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Approximate solution of Abel integral equation in Daubechies wavelet basis

Jyotirmoy Mouley[1]
M. M. Panja[2]
B. N. Mandal[3]

[1] *Department of Applied Mathematics, University of Calcutta, 92, A.P.C. Road, Kolkata-700 009, India.*
jyoti87.cu.wavelet@gmail.com

[2] *Department of Mathematics, Visva-Bharati, Santiniketan, West Bengal, 731235, India*
madanpanja2005@yahoo.co.in

[3] *Physics and Applied Mathematics Unit, Indian Statistical Institute, 203, B.T. Road, Kolkata-700108, India.*
bnm2006@rediffmail.com

ABSTRACT

This paper presents a new computational method for solving Abel integral equation (both first kind and second kind). The numerical scheme is based on approximations in Daubechies wavelet basis. The properties of Daubechies scale functions are employed to reduce an integral equation to the solution of a system of algebraic equations. The error analysis associated with the method is given. The method is illustrated with some examples and the present method works nicely for low resolution.

RESUMEN

Este artículo presenta un nuevo método computacional para resolver la ecuación integral de Abel (tanto de primer como de segundo tipo). El esquema numérico está basado en aproximaciones en la base de ondeletas de Daubechies. Se emplean las propiedades de las funciones de escala de Daubechies para reducir una ecuación integral a la solución de un sistema algebraico de ecuaciones. Se entrega el análisis de error asociado con el método. El método es ilustrado con algunos ejemplos donde el método presentado funciona bien en baja resolución.

Keywords and Phrases: Abel integral equation, Daubechies scale function, Daubechies wavelet, Gauss-Daubechies quadrature rule.

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1 Introduction

The theory of integral equations is a very important tool to deal with problems arising in mathematical physics. Abel integral equation appears in many physical problems of water waves, astrophysics, solid mechanics and in many applied sciences (see [1, 2, 3, 4]). In the year 1823, Abel integral equation was derived directly from the tautochorone problem in physics. In fact this gave birth to the topic known as integral equation.

Before 1930, the branch of mathematics which is related to wavelet began with Joseph Fourier with his theories of frequency analysis, now often referred to Fourier synthesis (see [5]). The concept of wavelet was first mentioned in an appendix of the thesis of A. Haar (see [6]), but the formulation of problems involving wavelets has been developed mostly over last 30 years. Grossman and Morelet [7] developed the continuous wavelet transform and the orthogonal one was developed by Lamarie and Meyer [8]. Daubechies (see [9, 10]) constructed a compactly supported orthogonal wavelet basis that can be generated from a single function with the aim to serve the multiresolution analysis (MRA of $L^2(\mathbb{R})$). Wavelets allow to represent variety of functions and operators very accurately. Furthermore, wavelets setup a connections with fast numerical algorithms [11]. Hence wavelets are used as an efficient tool to solve integral equations.

In this paper we consider the Abel integral equations in the form

$$\text{First kind :} \qquad \int_0^x \frac{y(t)\,\mathrm{d}t}{(x-t)^{\mu}} = f(x), \tag{1.1}$$

$$\text{Second kind :} \qquad y(x) + \lambda \int_0^x \frac{y(t)\,\mathrm{d}t}{(x-t)^{\mu}} = f(x). \tag{1.2}$$

Here $0<\mu<1$, $0 \le x \le 1$ and the forcing term $f(x) \in C[0,1]$ in order to confirm the existence and uniqueness of the solution $y(x) \in C[0,1]$, the space of all continuous function defined on $[0,1]$.

The Abel integral equation has been solved earlier analytically and numerically by various methods in the literature. For instance, Yousefi [12] constructed a numerical scheme based on Legendre multiwavelets to solve Abel integral equation. A system of generalized Abel integral equations was solved using Fractional calculus by Mandal et al [13]. Liu and Tao [14] applied mechanical quadrature methods for solving first kind Abel integral equation. Numerical solution of Abel integral equation is obtained using orthogonal functions by Derili and Sohrabi [15]. Alipour and Rostamy [16] used Bernstein polynomials to solve Abel integral equations. Shahsavaram [17] used Haar wavelet as the basis function in the collocation method to solve Volterra integral equation with weakly singular kernel.

In this paper, the unknown function in the integral equation is expanded by employing Daubechies wavelet basis with unknown coefficients. The integral equation is converted into a system algebraic equations utilizing the properties of Daubechies scale functions. After evaluating the unknown coefficients, the values of the unknown function in the integral equations can be determined at any dyadic point in $[0,1]$.

2 Preliminary concept of Daubechies scale function

Here some important properties of Daubechies scale function with a compact support are presented in a finite interval $[a,b] \subset \mathbb{R}$, where a and $b(> a)$ are integers.

2.1 Two-scale relations

Daubechies constructed a whole new class of orthogonal wavelets that can be generated from a single function $\phi(x)$, known as Daubechies scale function. This scale function has some interesting features like compact support, fractal nature, and unknown structure at all resolutions. Daubechies -K (Dau-K) scale function($K \in \mathbb{N}$) has $2K$ filter coefficients and compact support $[0, 2K-1]$. The two-scale relation of scale function is given by

$$\phi(\cdot) = \sqrt{2} H^T \Phi(\cdot), \tag{2.1}$$

where

$$H = [h_0, h_1, h_2, ..., h_{2K-1}]^T_{2K\times 1} \tag{2.2}$$

and

$$\Phi(\cdot) = [\phi(2\cdot), \phi(2\cdot -1), \phi(2\cdot -2), ..., \phi(2\cdot -2K+1)]^T_{2K\times 1} \tag{2.3}$$

with the normalization condition

$$\int_{\mathbb{R}} \phi(x)\mathrm{d}x = 1. \tag{2.4}$$

The elements h_l $(l = 0, 1, 2, ..., 2K-1)$ are known as filter coefficients or low pass filters. These filter coefficients satisfy the following algebraic relations

$$\sum_{l=0}^{2K-1} h_l = \sqrt{2} \quad ; \quad \sum_{l=0}^{2K-1} h_l h_{l-2m} = \delta_{m0}. \tag{2.5}$$

Here we define two operators, one is the translation operator T and other is the scale transformation operator D as

$$T^k \phi(x) = \phi_k(x) = \phi(x-k) \tag{2.6}$$

and

$$D^j \phi(x) = 2^{\frac{j}{2}} \phi(2^j x). \tag{2.7}$$

For a specific value of resolution j, the translate of scaling functions are orthonormal to each other viz.

$$\int_{\mathbb{R}} \phi_{jk_1}(x)\phi_{jk_2}(x)\mathrm{d}x = \delta_{k_1k_2} \tag{2.8}$$

where

$$\phi_{jk}(x) = 2^{\frac{j}{2}}\phi(2^j x - k). \tag{2.9}$$

It is evident that all the properties of scaling functions are applicable on $\mathbb{R}$. But in the finite interval $[a,b]$ the translation property (2.6) does not hold good for all $k \in \mathbb{Z}$ as well as the orthogonalization condition (2.8) cannot be applied for $\phi_{jk}(x)$. So in order to apply the machinery of Dau-K scale function on a finite interval $[a,b]$, we divide the translate of $\phi(x)$ for a specific resolution j into three classes (cf. Mouley et al. [18] and Panja et al. [19])

$$\begin{array}{rcll} \phi^L_{jk}(\cdot) & = & \phi_{jk}(\cdot)\chi_k(x) & \text{if } k \in \left\{a2^j - 2K + 2, ..., a2^j - 1\right\}, \\ \phi^I_{jk}(\cdot) & = & \phi_{jk}(\cdot)\chi_k(x) & \text{if } k \in \left\{a2^j, ..., b2^j - 2K + 1\right\}, \\ \phi^R_{jk}(\cdot) & = & \phi_{jk}(\cdot)\chi_k(x) & \text{if } k \in \left\{b2^j - 2K + 2, ..., b2^j - 1\right\}. \end{array} \tag{2.10}$$

Here $\chi_k(x)$ is the characteristic function assuming the value 1 or 0 according as $x \in [a,b]$ or $x \notin [a,b]$.

2.2 Scale function at dyadic points

A number of the form $\frac{m}{2^n}$ is known as a dyadic fraction or dyadic rational (m is an integer and n is a natural number). It has extensive application in measurement, the inch is normally subdivided in dyadic rather than decimal fraction. The ancient Egyptians also used dyadic fractions in measurement, with denominators up to 64 [20]. After knowing the value of scale function at integer points within support, it is possible to determine the scale function at any dyadic point with in the support [21] . Using the two-scale relation (2.1) the value of Dau-K scale function $\phi(x)$ at $x = \frac{m}{2^n}$ is calculated as

$$\phi\left(\frac{m}{2^n}\right) = \sum_{l_1=0}^{2K-1} \sqrt{2} h_{l_1} \phi\left(\frac{m - 2^{n-1}l_1}{2^{n-1}}\right). \tag{2.11}$$

Again using the two-scale relation (2.1) we get

$$\phi\left(\frac{m}{2^n}\right) = \sum_{l_1=0}^{2K-1}\sum_{l_2=0}^{2K-1} 2h_{l_1}h_{l_2}\phi\left(\frac{m - 2^{n-1}l_1 - 2^{n-2}l_2}{2^{n-2}}\right). \tag{2.12}$$

Repeating the two-scale relation (2.1) n times, we get

$$\phi\left(\frac{m}{2^n}\right) = \sum_{l_1=0}^{2K-1}\sum_{l_2=0}^{2K-1}...\sum_{l_n=0}^{2K-1} 2^{\frac{m}{2}} h_{l_1}h_{l_2}...h_{l_n}\phi(m - 2^{n-1}l_1 - 2^{n-2}l_2...2l_{n-1} - l_n). \tag{2.13}$$

3 Multiresolution analysis (MRA) and Daubechies wavelet

Basic concepts of MRA and Daubechies wavelet are discussed in most of the texts on wavelets (see [9, 10, 18, 19, 21]). Why wavelet has started to dominate in different applications such as technology, engineering and applied mathematics, one serious reason behind it is MRA. A MRA on $\mathbb{R}$ is defined as a sequence of nested subspaces V_j of function L^2 on $\mathbb{R}$ with scaling function $\phi(x)$ if the following properties hold,

$$\forall j \in \mathbb{Z}, \quad V_j \subseteq V_{j+1}, \tag{3.1}$$

$$Clos_{L^2}\left(\cup_{j\in\mathbb{Z}} V_j\right) = L^2\left(\mathbb{R}\right), \tag{3.2}$$

$$\cap_{j\in\mathbb{Z}}\, V_j = \{0\}\,, \tag{3.3}$$

$$\phi(x) \in V_j \Leftrightarrow \phi(2x) \in V_{j+1}, \quad \forall j \in \mathbb{Z}. \tag{3.4}$$

Here V_j's are called approximation spaces. The scale function $\phi(x)$ belongs to V_0 and the set $\{\phi(x-k) : k \in \mathbb{Z}\}$ is a Riesz basis of V_0. The scale function $\phi(x)$ satisfies the two-scale relation (2.1). Also the set $\{\phi_{jk}(x) : k \in \mathbb{Z}\}$ given by (2.9) is a Riesz basis of V_j. From the property (3.1), it is evident that each element of V_{j+1} can be uniquely written as the orthogonal sum of an element in V_j and an element in W_j that contains the complementary details i.e.

$$V_{j+1} = V_j \oplus W_j = V_0 \oplus W_0 \oplus W_1 \oplus W_2 \oplus ... \oplus W_j. \tag{3.5}$$

Let W_j be the span of $\psi_{jk}(x) = 2^{\frac{j}{2}}\psi(2^j x - k)$, which is called wavelet function. The wavelet function $\psi(x)$ satisfies the relation

$$\psi(\cdot) = \sqrt{2} G^T \Phi(\cdot) \tag{3.6}$$

where

$$\mathbf{G} = [g_0, g_1, g_2, ..., g_{2K-1}]^T_{2K\times 1}. \tag{3.7}$$

Here $\Phi(\cdot)$ is given by (2.3) and g_l $(l = 0, 1, 2, ..., 2K-1)$ are known as high pass filter coefficients and are given by

$$g_l = (-1)^l h_{2K-1-l}. \tag{3.8}$$

4 Method of approximation

We approximate the unknown function of the integral equations (1.1) and (1.2) in the approximation space V_j as

$$\begin{aligned} y(x) &\approx y_j^{MS}(x) \\ &= \sum_{k=0}^{2^j-1} c_{jk}\phi_{jk}(x) \\ &= \sum_{k=0}^{2^j-2K+1} c^I_{jk}\phi^I_{jk}(x) + \sum_{k=2^j-2K+2}^{2^j-1} c^R_{jk}\phi^R_{jk}(x) \\ &= \mathbf{C}^T\vec{\Phi}(x). \end{aligned} \tag{4.1}$$

As the support of $\phi(x)$ is $[0, 2K-1]$, so $y_j^{MS}(x)$ always vanishes at $x = 0$. The value of $y(x)$ at $x = 0$ for second kind Abel integral equation is obviously $f(0)$ but for the first kind Abel integral equation $y(x)$ cannot be evaluated at $x = 0$ but as $y(x)$ can be evaluated at any dyadic point in $(0, 1]$, it can be evaluated very close to $x = 0$ by making the resolution fairly large. Here $\mathbf{C}$ and $\vec{\Phi}(x)$ both are $2^j \times 1$ vectors, given by

$$\mathbf{C} = \left[c_{j0}^I, c_{j1}^I, ..., c_{j2^j-2K+1}^I, c_{j2^j-2K+2}^R, ..., c_{j2^j-1}^R\right]^T \tag{4.2}$$

and

$$\vec{\Phi}(x) = \left[\phi_{j0}^I(x), \phi_{j1}^I(x), ..., \phi_{j2^j-2K+1}^I(x), \phi_{j2^j-2K+2}^R(x), ..., \phi_{j2^j-1}^R(x)\right]^T. \tag{4.3}$$

Using the approximate form of $y(x)$ in (4.1) in both the first and second kind integral equations (1.1) and (1.2) we get,

$$\mathbf{C}^T \int_0^x \frac{\vec{\Phi}(t)\mathrm{d}t}{(x-t)^\mu} = f\,(x) \tag{4.4}$$

and

$$\mathbf{C}^T \left[\vec{\Phi}(x) + \lambda \int_0^x \frac{\vec{\Phi}(t)\mathrm{d}t}{(x-t)^\mu}\right] = f\,(x)\,. \tag{4.5}$$

We choose total 2^j number of points by $x_{jk'} = \frac{k'}{2^j}$ $(k' = 1, 2, 3, ..., 2^j)$ and substituting these points in both the equations (4.4) and (4.5) we get,

$$\mathbf{C}^T \mathbf{B}^{(k')} = f\left(\frac{k'}{2^j}\right) \tag{4.6}$$

and

$$\mathbf{C}^T \left[\mathbf{A}^{(k')} + \lambda \mathbf{B}^{(k')}\right] = f\left(\frac{k'}{2^j}\right) \tag{4.7}$$

where

$$\begin{aligned}\mathbf{A}^{(k')} &= \vec{\Phi}\left(\frac{k'}{2^j}\right) \\ &= \left[\phi_{j0}^I\left(\frac{k'}{2^j}\right), \phi_{j1}^I\left(\frac{k'}{2^j}\right), ..., \phi_{j2^j-2K+1}^I\left(\frac{k'}{2^j}\right), \phi_{j2^j-2K+2}^R\left(\frac{k'}{2^j}\right), ..., \phi_{j2^j-1}^R\left(\frac{k'}{2^j}\right)\right]^T\end{aligned} \tag{4.8}$$

and

$$\mathbf{B}^{(k')} = \left[\int_0^{\frac{k'}{2^j}} \frac{\phi_{j0}^I\,(t)\,\mathrm{d}t}{(\frac{k'}{2^j}-t)^\mu}, ..., \int_0^{\frac{k'}{2^j}} \frac{\phi_{j2^j-2K+1}^I\,(t)\,\mathrm{d}t}{(\frac{k'}{2^j}-t)^\mu}, \int_0^{\frac{k'}{2^j}} \frac{\phi_{j2^j-2K+2}^R\,(t)\,\mathrm{d}t}{(\frac{k'}{2^j}-t)^\mu}, ..., \int_0^{\frac{k'}{2^j}} \frac{\phi_{j2^j-1}^R\,(t)\,\mathrm{d}t}{(\frac{k'}{2^j}-t)^\mu}\right]^T. \tag{4.9}$$

As $k = 0, 1, 2, ..., 2^j - 1$ and $k' = 1, 2, 3, ..., 2^j$, each of the equation (4.6) and (4.7) represents a system of 2^j equations in 2^j variables c_{jk}^I and c_{jk}^R. Solving these systems the unknown coefficients c_{jk}^I and c_{jk}^R are obtained.

In the last part of this section, we explain the procedure for calculating the matrix elements of the matrix $\mathbf{B}^{(k^{'})}$. We use the notation

$$\mathbf{I}_{\mu\ j}(k^{'},k)=\int_0^{\frac{k^{'}}{2^j}}\frac{\phi_{jk}\,(t)\,\mathrm{d}t}{(\frac{k^{'}}{2^j}-t)^{\mu}}. \tag{4.10}$$

In the relation (4.10), for $0\leq k\leq 2^j-2K+1$, $\phi_{jk}\,(t)$ means $\phi^I_{jk}\,(t)$ and for $2^j-2K+2\leq k\leq 2^j-1$, $\phi_{jk}\,(t)$ means $\phi^R_{jk}\,(t)$. Using (2.9) we find

$$\mathbf{I}_{\mu\ j}(k^{'},k)=2^{\left(\mu-\dfrac{1}{2}\right)j}\mathcal{L}_{\mu}(k^{'}-k), \tag{4.11}$$

where

$$\mathcal{L}_{\mu}(k)=\int_0^k\frac{\phi\,(t)\,\mathrm{d}t}{(k-t)^{\mu}}. \tag{4.12}$$

As the support of Dau-K scale function $\phi(t)$ is $[0,2K-1]$, so if $k\leq 0$ the range of the integration in (4.12) is completely outside of the support. In this case $\mathcal{L}_{\mu}(k)$ vanishes. Again if $k\geq 2K, \mathcal{L}_{\mu}(k)$ has no singularity within the support $[0,2K-1]$. Using Gauss-Daubechies quadrature rule involving Daubechies scale function [22], $\mathcal{L}_{\mu}(k)$ is evaluated as

$$\mathcal{L}_{\mu}(k)=\sum_{i=1}^{M}\frac{w_i}{(k-t_i)^{\mu}},\quad (k\geq 2K). \tag{4.13}$$

Here w_i , t_i are weights are nodes of Gauss-Daubechies quadrature rule involving Daubechies scale function [22].

For $0<k\leq 2K-1,\ \mathcal{L}_{\mu}(k)$ has integrable singularity at the upper limit so that evaluation of such integrals by using the quadrature rule may not provide their approximate value with desired order of accuracy within less computational time. The two-scale relation (2.1) for $\phi(t)$, may be used to obtain a recurrence relation for $\mathcal{L}_{\mu}(k)$ as

$$\mathcal{L}_{\mu}(k)=2^{\mu-\frac{1}{2}}\sum_{l=0}^{2K-1}h_l\mathcal{L}_{\mu}(2k-l). \tag{4.14}$$

Using the symbols

$$\mathcal{H}_K=\begin{pmatrix} h_1 & h_0 & 0 & 0\cdots & 0 & 0\\ h_3 & h_2 & h_1 & h_0\cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots\ldots & \vdots & \vdots\\ 0 & 0 & 0 & 0\cdots & h_{2K-2} & h_{2K-3}\\ 0 & 0 & 0 & 0\cdots & 0 & h_{2K-1}\end{pmatrix} \tag{4.15}$$

and

$$\mathbf{b}_{\mu\ K}=\begin{pmatrix} 0\\ 0\\ \vdots\\ \sum_{l=0}^{2K-4}h_l\mathcal{L}_{\mu}(4K-4-l)\\ \sum_{l=0}^{2K-2}h_l\mathcal{L}_{\mu}(4K-2-l)\end{pmatrix} \tag{4.16}$$

the relation (4.14) can be put in the form

$$\left(I - 2^{\mu-\frac{1}{2}}\mathcal{H}_K\right)\mathcal{L}_\mu = \mathbf{b}_{\mu\ K}. \tag{4.17}$$

So, the singular integrals in $\mathcal{L}_\mu$ are found as

$$\mathcal{L}_\mu = \left(I - 2^{\mu-\frac{1}{2}}\mathcal{H}_K\right)^{-1}\mathbf{b}_{\mu\ K}. \tag{4.18}$$

Thus, evaluation of $\mathcal{L}_\mu(k)$ is summarized as

$$\mathcal{L}(k) = \begin{cases} 0 & k \leq 0, \\ \text{solution obtained by (4.18)} & 1 \leq k \leq 2K-1, \\ \sum_{i=0}^{M} \dfrac{w_i}{(k-t_i)^\mu} & k \geq 2K. \end{cases} \tag{4.19}$$

Table 1: Values of $\mathcal{L}(k)$

k	$\mu = \frac{1}{4}$	$\mu = \frac{1}{3}$	$\mu = \frac{1}{2}$
1	0.925995	1.098666	1.643812
2	1.064183	1.042183	0.954199
3	0.808341	0.759600	0.682604
4	0.748236	0.679445	0.560703
5	0.699178	0.620553	0.488824

In Table 1 the values of $\mathcal{L}(k)$ for $k = 1, 2, ..., 5$ are given taking Dau-3 scale function for $\mu = \frac{1}{4}, \frac{1}{3}, \frac{1}{2}$. For other values of μ $(0 < \mu < 1)$ these can be easily calculated.

Table 2: Accuracy of $\mathcal{L}(2K)$ for Dau-3 scale function

μ	Detemined by (4.13)	Detemined by (4.18)
1/4	0.662722	0.662722
1/3	0.577792	0.577792
1/2	0.439182	0.439182

In Table 2 the values of $\mathcal{L}(2K)$ for Dau-3 scale function are presented for $\mu = \frac{1}{4}, \frac{1}{3}, \frac{1}{2}$ using the relations (4.13) and (4.18) separately. For the two methods the values of $\mathcal{L}(2K)$ are found to be same. The values of $\mathcal{L}(2K)$ establish the efficiency of the relation (4.18) in the determination of $\mathcal{L}(k)$ $(k = 0, 1, 2, ..., 2K-1)$.

5 Error estimation

In this section, the error of the proposed method is estimated in detail. For this we need the following definitions and theorems.

Definition 5.1 ([23]). *In a σ-finite measure space $(X, \mathcal{F}, \mu^*)$ (X denotes underlying space, $\mathcal{F}$ is the σ-algebra of measurable sets and μ^* is the measure) the L^p-norm $(1 \leq p < \infty)$ of a function f is defined by*

$$\|f\|_{L^p(X,\ \mathcal{F},\ \mu^*)} = \left(\int_X |f(x)|^p \, d\mu^*(x)\right)^{\frac{1}{p}}.$$

The abbreviations $\|f\|_{L^p(X)}$, $\|f\|_{L^p}$, $\|f\|_p$ are also used to mean L^p- norm.

Definition 5.2 ([24]). *The inner product of two functions f and g on a measure space X is defined by*

$$< f, g >= \int_X f\bar{g}\,d\mu.$$

Theorem 5.3 (Minkowski [23]). *If $1 \leq p < \infty$ and $f, g \in L^p$ then $f + g \in L^p$ and $\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$.*

Theorem 5.4. *Let $\{\phi_{jk}(x) : k \in \mathbb{Z}\}$ and $\{\psi_{jk}(x) : k \in \mathbb{Z}\}$ be the Riesz bases of approximation space V_j and detail space W_j. If $N^B_{j:k,k'} = \int_a^b \phi^B_{jk}(x)\phi^B_{jk'}(x)dx$ and $T^B_{j:k,k'} = \int_a^b \psi^B_{jk}(x)\psi^B_{jk'}(x)dx$ (B stands for L or R) then*

$$T^B_{j:k,k'} = \sum_{l_1=0}^{2K-1}\sum_{l_2=0}^{2K-1} g_{l_1}g_{l_2}N^B_{j+1:2k+l_1,2k'+l_2}.$$

Proof. Here

$$N^B_{j:k,k'} = \int_a^b \phi^B_{jk}(x)\phi^B_{jk'}(x)\mathrm{d}x.$$

Now

$$\begin{aligned}
T^B_{j:k,k'} &= \int_a^b \psi^B_{jk}(x)\psi^B_{jk'}(x)\mathrm{d}x \\
&= 2^j \int_a^b \psi^B(2^j x - k)\psi^B(2^j x - k')\mathrm{d}x \quad \text{(using expression of } \psi_{j,k}(x)) \\
&= \int_{a2^j}^{b2^j} \psi^B(z - k)\psi^B(z - k')\mathrm{d}z \\
&= \sum_{l_1=0}^{2K-1}\sum_{l_2=0}^{2K-1} g_{l_1}g_{l_2}\int_{a2^{j+1}}^{b2^{j+1}} \phi^B(z - 2k - l_1)\phi^B(z - 2k' - l_2)\mathrm{d}z \qquad \text{(using equation (3.6))} \\
&= \sum_{l_1=0}^{2K-1}\sum_{l_2=0}^{2K-1} g_{l_1}g_{l_2}N^B_{j+1:2k+l_1,2k'+l_2}.
\end{aligned}$$

This completes the proof. ∎

So to evaluate $T^B_{j:k,k'}$, we need to evaluate $N^B_{j+1:2k+l_1,2k'+l_2}$ $(l_1, l_2 = 0, 1, 2, ..., 2K-1)$. The values of $N^B_{j:k,k'}$ are tabulated in Table 3 and Table 4 in [25].

In section 3 to find the approximate solution, the projection of the unknown function $y^{MS}_j(x)$ is used in the approximation space (the linear span of $\phi_{jk}(x), k = 0, 1, 2, ...2^j - 1$). To estimate the error of the unknown function $y(x) \in L^2([0, 1])$ satisfying both the integral equations (1.1) and (1.2), we employ the fact that the multiscale expansion of $y(x)$ (the projection of $y(x)$ into the approximation space V_j and detail space W_j) is

$$y(x) = \sum_{k=0}^{2^j-1} c_{jk}\phi_{jk}(x) + \sum_{j'=j}^{\infty}\sum_{k=0}^{2^{j'}-1} d_{j'k}\psi_{j'k}(x) \tag{5.1}$$

where

$$c_{jk} \approx \int_0^1 \phi_{jk}(x)y(x)\mathrm{d}x, \tag{5.2}$$

and

$$d_{jk} \approx \int_0^1 \psi_{jk}(x)y(x)\mathrm{d}x. \tag{5.3}$$

Using the two-scale relation (2.1) and the equation (3.6), (5.2) and (5.3) are reduced to

$$c_{jk} = \sum_{l=0}^{2K-1} h_l c_{j+1,2k+l}\ , \tag{5.4}$$

$$d_{jk} = \sum_{l=0}^{2K-1} g_l c_{j+1,2k+l}. \tag{5.5}$$

To evaluate c_{jk} and $d_{jk}, (k = 0, 1, 2, ..., 2^j - 1)$ at level j, we need the values of $c_{j+1,2k+l}$ and $d_{j+1,2k+l}$ at level $j+1$. If $0 \le k \le 2^j - 2K + 1$, c_{jk} and d_{jk} are denoted by c^I_{jk} and d^I_{jk} respectively. Again if $2^j - 2K + 2 \le k \le 2^j - 1$, c_{jk} and d_{jk} are denoted by c^R_{jk} and d^R_{jk} respectively.

Now using the expression for $y^{MS}_j(x)$ given by (4.1), (5.1) is reduced to

$$y(x) = y^{MS}_j(x) + \sum_{j'=j}^{\infty} \delta y_{j'} \tag{5.6}$$

where $\delta y_{j'}$ is given by

$$\begin{aligned} \delta y_{j'} &= \sum_{k=0}^{2^{j'}-1} d_{j'k}\psi_{j'k}(x) \\ &= \sum_{k=0}^{2^{j'}-2K+1} d^I_{j'k}\psi^I_{j'k}(x) + \sum_{k=2^{j'}-2K+2}^{2^{j'}-1} d^R_{j'k}\psi^R_{j'k}(x). \end{aligned} \tag{5.7}$$

The error in the multiscale approximation is given by

$$\begin{aligned} e(x) &= y(x) - y^{MS}_j(x) \\ &= \sum_{j'=j}^{\infty} \delta y_{j'}. \end{aligned} \tag{5.8}$$

Now

$$\begin{aligned}\|e(x)\|^2_{L^2[0,1]} &= \left\| \sum_{j'=j}^{\infty} \delta y_{j'} \right\|^2_{L^2[0,1]} \\ &\leq \sum_{j'=j}^{\infty} \|\delta y_{j'}\|^2_{L^2[0,1]} \\ &= \|\delta y_j\|^2_{L^2[0,1]} \left[1 + \frac{\|\delta y_{j+1}\|^2_{L^2[0,1]}}{\|\delta y_j\|^2_{L^2[0,1]}} + \frac{\|\delta y_{j+2}\|^2_{L^2[0,1]}}{\|\delta y_j\|^2_{L^2[0,1]}} + ...\right]\end{aligned} \tag{5.9}$$

We choose $\max\limits_{\eta} \dfrac{\|\delta y_{j+\eta}\|^2_{L^2[0,1]}}{\|\delta y_{j+\eta-1}\|^2_{L^2[0,1]}} = \tau$ for $\eta = 1, 2, 3, ...$ and τ is found to satisfy the condition $0 < \tau < 1$, which is verified by taking a few examples of Abel first kind and second kind integral equations. The values of τ are different for different examples. Then the expression in (5.9) becomes

$$\begin{aligned}\|\delta y_j\|^2_{L^2[0,1]} \left[1 + \frac{\|\delta y_{j+1}\|^2_{L^2[0,1]}}{\|\delta y_j\|^2_{L^2[0,1]}} + \frac{\|\delta y_{j+2}\|^2_{L^2[0,1]}}{\|\delta y_j\|^2_{L^2[0,1]}} + ...\right] &\leq \|\delta y_j\|^2_{L^2[0,1]} \left[1 + \tau + \tau^2 + \tau^3 + ...\right] \\ &= \|\delta y_j\|^2_{L^2[0,1]} \frac{1}{1-\tau}.\end{aligned} \tag{5.10}$$

The expression for $\|\delta y_j\|^2_{L^2[0,1]}$ is obtained by using orthonormality property of $\psi_{jk}(x)$ within its support and Theorem 5.4 for the partial support of $\psi_{jk}(x)$. This is given by

$$\begin{aligned}\|\delta y_j\|^2_{L^2[0,1]} &= \left\langle \sum_{k=0}^{2^j-1} d_{jk}\psi_{jk}(x), \sum_{k=0}^{2^j-1} d_{jk}\psi_{jk}(x) \right\rangle \\ &= \sum_{k=0}^{2^j-2K+1} \sum_{k'=0}^{2^j-2K+1} d^I_{jk} d^I_{jk'} \delta_{kk'} + \sum_{k=2^j-2K+2}^{2^j-1} \sum_{k'=2^j-2K+2}^{2^j-1} d^R_{jk} d^R_{jk'} T^R_{j:kk'}.\end{aligned} \tag{5.11}$$

As $\int_0^1 \psi^R_{jk}(x)\psi^I_{jk'}(x)\mathrm{d}x$ and $\int_0^1 \psi^I_{jk}(x)\psi^R_{jk'}(x)\mathrm{d}x$ vanish, so we neglect those terms in the expression (5.11) which contain these specific integrals.

The bound of L^2- norm of error $\|e(x)\|_{L^2[0,1]}$ can be estimated from the inequality (5.10).

6 Illustrative examples

Example 1

Consider the first kind Abel integral equation

$$\int_0^x \frac{y(t)\mathrm{d}t}{(x-t)^{\mu}} = B\left(1-\mu, 1+\nu\right) x^{1+\nu-\mu}, \quad 0 < \mu < 1, \quad \nu > 0$$

which has the exact solution $y(x) = x^{\nu}$. Here $B(m,n)$ is the beta function and defined by $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\mathrm{d}x,\ m > 0,\ n > 0.$

Table 3 shows the exact and approximate solutions of the example 1 at the points $x = \frac{i}{8}$ for $i = 1, 2, ..., 7$ taking Dau-3 scale function and $M = 5$. In this table, four sets of values of μ and ν are considered taking both fraction and integer values of ν.

Table 3: Comparison of exact and approximate solutions of Example 1

	x	Exact Solution	Approximate solution		
			$j = 4$	$j = 6$	$j = 8$
	1/8	0.353553	0.309319	0.352867	0.353554
	2/8	0.500000	0.486212	0.499995	0.500000
	3/8	0.612372	0.608044	0.612374	0.612373
$\mu = \frac{1}{4}, \nu = \frac{1}{2}$	4/8	0.707107	0.705733	0.707108	0.707107
	5/8	0.790569	0.790135	0.790570	0.790569
	6/8	0.866025	0.865890	0.866026	0.866025
	7/8	0.935414	0.935375	0.935415	0.935414
	1/8	0.001953	0.001805	0.001951	0.001953
	2/8	0.015625	0.015478	0.015623	0.015625
	3/8	0.052734	0.052588	0.052732	0.052734
$\mu = \frac{1}{4}, \nu = 3$	4/8	0.125000	0.124854	0.124998	0.125000
	5/8	0.244141	0.243995	0.244138	0.244141
	6/8	0.421875	0.421730	0.421873	0.421875
	7/8	0.669922	0.669776	0.669920	0.669922
	1/8	0.353553	0.358049	0.353775	0.353575
	2/8	0.500000	0.500476	0.500099	0.500009
	3/8	0.612372	0.613000	0.612433	0.612378
$\mu = \frac{3}{4}, \nu = \frac{1}{2}$	4/8	0.707107	0.707550	0.707149	0.707111
	5/8	0.790569	0.790915	0.790602	0.790572
	6/8	0.866025	0.866305	0.866051	0.866028
	7/8	0.935414	0.935647	0.935436	0.935416
	1/8	0.001953	0.001870	0.001951	0.001953
	2/8	0.015625	0.015525	0.015623	0.015625
	3/8	0.052734	0.052630	0.052732	0.052734
$\mu = \frac{3}{4}, \nu = 3$	4/8	0.125000	0.124892	0.124998	0.125000
	5/8	0.244141	0.244030	0.244139	0.244141
	6/8	0.421875	0.421763	0.421873	0.421875
	7/8	0.669922	0.669809	0.669920	0.669922

Table 4: Values of $\|\delta y_j\|^2_{L^2[0,1]}$ for different resolution j

	j	For d^I_{jk}	For both d^I_{jk} and d^R_{jk}
$\mu=\frac{1}{4},\nu=\frac{1}{2}$	4	5.75007×10^{-8}	1.14422×10^{-3}
	5	1.43777×10^{-8}	5.71189×10^{-4}
	6	3.59444×10^{-9}	2.85378×10^{-4}
	7	8.9861×10^{-10}	1.42637×10^{-4}
	8	2.24653×10^{-10}	7.13055×10^{-5}
	9	5.61631×10^{-11}	3.56496×10^{-5}
$\mu=\frac{1}{4},\nu=3$	4	3.92808×10^{-9}	1.15222×10^{-3}
	5	7.16154×10^{-11}	5.74173×10^{-4}
	6	1.19898×10^{-12}	2.86245×10^{-4}
	7	1.93591×10^{-14}	1.42869×10^{-4}
	8	3.07368×10^{-16}	7.13653×10^{-5}
	9	4.84076×10^{-18}	3.56647×10^{-5}
$\mu=\frac{3}{4},\nu=\frac{1}{2}$	4	1.28416×10^{-7}	1.14442×10^{-3}
	5	3.21065×10^{-8}	5.71226×10^{-4}
	6	2.85385×10^{-9}	2.86245×10^{-4}
	7	2.00666×10^{-9}	1.42638×10^{-4}
	8	5.01665×10^{-10}	7.13058×10^{-5}
	9	1.25416×10^{-10}	3.56496×10^{-5}
$\mu=\frac{3}{4},\nu=3$	4	3.92901×10^{-9}	1.15223×10^{-3}
	5	7.16226×10^{-11}	5.74174×10^{-4}
	6	1.9904×10^{-12}	2.86245×10^{-4}
	7	1.93595×10^{-14}	1.42869×10^{-4}
	8	3.07371×10^{-16}	7.13653×10^{-5}
	9	4.84079×10^{-18}	3.56648×10^{-5}

Table 5: Comparison of Sup error and bound of L^2-norm of error $\|e(x)\|_{L^2[0,1]}$

	j	Sup error	Bound of $\|e(x)\|_{L^2[0,1]}$	
			taking d^I_{jk}	taking d^I_{jk} and d^R_{jk}
$\mu=\frac{1}{4},\nu=\frac{1}{2}$	4	4.423400×10^{-2}	2.768973×10^{-4}	4.783764×10^{-2}
	6	6.867130×10^{-4}	6.923054×10^{-5}	3.389050×10^{-2}
	8	7.100990×10^{-7}	1.730766×10^{-5}	1.194198×10^{-2}

$\mu = \frac{1}{4}, \nu = 3$	4	1.48274×10^{-4}	6.324620×10^{-5}	4.800458×10^{-2}
	6	2.27344×10^{-6}	1.104969×10^{-6}	1.691878×10^{-2}
	8	3.55446×10^{-8}	1.769186×10^{-8}	1.194201×10^{-2}
$\mu = \frac{3}{4}, \nu = \frac{1}{2}$	4	4.49596×10^{-3}	4.165755×10^{-4}	4.784182×10^{-2}
	6	2.21534×10^{-4}	1.041480×10^{-4}	2.389079×10^{-2}
	8	2.13190×10^{-5}	2.603701×10^{-5}	1.194201×10^{-2}
$\mu = \frac{3}{4}, \nu = 3$	4	8.32861×10^{-5}	6.316013×10^{-5}	4.800479×10^{-2}
	6	1.68850×10^{-6}	1.105110×10^{-6}	2.023927×10^{-2}
	8	2.90795×10^{-8}	1.769375×10^{-8}	1.194699×10^{-2}

Example 2

Consider the second kind Abel integral equation [12]

$$y\left(x\right) = x^2 + \frac{16}{5} x^{\frac{5}{2}} - \int_0^x \frac{y(t)\mathrm{d}t}{\sqrt{x-t}}$$

which has the exact solution $y(x) = x^2$.

Table 6 shows the exact and approximate solutions of the example 2 at the points $x = \frac{i}{8}$ for $i = 0, 1, 2, ..., 7$ taking Dau-3 scale function and $M = 5$.

Table 6: Comparison of exact and approximate solutions of example 2

x	Exact Solution	Approximate solution		
		$j = 4$	$j = 6$	$j = 8$
0	0	0	0	0
1/8	0.015625	0.015508	0.015624	0.015625
2/8	0.062500	0.062463	0.062499	0.062500
3/8	0.140625	0.140603	0.140625	0.140625
4/8	0.250000	0.249984	0.250000	0.250000
5/8	0.390625	0.390613	0.390625	0.390625
6/8	0.562500	0.562490	0.562500	0.562500
7/8	0.765625	0.765617	0.765625	0.765625

Table 7: Values of $\|\delta y_j\|^2_{L^2[0,1]}$ for different resolution j

j	For d^I_{jk}	For both d^I_{jk} and d^R_{jk}
4	1.45784×10^{-12}	1.15094×10^{-3}
5	4.66357×10^{-14}	5.73210×10^{-4}
6	1.46128×10^{-15}	2.85926×10^{-4}
7	4.53812×10^{-17}	1.42779×10^{-4}
8	1.40555×10^{-18}	7.13417×10^{-5}
9	4.35409×10^{-20}	3.56587×10^{-5}

Table 8: Comparison of Sup error and bound of L^2- norm of error $\|e(x)\|_{L^2[0,1]}$

j	Sup error	Bound of $\|e(x)\|_{L^2[0,1]}$	
		taking d^I_{jk}	taking d^I_{jk} and d^R_{jk}
4	1.16723×10^{-4}	1.22720×10^{-6}	4.79779×10^{-2}
6	9.56852×10^{-7}	3.88534×10^{-8}	2.39134×10^{-2}
8	1.34341×10^{-8}	1.20500×10^{-9}	1.19450×10^{-2}

Example 3

Consider the second kind Abel integral equation [17]

$$y\left(x\right) = \frac{1}{\sqrt{x+1}} + \frac{\pi}{8} - \frac{1}{4}\sin^{-1}\left(\frac{1-x}{1+x}\right) - \frac{1}{4}\int_0^x \frac{y(t)\mathrm{d}t}{\sqrt{x-t}}$$

which has the exact solution $y(x) = \dfrac{1}{\sqrt{x+1}}$.

Table 9 shows the exact and approximate solutions of the example 3 at the points $x = \frac{i}{8}$ for $i = 0, 1, 2, ..., 7$ taking Dau-3 scale function and $M = 5$.

Table 9: Comparison of exact and approximate solutions of Example 3

x	Exact Solution	Approximate solution		
		$j=4$	$j=6$	$j=8$
0	1	1	1	1
1/8	0.942809	0.964541	0.947179	0.943883
2/8	0.894427	0.905166	0.897201	0.895110
3/8	0.852803	0.861371	0.854894	0.853318
4/8	0.816497	0.823468	0.818192	0.816914
5/8	0.784465	0.790355	0.785898	0.784818
6/8	0.755929	0.761042	0.757173	0.756236
7/8	0.730297	0.734819	0.731398	0.730568

Table 10: Values of $\|\delta y_j\|^2_{L^2[0,1]}$ for different resolution j

j	For d^I_{jk}	For both d^I_{jk} and d^R_{jk}
4	4.60915×10^{-6}	5.80526×10^{-4}
5	2.39761×10^{-6}	2.88967×10^{-4}
6	1.22773×10^{-6}	1.44416×10^{-4}
7	6.23369×10^{-7}	7.20037×10^{-5}
8	3.14886×10^{-7}	3.59832×10^{-5}
9	1.58536×10^{-7}	1.79872×10^{-5}

Table 11: Comparison of Sup error and bound of L^2- norm of error $\|e(x)\|_{L^2[0,1]}$

j	Sup error	Bound of $\|e(x)\|_{L^2[0,1]}$	
		taking d^I_{jk}	taking d^I_{jk} and d^R_{jk}
4	2.17315×10^{-2}	3.09877×10^{-3}	3.40742×10^{-2}
6	4.37017×10^{-3}	1.59930×10^{-3}	1.69951×10^{-2}
8	1.07361×10^{-3}	8.09946×10^{-4}	8.48330×10^{-3}

We present in Tables 4, 7 and 10 the values of $\|\delta y_j\|^2_{L^2[0,1]}$ ($j = 4, 5, 6, ..., 9$) given by equation (5.11) for the examples 1, 2 and 3 respectively. Second column of all tables present the values $\|\delta y_j\|^2_{L^2[0,1]}$ taking only d^I_{jk} i.e. taking only first term of (5.11), whereas third column presents the values $\|\delta y_j\|^2_{L^2[0,1]}$ taking both d^I_{jk} and d^R_{jk}. From these tables it appears that the values of $\|\delta y_j\|^2_{L^2[0,1]}$ gradually decrease if the resolution j increases. The presence of a few d^R_{jk} in (5.11) makes a lot of difference in calculating $\|\delta y_j\|^2_{L^2[0,1]}$ taking only d^I_{jk} and taking both d^I_{jk} and d^R_{jk}.

In Tables 5, 8 and 11, the Sup errors are compared with the bound of L^2-norm of error $\|e(x)\|_{L^2[0,1]}$ taking d^I_{jk} and taking both d^I_{jk} and d^R_{jk} for examples 1, 2 and 3 respectively. To evaluate bound of L^2- norm of error $\|e(x)\|_{L^2[0,1]}$, $\tau = 0.250044, \tau = 0.50$; $\tau = 0.250044, \tau = 0.50$; $\tau = 0.250044, \tau = 0.50$ and $\tau = 0.250044, \tau = 0.50$ are used for the four sets of values of μ and ν taking only d^I_{jk} and taking both d^I_{jk} and d^R_{jk} for example 1. Also to evaluate bound of L^2 norm of error $\|e(x)\|_{L^2[0,1]}$, $\tau = 0.032$, $\tau = 0.50$ and $\tau = 0.52$, $\tau = 0.50$ are used for Examples 2 and 3 respectively. Sup errors are calculated taking maximum absolute difference of exact and approximate solutions from Tables 3, 6 and 9.

Figures 1 to 6 display the exact and approximate solutions of examples 1, 2 and 3 for different resolutions ($j = 4, 6, 8$). We observe from these figures that as j increases, an approximate solution becomes closer to exact solution. This demonstrates efficiency of the proposed method.

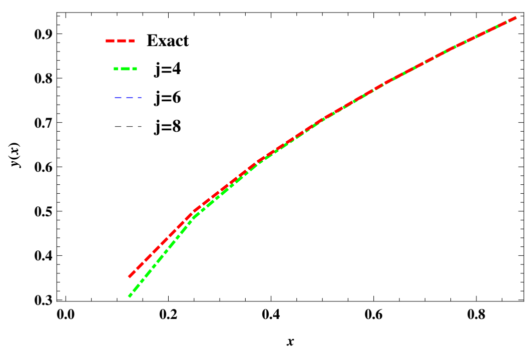

Figure 1: Example 1 ($\mu = \frac{1}{4}, \nu = \frac{1}{2}$)

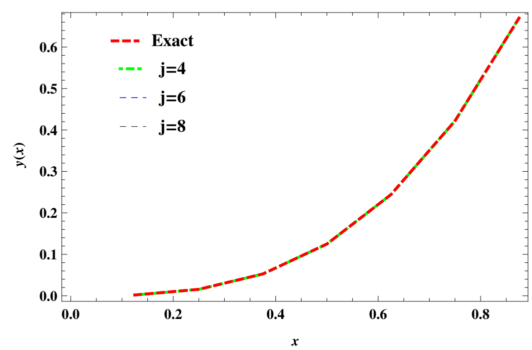

Figure 2: Example 1 ($\mu = \frac{1}{4}, \nu = 3$)

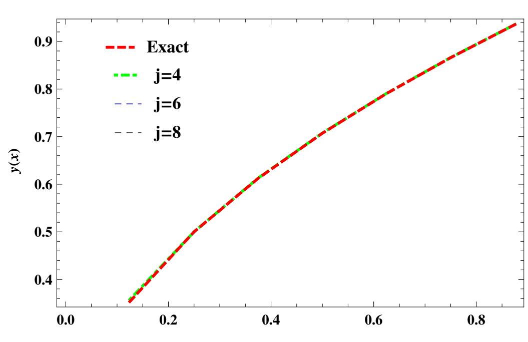

Figure 3: Example 1 ($\mu = \frac{3}{4}, \nu = \frac{1}{2}$)

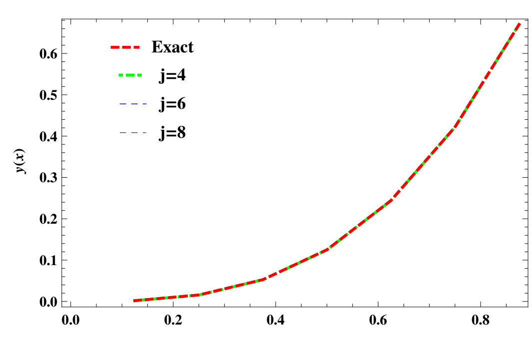

Figure 4: Example 1 ($\mu = \frac{3}{4}, \nu = 3$)

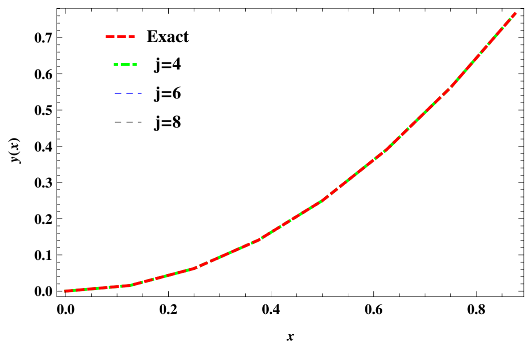

Figure 5: Example 2

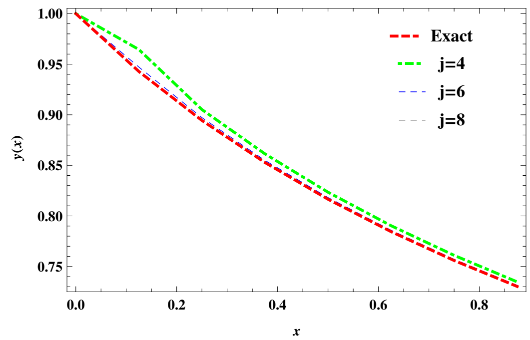

Figure 6: Example 3

7 Conclusion

The purpose of the present work is to develop an efficient and accurate numerical scheme based on Daubechies wavelet basis to solve Abel integral equation. As wavelets are orthogonal systems, they have different resolution capabilities. The detail error estimation shows that the bound of L^2-norm of error $\|e(x)\|_{L^2[0,1]}$ depends on resolution j. From Tables 3, 6 and 9 it appears that the present numerical scheme works nicely for low resolution ($j = 4, 6, 8$). The results can be further improved by taking larger resolution j.

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