Independent partial domination

L. Philo Nithya[1]

Joseph Varghese Kureethara[1]

[1] *Department of Mathematics, Christ University, Bengaluru, Karnataka, India.*
philo.nithya@res.christuniversity.in
frjoseph@christuniversity.in

ABSTRACT

For $p \in (0,1]$, a set $S \subseteq V$ is said to p-dominate or partially dominate a graph $G = (V, E)$ if $\frac{|N[S]|}{|V|} \geq p$. The minimum cardinality among all p-dominating sets is called the p-domination number and it is denoted by $\gamma_p(G)$. Analogously, the independent partial domination $(i_p(G))$ is introduced and studied here independently and in relation with the classical domination. Further, the partial independent set and the partial independence number $\beta_p(G)$ are defined and some of their properties are presented. Finally, the partial domination chain is established as $\gamma_p(G) \leq i_p(G) \leq \beta_p(G) \leq \Gamma_p(G)$.

RESUMEN

Para $p \in (0,1]$, un conjunto $S \subseteq V$ se dice que p-domina o parcialmente domina un grafo $G = (V, E)$ si $\frac{|N[S]|}{|V|} \geq p$. La cardinalidad mínima entre todos los conjuntos p-dominantes se llama el número de p-dominación y se denota por $\gamma_p(G)$. Análogamente, la dominación parcial independiente $(i_p(G))$ es introducida y estudiada independientemente y en relación con la dominación clásica. Más aún el conjunto independiente parcial y el número de independencia parcial $\beta_p(G)$ se definen y se presentan algunas de sus propiedades. Finalmente, se establece la cadena de dominación parcial como $\gamma_p(G) \leq i_p(G) \leq \beta_p(G) \leq \Gamma_p(G)$.

1　Introduction

The theory of domination is one of the profusely researched areas in graph theory. Recently a new domination parameter called partial domination number was introduced simultaneously in [3], [4] and [6], and studied in [12, 13, 14, 15]. We extend the concept of partial domination to independent domination in graphs. In [9], the concept of independent partial domination has been defined in the context of partial domination that was defined in [4]. But our work is based on the definition of partial domination in [3, 6] and we concentrate on partial domination chain. Domination addresses the issue of the number of vertices that are dominating all the vertices in a graph. As the set of all vertices of a graph dominates itself, the mathematical adventure is in finding the least number of vertices that can dominate the entire graph. This number is the domination number of a graph.

Finding the domination number of a graph is a well known NP-complete decision problem [11]. In the case of large graphs with a good number of small-degree vertices, the domination number shoots up. Hence, instead of finding the dominating set that dominates the entire graph, it might be convenient to study the set of vertices that dominates the graph partially. This also could be treated as the density problem. By identifying the vertices with large degrees, we can find dense structures in the graph. The vertices that are contributing to the high density neighbourhoods are likely to dominate the major section of vertices of a graph. Hence, domination problem and its variations could also be interpreted as density problems. We follow the popular nomenclature domination and study the structures that are partially dominating a graph.

Domination has been addressed in many different ways by imposing conditions on the dominating set or on its complement or on both. The relations between various domination parameters thus developed aroused mathematical curiosity. The domination chain proposed by Cockayne *et al.* is mathematically profound and aesthetically appealing (see Section 5). A recent survey by Bazgan *et al.* lists the most important results regarding the domination chain parameters [2]. In this paper, we partially address the problem raised by Case *et al.* in [3].

The paper is structured as follows. In Section 2, we present all the preliminary concepts required for this paper. In Section 3, we define independent partial domination number and study some of their properties. In Section 4, we explore some relations between independent dominating set and independent partial dominating set. In Section 5, we define partial independence number and investigate some of its properties which in turn lead to a part of the partial domination chain.

2　Preliminaries

Let G be a simple, finite and undirected graph with $V(G)$ as its set of vertices and $E(G)$ as its edge set. A set $S \subseteq V(G)$ is an independent set of vertices if no two vertices of S are adjacent to each

other. An independent set S of vertices is said to be maximal if no superset $T \supset S$ is independent. The maximum cardinality of an independent set in G is called its vertex independence number denoted by $\beta(G)$ and the corresponding vertex set is called the β-set of G. For every vertex u in G, the set $N(u)$ of all vertices adjacent to u is called the open neighbourhood of u. The set $N(u)$ taken together with $\{u\}$ is called the closed neighbourhood of u and is denoted by $N[u]$.

A set D of vertices is called a dominating set of G if every vertex outside D is adjacent to at least one vertex in D. A dominating set D is minimal if no proper subset of D is a dominating set. The minimum cardinality of a minimal dominating set is called the domination number of G denoted by $\gamma(G)$ and the maximum cardinality of a minimal dominating set is called the upper domination number denoted by $\Gamma(G)$. If a dominating set is independent, it becomes an independent dominating set and the minimum cardinality of such a set is called the independent domination number of G denoted by $i(G)$. For any graph $G = (V, E)$ and proportion $p \in (0, 1]$, a set $S \subseteq V$ is a p-dominating or partial dominating set if $\frac{|N[S]|}{|V|} \geq p$. The p-domination or partial domination number $\gamma_p(G)$ equals the minimum cardinality of a p-dominating set in G.
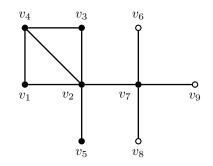

Figure 1: Partial domination by white vertices

In the light of definitions of the neighbourhoods, it is obvious that, for a dominating set S, $N[S] = V$. So a partial dominating set, when compared with a dominating set, dominates a proportion 'p' of the vertex set, which is not necessarily the whole set and hence partially dominates G. The set of all white vertices in Figure 1 dominates exactly 4 vertices and hence is a $\frac{4}{9}$-dominating set. As $\{v_7\}$ is enough to dominate 4 vertices, $\gamma_{\frac{4}{9}} = 1$ for the above graph. For all the other graph theoretic parameters and the notations that are used in this paper, one can refer to [11].

3 Independent partial domination

In this section, we define independent partial domination number and present some observations and some of the basic results. Since the observations are obvious, we present them without proofs.

Definition 3.1. *Suppose $G = (V, E)$ is a simple graph and $p \in (0, 1]$. A subset S of V is called*

an independent p-dominating set (IPD-set) if S is a p-dominating set and is independent.

Definition 3.2. *The minimum cardinality of an independent p-dominating set is called the independent p-domination number (IPD-number) and is denoted by $i_p(G)$.*

Observations:

(i) For $p \in (0, 1]$, $\gamma_p(G) \leq i_p(G)$.

(ii) For any n-vertex graph G and for $p \in (0, \frac{\Delta+1}{n}]$, $i_p(G) = 1$.

(iii) For all $p \in (0, 1]$, $i_p(G) = 1$ if and only if $i(G) = 1$.

(iv) For $p \in \left(\frac{n-1}{n}, 1\right]$, $i_p(G) = i(G)$.

(v) For all $p \in (0, 1]$, $i_p(G) \leq i(G)$.

We proceed to find the IPD-numbers of paths, cycles and complete bipartite graphs.

Proposition 3.3. *Suppose P_n and C_n are paths and cycles respectively on n-vertices. Then for $n \geq 3$, $i_p(C_n) = i_p(P_n) = \lceil \frac{np}{3} \rceil$.*

Proof. Consider C_n for $n \geq 3$. Let S be a γ_p-set of C_n. Then $|S| = \gamma_p = \lceil \frac{np}{3} \rceil$. If we can choose S in such a way that S is independent, then $i_p(C_n) = \lceil \frac{np}{3} \rceil$. For this, consider $C_n = (v_1, v_2, v_3, ..., v_{3r}, v_{3r+1}, v_{3r+2})$.

Here three cases arise *viz.*, (i) $n = 3r$, (ii) $n = 3r + 1$, (iii) $n = 3r + 2$ where $r \geq 1$.

Let $S_1 = \{v_2, v_5, ..., v_{3r-1}\}, S_2 = \{v_2, v_5, ..., v_{3r-1}, v_{3r+1}\}$ and $S_3 = \{v_2, v_5, ..., v_{3r-1}, v_{3r+2}\}$. We can see that $|S_1| = |S_2| = |S_3| = \lceil \frac{n}{3} \rceil$ and S_i is independent for $1 \leq i \leq 3$. For cases (i), (ii) and (iii) we can choose our set of $\lceil \frac{np}{3} \rceil$ vertices from S_1, S_2 and S_3. Hence, $i_p(C_n) = \lceil \frac{np}{3} \rceil$. This proof holds for P_n also. $\square$

Proposition 3.4. *For $m \leq n$, $i_p(K_{m,n}) = \begin{cases} 1, & \text{for } p \in (0, \frac{n+1}{m+n}] \\ i + 1, & \text{for } p \in (\frac{n+i}{m+n}, \frac{n+(i+1)}{m+n}] \text{ where } 1 \leq i \leq m-1. \end{cases}$*

Also $i_p(K_{m,n}) \leq m$.

Proof. Consider $K_{m,n}$ for $m \leq n$. Let $V_1 = \{v_1, v_2, ..., v_m\}$ and $V_2 = \{u_1, u_2, ..., u_n\}$ be the two partite sets of $K_{m,n}$, where each of V_1 and V_2 is an independent set.

Now, $v_1 \in V_1$ dominates $\frac{n+1}{m+n}$ vertices. Consequently, of the remaining $m - 1$ vertices in V_1, each $v_i \in V_1$ dominates $\frac{n+i}{m+n}$ vertices. Thus $i_p(K_{m,n}) \leq m$. $\square$

4   Independent domination and independent partial domination

Allan and Laskar described the relation between the domination number and the independent domination number of a graph in [1]. Partial domination is all about dominating a proportion p of the vertices of G. So a natural question which arises is that: whether this proportion p has any role in relating the partial domination and the original domination parameters. In this section, we do say 'yes' to that question by giving an upper bound for IPD-numbers in terms of p and independent domination numbers. We also give some results, which relate independent dominating sets [10] with that of partial independent dominating sets.

Theorem 4.1. *For any graph G with independent domination number $i(G)$ and $p \in (0,1]$, $i_p(G) \leq \lceil p.i(G) \rceil$.*

Proof. Let $D = \{v_1, v_2, ..., v_i\}$ be an i-set of G. Partition V into sets $V_1, V_2, ..., V_i$ such that for each $1 \leq j \leq i$, $V_j \subseteq N[v_j]$. Without loss of generality, let us assume that $|V_j| \geq |V_{j+1}|$ for $1 \leq j \leq i$.

Consider $D' = \{v_1, v_2, ..., v_{\lceil p.i \rceil}\}$.

Claim: D' is an IPD-set of G.

Proof of the Claim

Our construction yields,

$$\left| \bigcup_{j=1}^{i} V_j \right| = |V| \implies \left| \bigcup_{j=1}^{\lceil p.i \rceil} V_j \right| + \left| \bigcup_{j=\lceil p.i \rceil}^{i} V_j \right| = |V| \implies \frac{\left| \bigcup_{j=1}^{\lceil p.i \rceil} V_j \right|}{\lceil p.i \rceil} \geq \frac{\left| \bigcup_{j=1}^{i} V_j \right|}{i} = \frac{|V|}{i}$$

$$\implies \left| \bigcup_{j=1}^{\lceil p.i \rceil} V_j \right| \geq \frac{|V|.\lceil p.i \rceil}{i}.$$

Hence $|N[D']| \geq p.|V|$. We have thus proved the claim.

Thus using the claim, we have $i_p(G) \leq |D'| = \lceil p.i(G) \rceil$. $\square$

Proposition 4.2. *Let G be any graph with independent domination number $i(G)$ and $p \in (0,1]$. Then $i_p(G) + i_{1-p}(G) \leq i(G) + 1$.*

Proof. By Theorem 5.7, $i_p(G) \leq \lceil p.i \rceil < p.i + 1$ and $i_{1-p}(G) \leq \lceil (1-p).i \rceil < (1-p).i + 1$, then $i_p(G) + i_{1-p}(G) < i + 2 \leq i + 1$. $\square$

Proposition 4.3. *Let S be any independent dominating set of G. If $p = \frac{|N[H]|}{|V|}$, for some $H \subset S$, then $S - H$ is a $1 - p$ independent dominating set in G.*

Proof. It can be easily proved that, $N[S] - N[H] \subseteq N[S - H]$. Therefore, $\frac{|N[S-H]|}{|V|} \geq 1 - p$ since $N[S] = V$. $\hfill\square$

The following result provides us with an algorithm that develops a minimal independent dominating set from a minimal IPD-set.

Proposition 4.4. *Every minimal IPD-set can be extended to form a minimal independent dominating set.*

Proof. Let I be a minimal IPD-set for any $p \in (0, 1]$. The following algorithm extends I to I', a minimal independent dominating set and gives m, the cardinality of I'.

Procedure 1 Algorithm to construct I' from I

Input: $V(G), I, N[I], N[u] \forall u \in V(G) - N[I]$

Output: I', m

  1:   $I' = I, m = |I'|, M = \{\}$

  2:   $M = V(G) - N[I']$

  3: **if** $M = \phi$ **then**

  4:     return I', m

  5: **else**

  6:     $I' = I' \cup \{u\}$ for any $u \in M$

  7:     $N[I'] = N[I'] \cup N[u]$

  8:     $m = m + 1$

  9:     go to 2

10: **end if**

$\hfill\square$

When a graph is claw-free, it has been already proved in [1], that its domination number coincides with that of its independent domination number. We found that to be true in the context of partial domination also.

Proposition 4.5. *If a graph G is claw-free, then $\gamma_p(G) = i_p(G)$.*

Proof. Let S be a γ_p−set of G, for any $p \in (0, 1]$. Since G is claw-free, $< N[S] >$ is also claw-free. Hence, $\gamma(< N[S] >) = i(< N[S] >)$. This implies that $\gamma_p(G) \geq i_p(G)$. But in general, $\gamma_p(G) \leq i_p(G)$. Thus $\gamma_p(G) = i_p(G)$. $\hfill\square$

Corollary 4.6. *If $L(G)$ is the line graph of a graph G, then $\gamma_p(L(G)) = i_p(L(G))$.*

5 Partial domination chain

A chain of inequalities involving domination numbers, independence numbers and irredundance numbers of the form

$$ir(G) \leq \gamma(G) \leq i(G) \leq \beta(G) \leq \Gamma(G) \leq IR(G)$$

was first observed in 1978 (see [5]). This type of chain was observed in the case of many other domination parameters like α-domination [7] and k-dependent domination [8]. Also one of the open questions posed by Case *et al.* in [3] was to find out, what relationship the above parameters have amongst themselves in the context of partial domination. Hence we try to establish a similar kind of chain involving partial domination and partial independence parameters. Having already defined independent partial domination, we now define partial independence number of a graph.

Definition 5.1. *Suppose $G = (V, E)$ is a graph and $p \in (0, 1]$. A set S of independent vertices is called a p-independent set in G if $N[S] \subseteq V(H)$ for some induced subgraph H of G with $|V(H)| \geq np$. A p-independent set S is said to be p-maximal if S is a maximal independent set in $V(H)$. A maximal p-independent set S is said to be min-max p-independent set if $T \subset S$ is not p-maximal. Partial independence number or p-independence number is the maximum cardinality of a min-max p-independent set and is denoted by $\beta_p(G)$ and the associated induced subgraph H is denoted by H_p.*

For the graph in Figure 1, the set of all white vertices form a $\beta_{\frac{4}{9}}$-set. For the same graph, the set $\{v_5, v_8\}$ is a min-max $\frac{4}{9}$-independent set, but it is not of maximum cardinality and hence is not a $\beta_{\frac{4}{9}}$-set.

5.1 Partial independent sets

This section explores some of the properties of partial independent sets, thereby proceeding towards the suggested partial domination chain.

In light of the above definition, it may be noted that, for every maximal p-independent set S, $N[S] = V(H)$ of the proposed induced subgraph H of G and hence S is a p-dominating set. Thus independent p-domination number is the minimum cardinality of a maximal p-independent set and we have the following inequality.

Proposition 5.2. *For $p \in (0, 1]$, $\gamma_p(G) \leq i_p(G) \leq \beta_p(G)$.*

Proposition 5.3. *If $p_1 \leq p_2$, then $\beta_{p_1}(G) \leq \beta_{p_2}(G)$.*

Proof. Let $S \subseteq V$ be such that $|S| = \beta_{p_2}(G)$. Then S is also maximal p_1-independent set. Also the cardinality of every min-max p_1-independent set $\leq |S|$. Thus $\beta_{p_1}(G) \leq \beta_{p_2}(G)$. $\square$

We now proceed to relate partial independence number β_p with that of upper p-domination number, Γ_p which is the maximum cardinality of a minimal p-dominating set.

Proposition 5.4. *Every min-max p-independent set is a minimal p-dominating set.*

Proof. Let S be a min-max p-independent set and H_p be an induced subgraph associated with it. Then by definition, S is p-dominating in G.

Suppose S is not minimal p-dominating. Then $\exists\, u \in S$ such that $S - \{u\}$ is p-dominating. Then $\exists\, v \in S - \{u\}$, such that $uv \in E(H)$ which is a contradiction since S is an independent set.

Thus S is a minimal p-dominating set. $\hspace{2cm}\square$

Corollary 5.5. *For $p \in (0, 1]$, $\beta_p(G) \leq \Gamma_p(G)$.*

From Proposition 5.2 and Corollary 5.5 we obtain the following chain of inequalities:

For $p \in (0, 1]$, $\gamma_p(G) \leq i_p(G) \leq \beta_p(G) \leq \Gamma_p(G)$.

We present some more properties of independent sets, which in turn lead us to a method, by which one can deduce β_p-sets for some 'p' values from the existing β-set of a graph.

Lemma 5.6. *Suppose S is a β-set of a graph G and $T \subset S$. Then T is a min-max $\frac{|N[T]|}{n}$-independent set.*

Proof. By definition T is a maximal $\frac{|N[T]|}{n}$-independent set. It is also min-max since $R \subset T$ is not $\frac{|N[T]|}{n}$ maximal. Suppose R is maximal then $(S-T) \cup R$ is a dominating set which is a contradiction as S is a minimal dominating set of G. $\hspace{1cm}\square$

Theorem 5.7. *Let $\mathcal{B}_i$ denote the set of all i-element subsets of a β-set of a graph G for $1 \leq i \leq \beta(G)$. Let $B_i \in \mathcal{B}_i$ be such that $|N[B_i]| = min\{|N[X]|/X \in \mathcal{B}_i\}$. Then*

   (i) B_i is a β_p-set for $p = \frac{|N[B_i]|}{n}$.

   (ii) For $0 < p \leq \frac{|N[B_1]|}{n}$, $\beta_p = 1$ and B_1 is a β_p-set.

Proof. For $1 \leq i \leq \beta(G)$ let B_i be chosen by the given method. By the previous Lemma (5.6) B_i is a min-max $\frac{|N[B_i]|}{n}$ independent set. Suppose B_i is not of maximum cardinality amongst all $\frac{|N[B_i]|}{n}$ independent sets, then for $j > i$ there exists a $Y \in \mathcal{B}_j$ such that Y is a min-max $\frac{|N[B_i]|}{n}$ independent set. Also Y is min-max $\frac{|N[Y]|}{n}$ independent set and thus Y is a maximal independent set in both $< N[B_i] >$ and $< N[Y] >$ and also $|N[B_i]| = |N[Y]|$. But by the definition of B_is, $|N[Y]| \geq |N[B_j]|$ which implies that $|N[B_j]| \leq |N[B_i]|$ which is a contradiction since for $j > i$, $|N[B_j]| > |N[B_i]|$.

Suppose $|N[B_j]| \leq |N[B_i]|$ for some $j > i$, choose R such that $R \subset B_j$ and $|R| = i$. Then $|N[R]| < |N[B_j]|$ which implies that $|N[R]| < |N[B_i]|$ by our assumption. This contradicts our definition of B_i. □

6   Conclusion

Partial domination has a lot to promise. One of the striking features of the concept of partial domination is its nature of accommodation. Domination with conditions are studied extensively. In the case of partial domination, the imperfect situations are addressed. Hence, it is worth exploring the partial domination in all the numerous types of dominations. In this context we could establish the partial domination chain. Future beckons with great hope of the explorations of partial domination in the areas of distance domination, stratified domination, Roman domination etc., but not exclusively.

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