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On the conformally k-th Gauduchon condition and the conformally semi-Kähler condition on almost complex manifolds

Masaya Kawamura[1]

[1] *Department of General Education, National Institute of Technology, Kagawa College, 355, Chokushi-cho, Takamatsu, Kagawa, Japan 761-8058.*
kawamura-m@t.kagawa-nct.ac.jp

ABSTRACT

We introduce the k-th Gauduchon condition on almost complex manifolds. We show that if both the conformally k-th Gauduchon condition and the conformally semi-Kähler condition are satisfied, then it becomes conformally quasi-Kähler.

RESUMEN

Introducimos la k-ésima condición de Gauduchon en variedades casi complejas. Mostramos que si la k-ésima condición de Gauduchon conforme y la condición semi-Kähler conforme se satisfacen ambas, entonces la variedad es cuasi-Kähler conforme.

Keywords and Phrases: Almost Hermitian manifold, k-th Gauduchon metric, semi-Kähler metric.

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1 Introduction

S. Ivanov and G. Papadopoulous introduced the conditions on the Hermitian form such that $\omega^l \wedge \partial\bar{\partial}\omega^k = 0$ for $1 \leq k + l \leq n - 1$, which is called the $(l|k)$-SKT condition. They have proven that every compact conformally balanced $(l|k)$-SKT manifold, $k < n - 1$, $n > 2$, is Kähler (cf. [5]). J. Fu, Z. Wang and D. Wu introduced and investigated the generalization of Gauduchon metrics, which is called k-th Gauduchon. The k-th Gauduchon condition is the case $l = n - k - 1$, $1 \leq k \leq n-1$ of the $(l|k)$-SKT condition. By definition, $(n-1)$-th Gauduchon metrics are the usual Gauduchon metrics, astheno-Kähler metrics are examples of $(n - 2)$-th Gauduchon metrics, and pluriclosed metrics are in particular 1-st Gauduchon. They proved that there exists a non-Kähler 3-fold which can support a 1-Gauduchon metric and a balanced metric simultaneously (cf. [2]). Since K. Liu and X. Yang have shown that if a compact complex manifold is k-th Gauduchon for $1 \leq k \leq n - 2$ and also balanced, then it must be Kähler, a 1-Gauduchon metric and a balanced metric on a non-Kähler 3-fold which Fu, Wang and Wu discovered must be different Hermitian metrics. Liu and Yang also have shown that the conformally Kählerianity is equivalent to that both the conformally k-th Gauduchon for $1 \leq k \leq n - 2$, and the conformally balancedness are satisfied (cf. [7]). Our aim in this paper is to generalize the Liu-Yang's equivalence [7, Corollary 1.17] to almost Hermitian geometry.

Let (M^{2n}, J) be an almost complex manifold with $n \geq 3$ and let g be an almost Hermitian metric on M. Let $\{Z_r\}$ be an arbitrary local $(1, 0)$-frame around a fixed point $p \in M$ and let $\{\zeta^r\}$ be the associated coframe. Then the associated real $(1, 1)$-form ω with respect to g takes the local expression $\omega = \sqrt{-1}g_{r\bar{k}}\zeta^r \wedge \zeta^{\bar{k}}$. We will also refer to ω as to an almost Hermitian metric. We introduce the definition of a Gauduchon metric and we define a k-th Gauduchon metric as follows.

Definition 1.1. *Let (M^{2n}, J) be an almost complex manifold. A metric g is called a Gauduchon metric on M if g is an almost Hermitian metric whose associated real $(1, 1)$-form $\omega = \sqrt{-1}g_{i\bar{j}}\zeta^i \wedge \zeta^{\bar{j}}$ satisfies $d^*(Jd^*\omega) = 0$, where d^* is the adjoint of d with respect to g, which is equivalent to $d(Jd(\omega^{n-1})) = 0$, or $\partial\bar{\partial}(\omega^{n-1}) = 0$. When an almost Hermitian metric g is Gauduchon, the triple (M^{2n}, J, g) will be called a Gauduchon manifold. For $1 \leq k \leq n - 1$, an almost Hermitian metric ω is called k-th Gauduchon if it satisfies that $\partial\bar{\partial}\omega^k \wedge \omega^{n-k-1} = 0$.*

Notice that the condition $\partial\bar{\partial}\omega^k \wedge \omega^{n-k-1} = 0$ for $1 \leq k \leq n - 2$ is not equivalent to $d(Jd(\omega^k)) \wedge \omega^{n-k-1} = 0$ for $1 \leq k \leq n-2$ since there exist A and $\bar{A}$ parts of the exterior differential operator d in the almost complex setting (Note that these conditions are equivalent in the case of $k = n-1$ as we confirmed in Definition 1.1 since then we have $A(\omega^{n-1}) = \bar{A}(\omega^{n-1}) = 0$.). Hence the condition $\partial\bar{\partial}\omega^k \wedge \omega^{n-k-1} = 0$ for $1 \leq k \leq n - 1$ can be regarded as a natural extension of the Gauduchon condition on almost complex manifolds.

We next introduce the definition of a semi-Kähler metric.

Definition 1.2. *Let (M^{2n}, J) be an almost complex manifold. A metric g is called a semi-Kähler metric on M if g is an almost Hermitian metric whose associated real $(1,1)$-form $\omega = \sqrt{-1}g_{i\bar{j}}\zeta^i \wedge \zeta^{\bar{j}}$ satisfies $d(\omega^{n-1}) = 0$. When an almost Hermitian metric g is semi-Kähler, the triple (M^{2n}, J, ω) will be called a semi-Kähler manifold.*

Recall that on an almost Hermitian manifold (M, J, g), a quasi-Kähler structure is an almost Hermitian structure whose real $(1,1)$-form ω satisfies $(d\omega)^{(1,2)} = \bar{\partial}\omega = 0$, which is equivalent to the original definition of quasi-Kählerianity: $D_X J(Y) + D_{JX} J(JY) = 0$ for all vector fields X, Y (cf. [4]), where D is the Levi-Civita connection associated to g. It is important for us to study quasi-Kähler manifolds since they include the classes of almost Kähler manifolds and nearly Kähler manifolds. An almost Kähler or quasi-Kähler manifold with J integrable is a Kähler manifold. We define some conformally conditions.

Definition 1.3. *Let (M, J, ω) be an almost Hermitian manifold. We say ω is conformally k-th Gauduchon (resp. semi-Kähler, quasi-Kähler) if there exist a k-th Gauduchon (resp. semi-Kähler, quasi-Kähler) metric $\tilde{\omega}$ and a smooth function $F \in C^\infty(M, \mathbb{R})$ such that $\omega = e^F \tilde{\omega}$.*

Our main result is as follows.

Theorem 1.4. *On a compact almost Hermitian manifold (M, J, ω), the following are equivalent:*

(1) *(M, J, ω) is conformally quasi-Kähler.*

(2) *(M, J, ω) is conformally k-th Gauduchon for $1 \leq k \leq n-2$, and conformally semi-Kähler.*

In particular, the following are also equivalent:

(a) *(M, J, ω) is quasi-Kähler.*

(b) *(M, J, ω) is k-th Gauduchon for $1 \leq k \leq n-2$, and conformally semi-Kähler.*

This paper is organized as follows: in the second section, we recall some basic definitions and computations. In the last section, we will give a proof of the main result. Notice that we assume the Einstein convention omitting the symbol of sum over repeated indexes in all this paper.

2 Preliminaries

2.1 The Nijenhuis tensor of the almost complex structure

Let M be a $2n$-dimensional smooth differentiable manifold. An almost complex structure on M is an endomorphism J of TM, $J \in \Gamma(\mathrm{End}(TM))$, satisfying $J^2 = -Id_{TM}$. The pair (M, J) is called an almost complex manifold. Let (M, J) be an almost complex manifold. We define a bilinear map on $C^\infty(M)$ for $X, Y \in \Gamma(TM)$ by

$$4N(X,Y) := [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y], \tag{2.1}$$

which is the Nijenhuis tensor of J. The Nijenhuis tensor N satisfies $N(X,Y) = -N(Y,X)$, $N(JX,Y) = -JN(X,Y)$, $N(X,JY) = -JN(X,Y)$, $N(JX,JY) = -N(X,Y)$. For any $(1,0)$-vector fields W and V, $N(V,W) = -[V,W]^{(0,1)}$, $N(V,\bar{W}) = N(\bar{V},W) = 0$ and $N(\bar{V},\bar{W}) = -[\bar{V},\bar{W}]^{(1,0)}$ since we have $4N(V,W) = -2([V,W] + \sqrt{-1}J[V,W])$, $4N(\bar{V},\bar{W}) = -2([\bar{V},\bar{W}] - \sqrt{-1}J[\bar{V},\bar{W}])$. An almost complex structure J is called integrable if $N = 0$ everywhere on M. Giving a complex structure on a differentiable manifold M is equivalent to giving an integrable almost complex structure on M. Let (M, J) be an almost complex manifold. A Riemannian metric g on M is called J-invariant if J is compatible with g, *i.e.*, for any $X, Y \in \Gamma(TM)$, $g(X,Y) = g(JX, JY)$. In this case, the pair (J, g) is called an almost Hermitian structure. The fundamental 2-form ω associated to a J-invariant Riemannian metric g, *i.e.*, an almost Hermitian metric, is determined by, for $X, Y \in \Gamma(TM)$, $\omega(X,Y) = g(JX,Y)$. Indeed we have, for any $X, Y \in \Gamma(TM)$,

$$\omega(Y,X) = g(JY,X) = g(J^2Y, JX) = -g(JX,Y) = -\omega(X,Y) \tag{2.2}$$

and $\omega \in \Gamma(\bigwedge^2 T^*M)$. We will also refer to the associated real fundamental $(1,1)$-form ω as an almost Hermitian metric. The form ω is related to the volume form dV_g by $n!dV_g = \omega^n$. Let a local $(1,0)$-frame $\{Z_r\}$ on (M,J) with an almost Hermitian metric g and let $\{\zeta^r\}$ be a local associated coframe with respect to $\{Z_r\}$, i.e., $\zeta^i(Z_j) = \delta^i_j$ for $i, j = 1, \ldots, n$. Since g is almost Hermitian, its components satsfy $g_{ij} = g_{\bar{i}\bar{j}} = 0$ and $g_{i\bar{j}} = g_{\bar{j}i} = \bar{g}_{\bar{i}j}$.

We write $T^{\mathbb{R}}M$ for the real tangent space of M. Then its complexified tangent space is given by $T^{\mathbb{C}}M = T^{\mathbb{R}}M \otimes_{\mathbb{R}} \mathbb{C}$. By extending J $\mathbb{C}$-linearly and g, ω, $\mathbb{C}$-bilinearly to $T^{\mathbb{C}}M$, they are also defined on $T^{\mathbb{C}}M$ and we observe that the complexified tangent space $T^{\mathbb{C}}M$ can be decomposed as $T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$, where $T^{1,0}M$, $T^{0,1}M$ are the eigenspaces of J corresponding to eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively:

$$T^{1,0}M = \{X - \sqrt{-1}JX \big| X \in TM\}, \quad T^{0,1}M = \{X + \sqrt{-1}JX \big| X \in TM\}. \tag{2.3}$$

Let $\Lambda^r M = \bigoplus_{p+q=r} \Lambda^{p,q} M$ for $0 \le r \le 2n$ denote the decomposition of complex differential r-forms into (p,q)-forms, where $\Lambda^{p,q}M = \Lambda^p(\Lambda^{1,0}M) \otimes \Lambda^q(\Lambda^{0,1}M)$,

$$\Lambda^{1,0}M = \{\alpha + \sqrt{-1}J\alpha \big| \alpha \in \Lambda^1 M\}, \quad \Lambda^{0,1}M = \{\alpha - \sqrt{-1}J\alpha \big| \alpha \in \Lambda^1 M\} \tag{2.4}$$

and $\Lambda^1 M$ denotes the dual of TM. For any $\alpha \in \Lambda^1 M$, we define $J\alpha(X) = -\alpha(JX)$ for $X \in TM$.

Let (M^{2n}, J, g) be an almost Hermitian manifold. An affine connection D on TM is called almost Hermitian connection if $Dg = DJ = 0$. For the almost Hermitian connection, we have the following Lemma (*cf.* [3, 9, 11]).

Lemma 2.1. *Let (M, J, g) be an almost Hermitian manifold with $\dim_{\mathbb{R}} M = 2n$. Then for any given vector valued $(1,1)$-form $\Theta = (\Theta^i)_{1\le i\le n}$, there exists a unique almost Hermitian connection D on (M, J, g) such that the $(1,1)$-part of the torsion is equal to the given Θ.*

If the $(1,1)$-part of the torsion of an almost Hermitian connection vanishes everywhere, then the connction is called the second canonical connection or the Chern connection. We will refer the connection as the Chern connection and denote it by ∇.

Note that for any p-form ψ, there holds that

$$\begin{aligned} d\psi(X_1, \dots, X_{p+1}) &= \sum_{i=1}^{p+1}(-1)^{i+1}X_i(\psi(X_1, \dots, \widehat{X_i}, \dots, X_{p+1})) \\ &\quad + \sum_{i<j}(-1)^{i+j}\psi([X_i, X_j], X_1, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_{p+1}) \end{aligned} \tag{2.5}$$

for any vector fields $X_1, \dots, X_{p+1}$ on M (*cf.* [11]). We directly compute that

$$d\zeta^s = -\frac{1}{2}B^s_{kl}\zeta^k \wedge \zeta^l - B^s_{k\bar{l}}\zeta^k \wedge \zeta^{\bar{l}} + \frac{1}{2}N^s_{\bar{k}\bar{l}}\zeta^{\bar{k}} \wedge \zeta^{\bar{l}}. \tag{2.6}$$

According to the direct computation above, we may split the exterior differential operator $d : \Lambda^p M \otimes_{\mathbb{R}} \mathbb{C} \to \Lambda^{p+1} M \otimes_{\mathbb{R}} \mathbb{C}$, into four components

$$d = A + \partial + \bar{\partial} + \bar{A} \tag{2.7}$$

with

$$\partial : \Lambda^{p,q}M \to \Lambda^{p+1,q}M, \quad \bar{\partial} : \Lambda^{p,q}M \to \Lambda^{p,q+1}M, \tag{2.8}$$

$$A : \Lambda^{p,q}M \to \Lambda^{p+2,q-1}M, \quad \bar{A} : \Lambda^{p,q}M \to \Lambda^{p-1,q+2}M, \tag{2.9}$$

since we have

$$d(\Gamma(\Lambda^{r,s}M)) \subseteq \Gamma(\Lambda^{r+2,s-1}M \oplus \Lambda^{r+1,s}M \oplus \Lambda^{r,s+1}M \oplus \Lambda^{r-1,s+2}M). \tag{2.10}$$

In terms of these components, the condition $d^2 = 0$ can be written as

$$\begin{gathered} A^2 = 0, \quad \partial A + A\partial = 0, \quad \bar{\partial}\bar{A} + \bar{A}\bar{\partial} = 0, \quad \bar{A}^2 = 0, \\ A\bar{\partial} + \partial^2 + \bar{\partial}A = 0, \quad A\bar{A} + \partial\bar{\partial} + \bar{\partial}\partial + \bar{A}A = 0, \quad \partial\bar{A} + \bar{\partial}^2 + \bar{A}\partial = 0. \end{gathered} \tag{2.11}$$

Notice that J is integrable if and only if $A = 0$, equivalently, if and only if $\bar{\partial}^2 = 0$.

For any real $(1,1)$-form $\sigma = \sqrt{-1}\sigma_{i\bar{j}}\zeta^i \wedge \zeta^{\bar{j}}$, we have

$$\bar{\partial}\sigma = \frac{\sqrt{-1}}{2}\Big(Z_{\bar{j}}(\sigma_{k\bar{i}}) - Z_{\bar{i}}(\sigma_{k\bar{j}}) - B^s_{k\bar{i}}\sigma_{s\bar{j}} + B^s_{k\bar{j}}\sigma_{s\bar{i}} + B^{\bar{s}}_{\bar{i}\bar{j}}\sigma_{k\bar{s}}\Big)\zeta^k \wedge \zeta^{\bar{i}} \wedge \zeta^{\bar{j}},$$

$$\partial\sigma = \frac{\sqrt{-1}}{2}\Big(Z_i(\sigma_{j\bar{k}}) - Z_j(\sigma_{i\bar{k}}) - B^s_{ij}\sigma_{s\bar{k}} - B^{\bar{s}}_{i\bar{k}}\sigma_{j\bar{s}} + B^{\bar{s}}_{j\bar{k}}\sigma_{i\bar{s}}\Big)\zeta^i \wedge \zeta^j \wedge \zeta^{\bar{k}}.$$

From these computations above, we have

$$\bar{\partial}\omega = \frac{\sqrt{-1}}{2}\Big(Z_{\bar{j}}(g_{k\bar{i}}) - Z_{\bar{i}}(g_{k\bar{j}}) - B^s_{k\bar{i}}g_{s\bar{j}} + B^s_{k\bar{j}}g_{s\bar{i}} + B^{\bar{s}}_{\bar{i}\bar{j}}g_{k\bar{s}}\Big)\zeta^k \wedge \zeta^{\bar{i}} \wedge \zeta^{\bar{j}} = \frac{\sqrt{-1}}{2}T_{\bar{j}\bar{i}k}\zeta^k \wedge \zeta^{\bar{i}} \wedge \zeta^{\bar{j}} \quad (2.12)$$

and

$$\partial\omega = \frac{\sqrt{-1}}{2}\Big(Z_i(g_{j\bar{k}}) - Z_j(g_{i\bar{k}}) - B^s_{ij}g_{s\bar{k}} - B^{\bar{s}}_{i\bar{k}}g_{j\bar{s}} + B^{\bar{s}}_{j\bar{k}}g_{i\bar{s}}\Big)\zeta^i \wedge \zeta^j \wedge \zeta^{\bar{k}} = \frac{\sqrt{-1}}{2}T_{ij\bar{k}}\zeta^i \wedge \zeta^j \wedge \zeta^{\bar{k}}, \quad (2.13)$$

where T is the torsion of the Chern connection. For any $\varphi \in C^\infty(M, \mathbb{R})$, a direct computation yields

$$\sqrt{-1}\partial\bar{\partial}\varphi = \frac{1}{2}(dJd\varphi)^{(1,1)} = \sqrt{-1}(Z_i Z_{\bar{j}} - [Z_i, Z_{\bar{j}}]^{(0,1)})(\varphi)\zeta^i \wedge \zeta^{\bar{j}}, \quad (2.14)$$

so we write locally

$$\partial_i\partial_{\bar{j}}\varphi = (Z_i Z_{\bar{j}} - [Z_i, Z_{\bar{j}}]^{(0,1)})\varphi. \quad (2.15)$$

For basic definitions and computations about the torsion and the curvature on almost Hermitian manifolds, see [6, Section 2].

3 Proof of Theorem 1.4

We need the following lemmas in order to prove Theorem 1.4. Here we introduce the following characterizations of quasi-Kählerianity and semi-Kählerianity.

Lemma 3.1 (*cf.* [8, Lemma 2.4]). *An almost Hermitian manifold (M^{2n}, g, J) is quasi-Kähler if and only if $T^k_{\bar{i}\bar{j}} = 0$ for all i, j and k when a local unitary $(1,0)$-frame is fixed, where T is the torsion of the Chern connection ∇.*

Here, we define $w_r := T^i_{ri}$ and the torsion $(1,0)$-form $\eta := T^i_{ir}\zeta^r = -w_r\zeta^r$ (*cf.* [10]), where $T = (T^i)$ is the torsion of the Chern connection ∇.

Lemma 3.2 (*cf.* [6, Lemma 4.3]). *An almost Hermitian manifold (M^{2n}, J, ω) is semi-Kähler if and only if $\eta = 0$.*

Proof. We have $\partial\omega = \frac{\sqrt{-1}}{2}T_{ij\bar{k}}\zeta^i \wedge \zeta^j \wedge \zeta^{\bar{k}}$ as we see (2.12), (2.13). Then a direct calculation shows that

$$\partial\omega^{n-1} = (n-1)\partial\omega \wedge \omega^{n-2} = -\eta \wedge \omega^{n-1}, \quad (3.1)$$

where we used that $\eta = -w_i\zeta^i = -(n-1)\dfrac{\partial\omega \wedge \omega^{n-2}}{\omega^{n-1}}$. Similarly, we obtain that

$$\bar{\partial}\omega^{n-1} = (n-1)\bar{\partial}\omega \wedge \omega^{n-2} = -\bar{\eta} \wedge \omega^{n-1}, \quad (3.2)$$

since we have $\bar{\partial}\omega = \frac{\sqrt{-1}}{2}T_{\bar{j}\bar{i}k}\zeta^k \wedge \zeta^{\bar{i}} \wedge \zeta^{\bar{j}}$ and $\bar{\eta} = -w_{\bar{i}}\zeta^{\bar{i}} = -(n-1)\dfrac{\bar{\partial}\omega \wedge \omega^{n-2}}{\omega^{n-1}}$.

Recall that the metric g is said to be semi-Kähler if ω^{n-1} is closed. These identities (3.1), (3.2) show that g is semi-Kähler if and only if $\eta = 0$. $\square$

Proof of Theorem 1.4. Assume that ω is conformally k-th Gauduchon and conformally semi-Kähler. Then since ω is conformally semi-Kähler, there exist an semi-Kähler metric ω_B and a smooth function $F \in C^\infty(M, \mathbb{R})$ such that $\omega = e^F \omega_B$. By the conformally k-th Gauduchon condition, there exist a k-th Gauduchon ω_G and a smooth function $\tilde{F} \in C^\infty(M, \mathbb{R})$ such that $\omega = e^{\tilde{F}} \omega_G$. Set $f := F - \tilde{F}$, then we have $\omega_G = e^f \omega_B$. Since ω_G is k-th Gauduchon, we get

$$(e^f \omega_B)^{n-k-1} \wedge \partial\bar{\partial}(e^f \omega_B)^k = 0,$$

and then

$$\omega_B^{n-k-1} \wedge \partial\bar{\partial}(e^f \omega_B)^k = 0. \tag{3.3}$$

Since ω_B is semi-Kähler, $0 = d(\omega_B^{n-1}) = (\partial + A + \bar{A} + \bar{\partial})(\omega_B^{n-1}) = (\partial + \bar{\partial})(\omega_B^{n-1})$, which tells us that $\partial(\omega_B^{n-1}) = \bar{\partial}(\omega_B^{n-1}) = 0$, where we have used that $A(\omega_B^{n-1}) = \bar{A}(\omega_B^{n-1}) = 0$. Hence we see that

$$\omega_B^{n-k-1} \wedge \partial(\omega_B^k) = k\omega_B^{n-2} \wedge \partial\omega_B = \frac{k}{n-1}\partial(\omega_B^{n-1}) = 0. \tag{3.4}$$

Then from (3.3), we have

$$e^{kf}\omega_B^{n-k-1} \wedge \partial\bar{\partial}\omega_B^k + \omega_B^{n-1} \wedge \partial\bar{\partial}(e^{kf}) = 0. \tag{3.5}$$

Therefore, we obtain

$$\begin{aligned} \int_M e^{kf}\omega_B^{n-k-1} \wedge \partial\bar{\partial}(\omega_B^k) &= -\int_M \omega_B^{n-1} \wedge \partial\bar{\partial}(e^{kf}) \\ &= -\frac{1}{n}\int_M n \cdot \frac{\partial\bar{\partial}(e^{kf}) \wedge \omega_B^{n-1}}{\omega_B^n}\omega_B^n \\ &= -\frac{1}{n}\int_M \Delta_B(e^{kf})\omega_B^n = 0. \end{aligned} \tag{3.6}$$

Applying (3.4) and (3.6), we obtain

$$\begin{aligned} 0 &= \int_M e^{kf}\omega_B^{n-k-1} \wedge \partial\bar{\partial}(\omega_B^k) \\ &= \int_M \partial(e^{kf}\omega_B^{n-k-1} \wedge \bar{\partial}(\omega_B^k)) - \partial(e^{kf}) \wedge \omega_B^{n-k-1} \wedge \bar{\partial}(\omega_B^k) - e^{kf}\partial(\omega_B^{n-k-1}) \wedge \bar{\partial}(\omega_B^k) \\ &= \int_M d(e^{kf}\omega_B^{n-k-1} \wedge \bar{\partial}(\omega_B^k)) - e^{kf}\partial(\omega_B^{n-k-1}) \wedge \bar{\partial}(\omega_B^k) \\ &= -k(n-k-1)\int_M e^{kf}\omega_B^{n-3} \wedge \partial\omega_B \wedge \bar{\partial}\omega_B \\ &= -k(n-k-1)\int_M e^{kf}\frac{T'_B \wedge \bar{T}'_B \wedge \omega_B^{n-3}}{\omega_B^n}\omega_B^n \\ &= -\frac{k(n-k-1)}{6n(n-1)(n-2)}\int_M e^{kf}\mathrm{Tr}(T'_B \wedge \bar{T}'_B)\omega_B^n \\ &= -\frac{k(n-k-1)}{6n(n-1)(n-2)}\int_M e^{kf}(6|w_B|^2 - 3|T'_B|^2)\omega_B^n, \end{aligned} \tag{3.7}$$

which gives that $2\int_M e^{kf}|w_B|^2\omega_B^n = \int_M e^{kf}|T'_B|^2\omega_B^n$, where we used $\omega_B^{n-k-1}\wedge\bar\partial(\omega_B^k) = 0$ from (3.4), $(\bar\partial + A + \bar A)(e^{kf}\omega_B^{n-k-1}\wedge\bar\partial(\omega_B^k)) = 0$, and that $\partial\omega_B = T'_B$ since we have $(\partial\omega_B)_{jl\bar k} = \partial_j(g_B)_{l\bar k} - \partial_l(g_B)_{j\bar k} = (T_B)_{jl\bar k}$ from (2.13). Note that as in [1, Chapter 2],

$$\begin{aligned}(T'_B\wedge\bar T'_B)_{ikm\bar j\bar l\bar n} &= (T_B)_{im\bar j}(T_B)_{\bar l\bar n k} + (T_B)_{im\bar l}(T_B)_{\bar n\bar j k} + (T_B)_{im\bar n}(T_B)_{\bar j\bar l k}\\ &\quad+(T_B)_{mk\bar j}(T_B)_{\bar l\bar n i} + (T_B)_{mk\bar l}(T_B)_{\bar n\bar j i} + (T_B)_{mk\bar n}(T_B)_{\bar j\bar l i}\\ &\quad+(T_B)_{ki\bar j}(T_B)_{\bar l\bar n m} + (T_B)_{ki\bar l}(T_B)_{\bar n\bar j m} + (T_B)_{ki\bar n}(T_B)_{\bar j\bar l m}\end{aligned}$$

and

$$\mathrm{Tr}(T'_B\wedge\bar T'_B) = g^{i\bar j}g^{k\bar l}g^{m\bar n}(T'_B\wedge\bar T'_B)_{ikm\bar j\bar l\bar n} = 6|w_B|^2 - 3|T'_B|^2,$$

where $(w_B)_r = (T_B)^i_{ri}$ and $\eta_B = (T_B)^i_{ir}\zeta^r = -(w_B)_r\zeta^r$ is the torsion $(1,0)$-form of ω_B. Since the metric ω_B is semi-Kähler, which is equivalent to that $\eta_B = 0$ from Lemma 3.2. Since $\eta_B = 0$ implies that $(w_B)_r = 0$ for all $r = 1,\ldots,n$, we get

$$\int_M e^{kf}|T'_B|^2\omega_B^n = 2\int_M e^{kf}|w_B|^2\omega_B^n = 0.$$

Hence we have $T'_B = 0$, which is equivalent to the quasi-Kählerianity from Lemma 3.1. Notice that since ω_B is now quasi-Kähler, we have that from (3.5),

$$\Delta_B(e^{kf}) = n\cdot\frac{\partial\bar\partial(e^{kf})\wedge\omega_B^{n-1}}{\omega_B^n} = 0,$$

which implies that f is constant. The converse is obvious. The equivalence of (a) and (b) in the statement of Theorem 1.4 follows by the same argument under the condition $\omega = \omega_G$ and $f = F$. $\square$

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