the equation (2.8) we have

$$\beta h(X, Y) - \alpha h(\phi X, Y) = -\alpha g(\phi X, Y)h(\xi, \xi) + \beta h(\xi, \xi)g(X, Y). \tag{3.5}$$

Interchanging X by Y we have

$$\beta h(X, Y) - \alpha h(X, \phi Y) = -\alpha g(X, \phi Y)h(\xi, \xi) + \beta h(\xi, \xi)g(X, Y). \tag{3.6}$$

Then adding the above two equations we get

$$\beta h(X, Y) - \frac{\alpha}{2}[h(\phi X, Y) + h(X, \phi Y)] = \beta h(\xi, \xi)g(X, Y). \tag{3.7}$$

We see that $\beta h(X, Y) - \frac{\alpha}{2}[h(\phi X, Y) + h(X, \phi Y)]$ is a symmetric tensor of type $(0, 2)$. Since $\mathcal{L}_\xi g(X, Y)$, $S(X, Y)$, $\eta(X) = g(X, \xi)$ and $\eta(Y) = g(Y, \xi)$ are symmetric tensors of type $(0, 2)$ and λ, μ are real constants, the sum $\mathcal{L}_\xi g(X, Y) + 2S(X, Y) + 2\mu\eta(X)\eta(Y)$ is a symmetric tensor of type $(0, 2)$.

Therefore, we can take the sum as an another symmetric tensor field of type $(0,2)$. Hence for we can assume that $\beta h(X,Y) - \frac{\alpha}{2}[h(\phi X,Y) + h(X,\phi Y)] = \mathcal{L}_\xi g(X,Y) + 2S(X,Y) + 2\mu\eta(X)\eta(Y)$.
Then we compute

$$\beta h(\xi,\xi)g(X,Y) = \mathcal{L}_\xi g(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y).$$

As h is parallel so, $h(\xi,\xi)$ is constant. Hence, we can write $h(\xi,\xi) = -\frac{2}{\beta}\lambda$ where β is constant and $\beta \neq 0$.
So, from the equation (3.7) we have

$$\beta h(X,Y) - \frac{\alpha}{2}[h(\phi X,Y) + h(X,\phi Y)] = -2\lambda g(X,Y), \tag{3.8}$$

for any $X, Y \in \chi(M)$. Therefore $\mathcal{L}_\xi g(X,Y) + 2S(X,Y) + 2\mu\eta(X)\eta(Y) = -2\lambda g(X,Y)$ and so (g,ξ,μ) becomes an η-Ricci soliton.

Corollary 3.2: Let $(M, g, \phi, \eta, \xi, \alpha, \beta)$ be a 3-dimensional trans-Sasakian manifold with α, β constants $(\beta \neq 0)$. If the symmetric $(0,2)$ tensor field h admitting the condition $\beta h(X,Y) - \frac{\alpha}{2}[h(\phi X,Y) + h(X,\phi Y)] = \mathcal{L}_\xi g(X,Y) + 2S(X,Y)$ is parallel with respect to the Levi-Civita connection associated to g with $\lambda = 2n$. Then (g,ξ) becomes a Ricci soliton.

Next theorem shows the necessary condition for the existence of η-Ricci soliton on 3-dimensional trans-Sasakian manifolds.
Theorem 3.3: If 3-dimensional trans-Sasakian manifold satisfies an η-Ricci soliton then the manifold becomes η-Einstein manifold with α and β constants.

Proof: From the equation (1.1) we get

$$2S(X,Y) = -g(\nabla_X\xi, Y) - g(X, \nabla_Y\xi) - 2\lambda g(X,Y) - 2\mu\eta(X)\eta(Y). \tag{3.9}$$

By using the equation (2.8) we get

$$S(X,Y) = -(\beta+\lambda)g(X,Y) + (\beta-\mu)\eta(X)\eta(Y) \tag{3.10}$$

and

$$S(X,\xi) = -(\lambda+\mu)\eta(X). \tag{3.11}$$

Also from (2.11) we have

$$\lambda + \mu = 2(\beta^2 - \alpha^2). \tag{3.12}$$

The Ricci operator Q is defined by $g(QX,Y) = S(X,Y)$. Then we get

$$QX = (\mu - \beta + 2(\alpha^2 - \beta^2))X + (\beta - \mu)\eta(X)\xi. \tag{3.13}$$

Then we can easily see that the manifold is an η-Einstein manifold.

We know a manifold is ϕ-Ricci symmetric if $\phi^2 \circ \nabla Q = 0$. Now we prove the next theorem.

Theorem 3.4: If a ϕ-Ricci symmetric trans-Sasakian manifold (M, g) satisfies an η-Ricci soliton then $\mu = \beta$, $\lambda = 2(\beta^2 - \alpha^2) - \beta$ and (M, g) is an Einstein manifold.

Proof: From the equation (3.13) we have

$$(\nabla_X Q)Y = \nabla_X QY - Q(\nabla_X Y)$$

$$= -\alpha(\beta - \mu)\eta(Y)\phi X + \beta(\beta - \mu)\eta(Y)X - (\beta - \mu)\eta(Y)\eta(X)\xi$$

$$+(\beta - \mu)[-\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y)]\xi.$$

Now applying ϕ^2 both sides we have $\mu = \beta$, $\lambda = 2(\beta^2 - \alpha^2) - \beta$ and (M, g) is an Einstein manifold.

We construct an example of η-Ricci soliton on 3-dimensional trans-Sasakian manifolds in the The next section.

4 Example of η-Ricci solitons on 3-dimensional trans-Sasakian manifolds

We consider the three dimensional manifold $M = \{(x, y, z) \in R^3 : y \neq 0\}$ where (x, y, z) are the standard coordinates in R^3. The vector fields

$$e_1 = e^{2z}\frac{\partial}{\partial x}, e_2 = e^{2z}\frac{\partial}{\partial y}, e_3 = \frac{\partial}{\partial z}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g_{ij} = \begin{cases} 1 & \text{for} \quad i = j, \\ 0 & \text{for} \quad i \neq j. \end{cases}$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M^3)$. Let ϕ be the $(1, 1)$ tensor field defined by $\phi(e_1) = e_2, \phi(e_2) = -e_1, \phi(e_3) = 0$. Then using the linearity property of ϕ and g we have

$$\eta(e_2) = 1, \ \phi^2(Z) = -Z + \eta(Z)e_2, \ g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M^3)$. Thus for $e_2 = \xi, (\phi, \xi, \eta, g)$ defines an almost contact metric structure on M. Now, after some calculation we have,

$$[e_1, e_3] = -2e_1, [e_2, e_3] = -2e_2, [e_1, e_2] = 0.$$

The Riemannian connection ∇ of the metric is given by the Koszul's formula which is

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).$$

By Koszul's formula we get,

$$\nabla_{e_1} e_1 = 2e_3, \nabla_{e_2} e_1 = 0, \nabla_{e_3} e_1 = 0, \nabla_{e_1} e_2 = 0, \nabla_{e_2} e_2 = 2e_3,$$

$$\nabla_{e_3} e_2 = 0, \nabla_{e_1} e_3 = -2e_1, \nabla_{e_2} e_3 = -2e_2, \nabla_{e_3} e_3 = 0.$$

From the above it can be easily shown that $M^3(\phi, \xi, \eta, g)$ is a trans-Sasakian manifold of type $(0, -2)$.
Here

$$R(e_1, e_2)e_2 = -4e_1, R(e_3, e_2)e_2 = 4e_2, R(e_1, e_3)e_3 = -4e_1, R(e_2, e_3)e_3 = -4e_2,$$

$$R(e_3, e_1)e_1 = -4e_2, R(e_2, e_1)e_1 = 4e_3.$$

So, we have

$$S(e_1, e_1) = 0, S(e_2, e_2) = 0, , S(e_3, e_3) = -8. \tag{4.1}$$

From the equation (1.1) we get $\lambda = -2$ and $\mu = 6$. Therefore, (g, ξ, λ, μ) is an η-Ricci soliton on $M^3(\phi, \xi, \eta, g)$.

In the next sections we consider η-Ricci Solitons on 3-dimensional trans-Sasakian manifolds satisfying some curvature conditions.

5 η-Ricci solitons on 3-dimensional trans-Sasakian manifolds satisfying $R(\xi, X) \cdot S = 0$

First we suppose that 3-dimensional trans-Sasakian manifolds with η-Ricci solitons satisfy the condition

$$R(\xi, X) \cdot S = 0.$$

Then we have

$$S(R(\xi, X)Y, Z) + S(Y, R(\xi, X)Z) = 0$$

for any $X, Y, Z \in \chi(M)$.

Using the equations (2.12), (3.10), (3.11) we get

$$(\beta-\mu)g(X,Y)\eta(Z)+(\beta-\mu)g(X,Z)\eta(Y)-2(\beta-\mu)\eta(X)\eta(Y)\eta(Z)=0.$$

Put $Z=\xi$ we have

$$(\beta-\mu)g(X,Y)-(\beta-\mu)\eta(X)\eta(Y)=0.$$

Setting $X=\phi X$ and $Y=\phi Y$ in the above equation we get

$$(\beta-\mu)g(\phi X,\phi Y)=0.$$

Again using the equation (3.12) we have

$$\mu=\beta, \quad \lambda=2(\beta^2-\alpha^2)-\beta.$$

Also we can easily see that M is an Einstein manifold. So we have the following theorem.

Theorem 5.1: If a 3-dimensional trans-Sasakian manifold $(M, g, \phi, \eta, \xi, \alpha, \beta)$ with α, β constants admitting an η-Ricci soliton satisfies the condition $R(\xi, X)\cdot S=0$ then $\mu=\beta$, $\lambda=2(\beta^2-\alpha^2)-\beta$ and M is an Einstein manifold.

Corollary 5.2: A 3-dimensional trans-Sasakian manifold with α, β constants satisfies the condition $R(\xi, X)\cdot S=0$, there is no Ricci soliton with the potential vector field ξ.

6 η-Ricci solitons on 3-dimensional trans-Sasakian manifolds satisfying $S(\xi, X)\cdot R=0$

We consider 3-dimensional trans-Sasakian manifolds with η-Ricci solitons satisfying the condition

$$S(\xi, X)\cdot R=0.$$

So we have

$$S(X,R(Y,Z)W)\xi - S(\xi,R(Y,Z)W)X + S(X,Y)R(\xi,Z)W - S(\xi,Y)R(X,Z)W$$
$$+S(X,Z)R(Y,\xi)W - S(\xi,Z)R(Y,X)W + S(X,W)R(Y,Z)\xi - S(\xi,W)R(Y,Z)X = 0.$$

Taking inner product with ξ then the above equation becomes

$$\begin{aligned} & S(X,R(Y,Z)W) - S(\xi,R(Y,Z)W)\eta(X) + S(X,Y)\eta(R(\xi,Z)W) \\ & -S(\xi,Y)\eta(R(X,Z)W) + S(X,Z)\eta(R(Y,\xi)W) - S(\xi,Z)\eta(R(Y,X)W) \\ & + S(X,W)\eta(R(Y,Z)\xi) - S(\xi,W)\eta(R(Y,Z)X) = 0. \end{aligned} \tag{6.1}$$

Put $W = \xi$ and using the equations (2.10), (2.12), (3.10), (3.11) we get

$$-(\beta+\lambda)g(X,R(Y,Z)\xi) + (\lambda+\mu)\eta(R(Y,Z)X) = 0. \tag{6.2}$$

Also we have

$$\eta(R(Y,Z)X) = -g(X,R(Y,Z)\xi).$$

So from the equation (6.2) we get

$$(\beta+2\lambda+\mu)g(X,R(Y,Z)\xi) = 0.$$

Again using the equation (3.12) we have

$$\mu = \beta + 4(\beta^2-\alpha^2), \quad \lambda = -[2(\beta^2-\alpha^2)+\beta].$$

So we have the following theorem.

Theorem 6.1: If a 3-dimensional trans-Sasakian manifold $(M,g,\phi,\eta,\xi,\alpha,\beta)$ with $\alpha,\ \beta$ constants admitting an η-Ricci soliton satisfies the condition $S(\xi,X)\cdot R = 0$ then $\mu = \beta + 4(\beta^2-\alpha^2), \quad \lambda = -[2(\beta^2-\alpha^2)+\beta]$.

Corollary 6.2: A 3-dimensional trans-Sasakian manifold with α, β constants satisfies the condition $S(\xi, X) \cdot R = 0$, there is no Ricci soliton with the potential vector field ξ.

7 η-Ricci solitons on 3-dimensional trans-Sasakian manifolds satisfying $W_2(\xi, X) \cdot S = 0$

Definition 7.1. Let M be 3-dimensional trans-Sasakian manifold with respect to semi-Symmetric metric connection. The W_2-curvature tensor of M is defined by [16]

$$W_2(X,Y)Z = R(X,Y)Z + \frac{1}{2}(g(X,Z)QY - g(Y,Z)QX). \tag{7.1}$$

We assume 3-dimensional trans-Sasakian manifolds with η-Ricci solitons satisfying the condition

$$W_2(\xi, X) \cdot S = 0.$$

Then we have

$$S(W_2(\xi, X)Y, Z) + S(Y, W_2(\xi, X)Z) = 0$$

for any $X, Y, Z \in \chi(M)$.
Using the equations (2.12), (3.10), (3.11), (7.1) we get

$$\begin{aligned}
&[-\frac{(\beta+\lambda)}{2}(\lambda+\mu) + \frac{(\beta+\lambda)^2}{2} + (\beta-\mu)(\alpha^2-\beta^2) + (\lambda+\mu)\frac{(\beta-\mu)}{2}]g(X,Y)\eta(Z) \\
&+[\frac{(\beta+\lambda)^2}{2} - \frac{(\beta+\lambda)}{2}(\lambda+\mu) + (\beta-\mu)(\alpha^2-\beta^2) + (\lambda+\mu)\frac{(\beta-\mu)}{2}]g(X,Z)\eta(Y) \\
&+[-(\beta+\lambda)(\beta-\mu) - 2(\beta-\mu)(\alpha^2-\beta^2) - (\beta-\mu)(\lambda+\mu)]\eta(X)\eta(Y)\eta(Z) = 0.
\end{aligned}$$

Put $Z = \xi$ in the above equation we get

$$\begin{aligned}
&[-\frac{(\beta+\lambda)}{2}(\lambda+\mu) + \frac{(\beta+\lambda)^2}{2} + (\beta-\mu)(\alpha^2-\beta^2) + (\lambda+\mu)\frac{(\beta-\mu)}{2}]g(X,Y) \\
&+[\frac{(\beta+\lambda)^2}{2} - \frac{(\beta+\lambda)}{2}(\lambda+\mu) + (\beta-\mu)(\alpha^2-\beta^2) + (\lambda+\mu)\frac{(\beta-\mu)}{2}
\end{aligned}$$

$$-(\beta+\lambda)(\beta-\mu)-2(\beta-\mu)(\alpha^2-\beta^2)-(\beta-\mu)(\lambda+\mu)]\eta(X)\eta(Y)=0.$$

Setting $X=\phi X$ and $Y=\phi Y$ in the above equation we get

$$(\beta-\mu)(\frac{(\beta+2\lambda+\mu+2(\alpha^2-\beta^2))}{2})g(\phi X,\phi Y)=0.$$

Again using the equation (3.12) we have

$$\mu=\beta,\quad \lambda=2(\beta^2-\alpha^2)-\beta$$

or

$$\mu=2(\beta^2-\alpha^2)+\beta,\quad \lambda=-\beta.$$

So we have the following theorem.

Theorem 7.1: If a 3-dimensional trans-Sasakian manifold $(M,g,\phi,\eta,\xi,\alpha,\beta)$ with $\alpha,\ \beta$ constants admitting an η-Ricci soliton satisfies the condition $W_2(\xi,X)\cdot S=0$ then $\mu=\beta,\ \ \lambda=2(\beta^2-\alpha^2)-\beta$ or $\mu=2(\beta^2-\alpha^2)+\beta,\ \ \lambda=-\beta$.

Corollary 7.2: A 3-dimensional trans-Sasakian manifold with $\alpha,\ \beta$ constants satisfies the condition $W_2(\xi,X)\cdot S=0$, there is no Ricci soliton with the potential vector field ξ.

8 η-Ricci solitons on 3-dimensional trans-Sasakian manifolds satisfying $S(\xi,X)\cdot W_2=0$

Suppose that 3-dimensional trans-Sasakian manifolds with η-Ricci solitons satisfy the condition

$$S(\xi,X)\cdot W_2=0.$$

So we have

$$S(X,W_2(Y,Z)V)\xi-S(\xi,W_2(Y,Z)V)X+S(X,Y)W_2(\xi,Z)V-S(\xi,Y)W_2(X,Z)V$$

$$+S(X,Z)W_2(Y,\xi)V-S(\xi,Z)W_2(Y,X)V+S(X,V)W_2(Y,Z)\xi-S(\xi,V)W_2(Y,Z)X=0.$$

Taking inner product with ξ then the above equation becomes

$$\begin{aligned} &S(X, W_2(Y,Z)V) - S(\xi, W_2(Y,Z)V)\eta(X) + S(X,Y)\eta(W_2(\xi,Z)V) \\ &-S(\xi,Y)\eta(W_2(X,Z)V) + S(X,Z)\eta(W_2(Y,\xi)V) - S(\xi,Z)\eta(W_2(Y,X)V) \\ &+ S(X,V)\eta(W_2(Y,Z)\xi) - S(\xi,V)\eta(W_2(Y,Z)X) = 0. \end{aligned} \tag{8.1}$$

Put $V = \xi$ and using the equations (2.10), (2.12), (3.10), (3.11), (7.1) we get

$$-(\beta+\lambda)g(X, W_2(Y,Z)\xi) + (\lambda+\mu)\eta(W_2(Y,Z)X) = 0. \tag{8.2}$$

Using the equations (3.10), (3.11), (7.1) then the equation (8.2) becomes

$$[(\beta+\lambda)^2 + (\lambda+\mu)^2 + 2(\alpha^2-\beta^2)(\beta+2\lambda+\mu)]g(X, R(Y,Z)\xi) = 0.$$

Using the equation (3.12) we have

$$\mu = \beta, \quad \lambda = 2(\beta^2-\alpha^2) - \beta$$

or

$$\mu = 2(\beta^2-\alpha^2) + \beta, \quad \lambda = -\beta.$$

So we have the following theorem.

Theorem 8.1: If Let a 3-dimensional trans-Sasakian manifold $(M, g, \phi, \eta, \xi, \alpha, \beta)$ with α, β constants admitting an η-Ricci soliton satisfies the condition $S(\xi, X) \cdot W_2 = 0$ then $\mu = \beta$, $\lambda = 2(\beta^2-\alpha^2) - \beta$ or $\mu = 2(\beta^2-\alpha^2) + \beta$, $\lambda = -\beta$.

Corollary 8.2: A 3-dimensional trans-Sasakian manifold with α, β constants satisfies the condition $S(\xi, X) \cdot W_2 = 0$, there is no Ricci soliton with the potential vector field ξ.

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