On the periodic solutions for some retarded partial differential equations by the use of semi-Fredholm operators

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ABSTRACT

The main goal of this work is to examine the periodic dynamic behavior of some retarded periodic partial differential equations (PDE). Taking into consideration that the linear part realizes the Hille-Yosida condition, we discuss the Massera's problem to this class of equations. Especially, we use the perturbation theory of semi-Fredholm operators and the Chow and Hale's fixed point theorem to study the relation between the boundedness and the periodicity of solutions for some inhomogeneous linear retarded PDE. An example is also given at the end of this work to show the applicability of our theoretical results.

RESUMEN

El principal objetivo de este trabajo es examinar el comportamiento dinámico periódico de algunas ecuaciones diferenciales parciales (EDP) periódicas con retardo. Tomando en consideración que la parte lineal cumple la condición de Hille-Yosida, discutimos el problema de Massera para esta clase de ecuaciones. Especialmente usamos la teoría de perturbaciones de operadores semi-Fredholm y el teorema de punto fijo de Chow y Hale para estudiar la relación entre el acotamiento y la periodicidad de soluciones para algunas EDP no homogéneas lineales con retardo. Se entrega un ejemplo al final de este trabajo para mostrar la aplicabilidad de los resultados teóricos.

1 Introduction

Along this work, we investigate the periodicity of solutions of the following inhomogeneous linear retarded PDE

$$\begin{cases} \dfrac{d}{dt}y(t) &=& \mathcal{A}y(t) + L(y_t) + h(t) \qquad \text{for} \quad t \geq 0 \\ y(t) &=& \psi(t) \qquad\qquad\qquad\quad \text{for} \quad -r \leq t \leq 0. \end{cases} \tag{1.1}$$

We assume that the generator $\mathcal{A}$ is not necessarily dense on a Banach space $\mathbb{E}$ and realizes the following Hille-Yosida condition:

(i) there exist $\overline{M} \geq 1$, $\widehat{\omega} \in \mathbb{R}$ such that $(\widehat{\omega}, \infty) \subset \rho(\mathcal{A})$,

(ii) the operator $\mathcal{A}$ satisfies for $n \in \mathbb{N}$, $\lambda > \widehat{\omega}$, the following inequality

$$\mid (\lambda I - \mathcal{A})^{-n} \mid \quad \leq \frac{\overline{M}}{(\lambda - \widehat{\omega})^n},$$

where $\rho(\mathcal{A})$ is the resolvent set of $\mathcal{A}$. The history function $y_t : [-r, 0] \to \mathbb{E}$ defined for each $\theta \in [-r, 0]$ by $y_t(\theta) = y(t + \theta)$, belongs to $\mathcal{C}([-r, 0], \mathbb{E})$ the space of continuous functions equipped with the supremum norm. $L : \mathcal{C} \to \mathbb{E}$ is a linear bounded operator and h is a continuous function from $\mathbb{R}$ to $\mathbb{E}$.

Almost periodic and periodic solutions remain the most interesting subject in the qualitative analysis of PDE in view of their important applications in many real phenomena and fields. Recall that the concept of almost periodic is more general than the one of periodicity. It was introduced by Bochner and studied by many authors. For more details on almost periodic function we refer to [9, 16, 17, 18]. For the periodicity, there is an extensive literature related to this topic, see for example [10, 11, 25] for more details. Moreover, the choice of a suitable fixed point theorem is a fundamental tool to establish the periodicity of solutions for different classes of differential equations, in fact, to find a fixed point of the well known Poincaré map is equivalent to find the initial data of the periodic solution of the equation. After a long period of research and development, Massera's theorem [24] is the first result explaining the relation between the existence of bounded and periodic solutions for periodic differential equations. In finite dimensional spaces, several works have been developed on this subject. The authors in [4, 12] proved the periodicity of solutions when the solutions of periodic system are just bounded and ultimately bounded by the use of the Horn's fixed point theorem. Especially, in infinite dimensional spaces, the authors in [8], used the Poincaré map approach to get the periodicity of solutions for a class of retarded differential equation, they supposed that the infinitesimal generator satisfies the Hille-Yosida condition and generates a compact semigroup $(\mathcal{T}(t))_{t\geq 0}$ by applying an appropriate fixed point theorem. In [22], the authors proved the periodicity for a nonhomogeneous linear differential equation when the linear part generates a C_0-semigroup on $\mathbb{E}$ and they obtained the existence and uniqueness of periodic solutions for this class of equations.

The present work would be a continuation and extension of the work [8] for inhomogeneous linear retarded PDE, we establish the periodicity of solution for Equation (1.1) by using the perturbation theory of semi-Fredholm operators and without considering the compactness condition of $(\mathcal{T}(t))_{t\geq 0}$. To achieve this goal, we suppose that Equation (1.1) admits a bounded solution on the positive real line and under suitable estimations on the norm of the operator L, we derive periodic solution of Equation (1.1) from bounded ones on the positive real line by using the perturbation theory of semi-Fredholm operators and the Chow and Hale's fixed point theorem.

This work is treated as follows, in Section 2, we give some definitions and results about integral solutions of Equation (1.1). Moreover, we give some definitions and properties concerning the semi-Fredholm operators. Section 3 is devoted to prove and introduce some useful estimations on the integral solutions of Equation (1.1). In Section 4, we discuss the problem of existence of periodic solutions of Equation (1.1). Finally, we apply our theoretical results to an equation appearing in physical systems.

2 Preliminary results

In this article, we assume that:

$(\mathbf{H_0})$ $\mathcal{A}$ satisfies the Hille-Yosida condition.

We consider the following definition and results.

Definition 2.1 ([1])**.** *A continuous function $y : [0, +\infty) \to \mathbb{E}$ is said to be an integral solution of Equation (1.1) if the following conditions hold:*

(i) $y : [0, +\infty) \to \mathbb{E}$ is continuous, such that $y_0 = \psi$,

(ii) $\displaystyle\int_0^t y(s)\, ds \in D(\mathcal{A})$ for $t \geq 0$,

(iii) $y(t) = \psi(0) + \mathcal{A} \displaystyle\int_0^t y(s)\, ds + \int_0^t (L(y_s) + h(s))\, ds$ for $t \geq 0$.

We can deduce from the continuity of the integral solution y that $y(t) \in \overline{D(\mathcal{A})}$, for all $t \geq 0$. Moreover $\psi(0) \in \overline{D(\mathcal{A})}$. In the next we define the part $\mathcal{A}_0$ of the operator $\mathcal{A}$ in $\overline{D(\mathcal{A})}$ as follows

$$D(\mathcal{A}_0) = \{y \in D(\mathcal{A}) : \mathcal{A}y \in \overline{D(\mathcal{A})}\},$$

and

$$\mathcal{A}_0 y = \mathcal{A}y \qquad \text{for} \quad y \in D(\mathcal{A}_0).$$

Lemma 2.2 ([2])**.** *The operator $\mathcal{A}_0$ is the infinitesimal generator of a strongly continuous semi-group denoted by $(\mathcal{T}_0(t))_{t\geq 0}$ on $\overline{D(\mathcal{A})}$.*

Theorem 2.3 ([1]). *Under the assumption* ($\mathbf{H_0}$). *For all* $\psi \in \mathcal{C}$ *such that* $\psi(0) \in \overline{D(\mathcal{A})}$, *Equation (1.1) has a unique integral solution* $y(.)$ *on* $[-r, +\infty)$. *Furthermore,* $y(.)$ *is given by*

$$y(t) = \mathcal{T}_0(t)\psi(0) + \lim_{\lambda \to +\infty} \int_0^t \mathcal{T}_0(t-s)\lambda R(\lambda, \mathcal{A}) \left(L(y_s) + h(s)\right) ds \qquad for \quad t \geq 0.$$

Through this work, the integral solutions of Equation (1.1) are called plainly solutions. Let $y(., \psi, L, h)$ be the solution of Equation (1.1).

We define $\mathcal{C}_0$ the phase space of Equation (1.1) as $\mathcal{C}_0 = \{\psi \in \mathcal{C} : \psi(0) \in \overline{D(\mathcal{A})}\}$.

Let $\mathcal{X}(t)$ be the linear operator defined on $\mathcal{C}_0$ for each $t \geq 0$, by

$$\mathcal{X}(t)\psi = y_t(., \psi, 0, 0),$$

where $y_t(., \psi, 0, 0)$ is the solution of the following equation

$$\begin{cases} \dfrac{d}{dt}y(t) & = & Ay(t) \quad for \quad t \geq 0, \\ y_0 & = & \psi. \end{cases}$$

Theorem 2.4 ([1]). $(\mathcal{X}(t))_{t \geq 0}$ *is a linear strongly continuous semigroup on* $\mathcal{C}_0$:

(i) *for all* $t \geq 0$, $\mathcal{X}(t)$ *is a bounded linear operator on* $\mathcal{C}_0$ *such that* $\mathcal{X}(0) = I$ *and* $\mathcal{X}(t+s) = \mathcal{X}(t)\mathcal{X}(s)$ *for all* $t, s \geq 0$,

(ii) *for* $t \geq 0$ *and* $\theta \in [-r, 0]$, $(\mathcal{X}(t))_{t \geq 0}$ *satisfies:*

$$[\mathcal{X}(t)\phi](\theta) = \begin{cases} [\mathcal{X}(t+\theta)\phi](0), & if \quad t+\theta \geq 0, \\ \phi(t+\theta), & if \quad t+\theta \leq 0. \end{cases}$$

(iii) *for all* $\psi \in \mathcal{C}_0$ *and* $t \geq 0$, $\mathcal{X}(t)\psi$ *is a continuous function with values in* $\mathcal{C}_0$.

Theorem 2.5 ([1]). *Under the assumption* ($\mathbf{H_0}$). *The solution* $\mathcal{Y}(t)\psi = y_t(., \psi, L, 0)$ *of Equation (1.1) with* $h = 0$ *can be decomposed as follows:*

$$\mathcal{Y}(t)\psi = \mathcal{X}(t)\psi + \mathcal{Z}(t)\psi \qquad for \quad t \geq 0,$$

where $\mathcal{Z}(t)$, *is a bounded linear operator defined on* $\mathcal{C}_0$, *by*

$$[\mathcal{Z}(t)\psi](\theta) = \begin{cases} \displaystyle\lim_{\lambda \to +\infty} \int_0^{t+\theta} \mathcal{T}_0(t+\theta-s)\lambda R(\lambda, \mathcal{A})L(y_s(\psi)) ds & t+\theta \geq 0, \\ 0 & t+\theta \leq 0. \end{cases} \qquad (2.1)$$

for each $t \geq 0$.

To discuss the existence of periodic solutions, we use the theory of semi-Fredholm operators. We consider some definitions and propositions which are taken from [21].

Definition 2.6. *Let $\mathbb{E}$, $\mathbb{F}$ be two Banach spaces. A bounded linear operator F from $\mathbb{E}$ to $\mathbb{F}$, denoted by $F \in \mathcal{L}(\mathbb{E}, \mathbb{F})$, is said to be semi-Fredholm from $\mathbb{E}$ to $\mathbb{F}$ if*

(i) $\dim \ker(F) < \infty$, *where* $\ker(F)$ *is the null space.*

(ii) $Im(F)$ *the range of F is closed in $\mathbb{F}$.*

We designate by $\Phi_+(\mathbb{E}, \mathbb{F})$ the set of all semi-Fredholm operators and $\Phi_+(\mathbb{E}) = \Phi_+(\mathbb{E}, \mathbb{E})$. Now we recall some well known theorems for the closed range operators. Let $F \in \mathcal{B}(\mathbb{E}, \mathbb{F})$. Then the quotient space $\mathbb{E}/\ker(F)$ is a Banach space equipped with the norm

$$||[x]|| = \inf\{|x + m| : m \in \ker(F)\},$$

where

$$[x] = x + \ker(F) := \{x + m : m \in \ker(F)\}.$$

Furthermore, if $\dim \ker(F) < \infty$, then there exists a closed subspace M of $\mathbb{E}$ such that

$$\mathbb{E} = \ker(F) \oplus M.$$

Theorem 2.7 ([21]). *Let F be a bounded linear operator in $\mathbb{E}$. Then, $Im(F)$ is closed if and only if there exists a constant $\widetilde{c}$ such that*

$$||[x]|| \leq \widetilde{c}\,||Fx|| \quad \text{for all} \quad x \in \mathbb{E}.$$

Theorem 2.8 ([21]). *Let F be a bounded linear operator in $\mathbb{E}$ such that $\dim \ker(F) < \infty$. Then, the following assertions are equivalent.*

(i) $F \in \Phi_+(\mathbb{E})$.

(ii) *There exists a constant $\widetilde{c}$ such that*

$$||[x]|| \leq \widetilde{c}\,||Fx|| \quad \text{for all} \quad x \in \mathbb{E}.$$

(iii) *There exists a constant b such that*

$$||(I - P)x|| \leq b\,||Fx|| \quad \text{for all} \quad x \in \mathbb{E},$$

where P is the projection operator onto $\ker(F)$ along M.

We present now a result for bounded perturbation of Semi-Fredholm operators.

Theorem 2.9 ([21]). *Let F be an operator in $\Phi_+(\mathbb{E}, \mathbb{F})$. If $S \in \mathbf{L}(\mathbb{E})$ satisfying*

$$||S|| < \frac{1}{2b},$$

where b is the constant given in Theorem 2.8. Then,

$$F + S \in \Phi_+(\mathbb{E}, \mathbb{F}) \quad \text{with} \quad \dim \ker(F + S) \leq \dim \ker(F).$$

Now, we need to introduce some well known definitions and results about the spectral theory. Let $\mathcal{J}$ be a linear bounded operator on $\mathbb{F}$, we define the measure of Kuratowski of noncompactness of the operator $\mathcal{J}$ as follows

$$|\mathcal{J}|_\alpha = \inf\{\epsilon > 0 : \alpha(\mathcal{J}(B)) \leq \epsilon\alpha(B), \text{ for every bounded set } B \subset \mathbb{F}\},$$

where $\alpha(.)$ is the measure of Kuratowski of noncompactness of bounded sets $B \subset \mathbb{F}$ defined by

$$\alpha(J) = \inf\{\epsilon > 0 : B \text{ has a finite cover of ball of diameter } < \epsilon\}.$$

The essential radius $r_{ess}(\mathcal{J})$ is characterized by the following Nussbaum Formula introduced in [19]:

$$r_{ess}(\mathcal{J}) = \lim_{n\to+\infty} [|\mathcal{J}^n|_\alpha]^{1/n}.$$

Moreover, if $\mathcal{J}$ is bounded and $r_{ess}(\mathcal{J}) < 1$, then $I - \mathcal{J} \in \Phi_+(\mathbb{F})$.

Let us define the essential growth bound of a strongly continuous semigroup $\mathcal{S} := (\mathcal{S}(t))_{t\geq 0}$ on a Banach space $\mathbb{F}$ as

$$\omega_{ess}(\mathcal{S}) := \lim_{t\to+\infty} \frac{1}{t} \log |\mathcal{S}(t)|_\alpha.$$

It is well know that

$$r_{ess}(\mathcal{S}(t)) = \exp(t\omega_{ess}(\mathcal{S})), \quad t > 0.$$

3   Several estimates

Before discussing the periodicity of solution of Equation (1.1), we need some preparatory estimates.

Proposition 3.1. *Suppose that* $|\mathcal{T}_0(t)| \leq M_0\, e^{\omega_0 t}$ *for* $t \geq 0$. *Then*

$$\|\mathcal{Z}(t)\|_\mathcal{C} \leq M_0\, e^{\omega_0^+ t}(e^{M_0|L|\overline{M}\, t} - 1) \text{ for } t \geq 0,$$

where $\omega_0^+ = \max\{\omega_0, 0\}$.

To prove the above Proposition, we need to introduce the following Lemma.

Lemma 3.2. *Let* $|\mathcal{T}_0(t)| \leq M_0 e^{\omega_0 t}$ *for* $t \geq 0$. *Then, the solution of Equation (1.1) in the case where* $h = 0$ *is estimated as*

$$|\mathcal{Y}(t)| \leq M_0\, e^{(\omega_0^+ + M_0|L|\overline{M})t}.$$

Proof. For $t \geq 0$, $\theta \in [-r, 0]$, one has

$$\|\mathcal{Y}(t)\psi\|_\mathcal{C} = \sup_{\theta\in[-r,0]} |y(t+\theta, \psi)| = \sup_{\xi\in[t-r,t]} |y(\xi, \psi)|.$$

Then for $t \geq r$,

$$
\begin{aligned}
\sup_{\xi \in [t-r,t]} |y(\xi, \psi)| & \leq \sup_{\xi \in [t-r,t]} |\mathcal{T}_0(\xi)\psi(0)| + \sup_{\xi \in [t-r,t]} \left| \lim_{\lambda \to +\infty} \lambda \int_0^\xi \mathcal{T}_0(\xi - s) R(\lambda, \mathcal{A}) L(\mathcal{Y}(s)\psi) \, ds \right| \\
& \leq M_0 e^{\omega_0^+ t} \|\psi\|_{\mathcal{C}} + M_0 |L| \overline{M} \left(\int_0^t e^{\omega_0^+ (t-s)} \|\mathcal{Y}(s)\psi\|_{\mathcal{C}} \, ds \right).
\end{aligned}
$$

For $t < r$,

$$
\begin{aligned}
\sup_{\xi \in [t-r,t]} |y(\xi, \psi)| & = \max \left\{ \sup_{\xi \in [t-r,0]} |y(\xi, \psi)|; \sup_{\xi \in [0,t]} |y(\xi, \psi)| \right\} \\
& \leq \max \left\{ \|\psi\|_{\mathcal{C}}; \sup_{\xi \in [0,t]} |y(\xi, \psi)| \right\},
\end{aligned}
$$

and

$$
\sup_{\xi \in [0,t]} |y(\xi, \psi)| \leq M_0 e^{\omega_0^+ t} \|\psi\|_{\mathcal{C}} + M_0 |L| \overline{M} \left(\int_0^t e^{\omega_0^+ (t-s)} \|\mathcal{Y}(s)\psi\|_{\mathcal{C}} ds \right).
$$

Finally, we obtain that

$$
\|\mathcal{Y}(t)\psi\|_{\mathcal{C}} \leq M_0 e^{\omega_0^+ t} |\psi|_{\mathcal{C}} + M_0 |L| \overline{M} \int_0^t e^{\omega_0^+ (t-s)} \|\mathcal{Y}(s)\psi\|_{\mathcal{C}} \, ds.
$$

Hence,

$$
\|e^{-\omega_0^+ t} \mathcal{Y}(t)\psi\|_{\mathcal{C}} \leq M_0 \|\psi\|_{\mathcal{C}} + M_0 |L| \overline{M} \int_0^t e^{-\omega_0^+ s} \|\mathcal{Y}(s)\psi\|_{\mathcal{C}} \, ds.
$$

Gronwall's inequality implies that

$$
\|e^{-\omega_0^+ t} \mathcal{Y}(t)\psi\|_{\mathcal{C}} \leq M_0 e^{M_0 |L| \overline{M} t} \|\psi\|_{\mathcal{C}},
$$

and then

$$
\|\mathcal{Y}(t)\psi\|_{\mathcal{C}} \leq M_0 e^{(\omega_0^+ + M_0 |L| \overline{M}) t} \|\psi\|_{\mathcal{C}} \qquad \qquad \square
$$

Proof of Proposition 3.1. Let $t \geq 0$, $t + \theta \geq 0$. Then

$$
\begin{aligned}
\|\mathcal{Z}(t)\psi\|_{\mathcal{C}} & = \sup_{\theta \in [-r,0]} |(\mathcal{Z}(t)\psi)(\theta)| \\
& \leq M_0 |L| \overline{M} \left(\int_0^t e^{\omega_0^+ (t-s)} \|\mathcal{Y}(s)\psi\|_{\mathcal{C}} \, ds \right).
\end{aligned}
$$

From Lemma 3.2, we have

$$
\begin{aligned}
\|\mathcal{Z}(t)\psi\|_{\mathcal{C}} & \leq M_0^2 |L| \overline{M} e^{\omega_0^+ t} \left(\int_0^t e^{M_0 |L| \overline{M} s} \, ds \right) \|\psi\|_{\mathcal{C}} \\
& \leq M_0 e^{\omega_0^+ t} (e^{M_0 |L| \overline{M} t} - 1) \|\psi\|_{\mathcal{C}}.
\end{aligned}
$$

This implies our inequality. $\qquad \square$

Proposition 3.3. *A function $\phi \in \ker(I - \mathcal{X}(\omega))$ if and only if $\phi(0) \in \ker(I - \mathcal{T}_0(\omega))$, furthermore*

$$
\dim \ker(I - \mathcal{X}(\omega)) = \dim \ker(I - \mathcal{T}_0(\omega)).
$$

Proof. Let $\phi \in \ker(I - \mathcal{X}(\omega))$. Then,

$$\mathcal{X}(\omega)\phi = \phi \quad \text{and} \quad (\mathcal{X}(\omega)\phi)(\theta) = \phi(\theta) \quad \text{for} \quad \theta \in [-r, 0].$$

Since

$$(\mathcal{X}(\omega)\phi)(0) = \mathcal{T}_0(\omega)\phi(0),$$

then

$$\phi(0) = \mathcal{T}_0(\omega)\phi(0),$$

and hence

$$\phi(0) \in \ker(I - \mathcal{T}_0(\omega)).$$

Conversely, let $x \in \ker(I - \mathcal{T}_0(\omega))$ and $\phi_n(\theta) = \mathcal{T}_0(n\omega + \theta)x$ for $n \geq [\frac{r}{\omega}] + 1$, where $[.]$ denotes the integer part. Then,

$$\mathcal{T}_0(t + \omega)x = \mathcal{T}_0(t)\mathcal{T}_0(\omega)x = \mathcal{T}_0(t)x \quad \text{for} \quad t \geq 0,$$

and $\phi_n(\theta)$ is independent of the integer n and then

$$\phi_n(\theta) = \mathcal{T}_0(\omega + \theta)x = \phi(\theta) \quad \text{and} \quad \phi(0) = x.$$

If $\omega + \theta \geq 0$, then

$$(\mathcal{X}(\omega)\phi)(\theta) = \mathcal{T}_0(\omega + \theta)\phi(0) = \mathcal{T}_0(\omega + \theta)x = \phi(\theta).$$

If $\omega + \theta \leq 0$, then

$$\begin{aligned}
(\mathcal{X}(\omega)\phi)(\theta) &= \phi(\omega + \theta) = \phi_n(\omega + \theta) \\
&= \mathcal{T}_0(\theta + \omega + n\omega)x = \mathcal{T}_0(n\omega + \theta)x \\
&= \phi_n(\theta) = \phi(\theta),
\end{aligned}$$

hence,

$$\mathcal{X}(\omega)\phi = \phi,$$

which implies that $\phi \in \ker(I - \mathcal{X}(\omega))$. Moreover, $\ker(I - \mathcal{T}_0(\omega))$ is mapped bijectively onto the space $\ker(I - \mathcal{X}(\omega))$. Therefore,

$$\dim \ker(I - \mathcal{X}(\omega)) = \dim \ker(I - \mathcal{T}_0(\omega)). \qquad \square$$

Let us define the constant m_ω by

$$m_\omega = \sup_{0 \leq t \leq \omega} |\mathcal{T}_0(t)|.$$

As it is shown in [22], the proof is omitted here, if $I - \mathcal{T}_0(\omega)$ is semi-Fredholm on $\overline{D(\mathcal{A})}$, then, the operator defined by

$$S_M := I - \mathcal{T}_0(\omega) : M \to Im(I - \mathcal{T}_0(\omega)),$$

is bijective, such that M is a subset of $\mathbb{E}$ and $\overline{D(\mathcal{A})}$ is decomposed as

$$\overline{D(\mathcal{A})} = \ker(I - \mathcal{T}_0(\omega)) \oplus M.$$

Let S_M^{-1} be the inverse operator of S_M and let $k \in \mathbb{N}^*$ such that

$$(k-1)\omega < r \leq k\omega.$$

We set

$$I_k = [-r, -(k-1)\omega) \quad \text{and} \quad I_p = [-p\omega, -(p-1)\omega) \quad \text{for} \quad p = 1, 2, \ldots, k-1 \quad \text{with} \quad k \geq 2.$$

Let $\mathcal{G}$ be a linear operator defined from $D(\mathcal{G})$ to $\mathcal{C}_0$ by

$$(\mathcal{G}\phi)(\theta) = \sum_{j=0}^{p-1} \phi(\theta + j\omega) + \mathcal{T}_0(\theta + p\omega)S_M^{-1}\phi(0) \quad \text{for} \quad \theta \in I_p,$$

with

$$D(\mathcal{G}) = \{\phi \in \mathcal{C}_0 : \phi(0) \in Im(I - \mathcal{T}_0(\omega))\}.$$

Clearly, for $\theta \in I_p, p = 1, 2, \ldots, k,$

$$\|\mathcal{G}\phi\|_{\mathcal{C}} = \sup_{\theta \in [-r,0]} |(\mathcal{G}\phi)(\theta)| \leq \sum_{j=0}^{p-1} \|\phi\|_{\mathcal{C}} + \sup_{s \in [0,\omega]} |\mathcal{T}_0(s)| \|S_M^{-1}\| |\phi(0)|.$$

Then

$$\|\mathcal{G}\phi\|_{\mathcal{C}} \leq \left(k + m_\omega |S_M^{-1}|\right) \|\phi\|_{\mathcal{C}}. \tag{3.1}$$

Consequently, we have the following Theorem.

Theorem 3.4. *$I - \mathcal{T}_0(\omega)$ is semi-Fredholm on $\overline{D(\mathcal{A})}$ if and only if $I - \mathcal{X}(\omega)$ is semi-Fredholm on $\mathcal{C}_0$.*

To prove Theorem 3.4, we need the following Lemma

Lemma 3.5 ([14])**.**

$$Im(I - \mathcal{X}(\omega)) = D(\mathcal{G}).$$

Proof of Theorem 3.4. Suppose that $Im(I - \mathcal{T}_0(\omega))$ is closed, let $\phi_n \in D(\mathcal{G})$ such that $\phi_n \to \phi$ as $n \to \infty$. Then

$$\phi_n(0) \in Im(I - \mathcal{T}_0(\omega)) \quad \text{and} \quad \phi_n(0) \to \phi(0) \in Im(I - \mathcal{T}_0(\omega)),$$

which implies that $\phi \in D(\mathcal{G})$ and $D(\mathcal{G})$ is closed. Lemma 3.5 implies that $Im(I - \mathcal{X}(\omega))$ is closed. Now, we suppose that $Im(I - \mathcal{X}(\omega))$ is closed and $x_n \in Im(I - \mathcal{T}_0(\omega))$ such that $x_n \to x$ as $n \to \infty$. Let $\phi_n, \phi \in \mathcal{C}_0$ be such that $\phi_n(\theta) = x_n$ and $\phi(\theta) = x$. It is clear that $\phi_n \to \phi$ as $n \to \infty$ and by Lemma 3.5 we have that $(\phi_n) \in Im(I - \mathcal{X}(\omega))$. Then $\phi \in Im(I - \mathcal{X}(\omega))$ and $\phi(0) = x \in Im(I - \mathcal{T}_0(\omega))$. Consequently, $Im(I - \mathcal{T}_0(\omega))$ is closed. Therefore, by the use of Proposition 3.3 we obtain the desired result. $\square$

In the nondensely defined case, we can prove the following result as in [22], the proof is omitted here.

Theorem 3.6. *Suppose that $I - \mathcal{T}_0(\omega)$ is semi-Fredholm on $\overline{D(\mathcal{A})}$ such that $\dim \ker(I - \mathcal{T}_0(\omega)) = n$. If*

$$|\mathcal{Z}(\omega)| < \frac{1}{2\widetilde{c}\,(1 + \sqrt{n})}.$$

Then,

$$I - \mathcal{Y}(\omega) \in \Phi_+(\mathcal{C}_0) \quad and \quad \dim \ker(I - \mathcal{Y}(\omega)) \leq n.$$

Proposition 3.7. *Suppose that $I - \mathcal{T}_0(\omega)$ is semi-Fredholm on $\overline{D(\mathcal{A})}$. If $|L|$ satisfies*

$$|L| < \frac{\log\left(\dfrac{e^{-\omega_0^+ \omega}}{2M_0\widetilde{c}\,(1 + \sqrt{n})} + 1\right)}{M_0\overline{M}\omega}. \tag{3.2}$$

Then,

$$I - \mathcal{Y}(\omega) \in \Phi_+(\mathcal{C}_0) \quad and \quad \dim \ker(I - \mathcal{Y}(\omega)) \leq n.$$

Proof. By the inequality (3.2), it follows that

$$M_0\, e^{\omega_0^+ \omega}(e^{M_0\overline{M}|L|\,\omega} - 1) < \frac{1}{2\widetilde{c}\,(1 + \sqrt{n})},$$

and by Proposition 3.1, one has

$$|\mathcal{Z}(\omega)| < \frac{1}{2\widetilde{c}\,(1 + \sqrt{n})}.$$

Theorem 3.6 gives that

$$I - \mathcal{Y}(\omega) \in \Phi_+(\mathcal{C}_0) \quad \text{and} \quad \dim \ker(I - \mathcal{Y}(\omega)) \leq n. \qquad \square$$

4   Periodic solutions for Equation (1.1)

To discuss the existence of periodic solutions of Equation (1.1), we introduce the following fixed point Theorem for a linear affine map T defined from $\mathbb{E}$ to $\mathbb{E}$ by

$$Tu = \overline{T}u + v \quad \text{for} \quad u \in \mathbb{E},$$

where $\overline{T} \in B(\mathbb{E})$ and $v \in \mathbb{E}$ is fixed. Let $\mathcal{F}_T$ be the set of all fixed points of the map T.

Theorem 4.1 ([5]). *Let T be a linear affine map on a Banach space $\mathbb{E}$ such that the range $Im(I - \overline{T})$ is closed. If there is an $u_0 \in \mathbb{E}$ such that $\{T^k u_0, k \in \mathbb{N}\}$ is bounded in $\mathbb{E}$, then $\mathcal{F}_T \neq \varnothing$.*

If there exists some $v \in \mathcal{F}_T$, then

$$\mathcal{F}_T = v + \ker(I - \overline{T}).$$

$\dim \mathcal{F}_T$ is defined as

$$\dim \mathcal{F}_T = \dim \ker(I - \overline{T}).$$

If $I - \overline{T} \in \Phi_+(\mathbb{X})$. Then, Theorem 4.1 is refined as follows

Theorem 4.2 ([22]). *Let T be a linear affine map on a Banach space $\mathbb{E}$. If $I - \overline{T} \in \Phi_+(\mathbb{E})$ and if there exists an $u_0 \in \mathbb{E}$ such that $\{T^k u_0, k \in \mathbb{N}\}$ is bounded, then $\mathcal{F}_T \neq \varnothing$ and $\dim \mathcal{F}_T$ is finite.*

Through the rest of this work, we suppose that

(**H$_1$**) h is an ω-periodic function.

Furthermore, by property $(\mathcal{R})$ we mean the following equivalence:

Equation (1.1) has an $\omega-$periodic solution if and only if it has a bounded one on the positive real line. Then, we have the following result.

Theorem 4.3. *Under assumptions (**H$_0$**) and (**H$_1$**). If $I - \mathcal{T}_0(\omega)$ is semi-Fredholm on $\overline{D(\mathcal{A})}$ and if the operator L satisfies the following estimate*

$$|L| < \frac{\log\left(\dfrac{e^{-\omega_0^+ \omega}}{2M_0 \widetilde{c}\,(1+\sqrt{n})} + 1\right)}{M_0 \overline{M} \omega},$$

where $\widetilde{c}$ and n are the constants given in Theorem 3.6. Then, Equation (1.1) satisfies the property $(\mathcal{R})$.

Proof. Let $y(.,\psi,h)$ be the solution of Equation (1.1). We introduce the Poincaré map $\mathcal{P}$ defined from $\mathcal{C}_0$ to $\mathcal{C}_0$ as follows

$$\mathcal{P}_\omega(\psi) = y_\omega(.,\psi,h),$$

Then,

$$\mathcal{P}_\omega \psi = y_\omega(.,\psi,0) + y_\omega(.,0,h),$$

and hence $\mathcal{P}_\omega$ is an affine map such that

$$\mathcal{P}_\omega \psi = \overline{P}\psi + \varphi,$$

with

$$\overline{P}\psi = y_\omega(.,\psi,0) \quad \text{and} \quad \varphi = y_\omega(.,0,h).$$

According to the second section, $\overline{P}$ is decomposed as

$$\overline{P} = \mathcal{X}(\omega) + \mathcal{Z}(\omega).$$

Moreover, Proposition 3.7 gives that

$$I - \overline{P} \in \Phi_+(\mathcal{C}_0).$$

Now, let $y(., \psi, h)$ denote the bounded solution of Equation (1.1) on $\mathbb{R}^+$. Thus, for each $n \in \mathbb{N}$, we have

$$\mathcal{P}_\omega^n \psi = y_{n\omega}(., \psi, h),$$

and then $(\mathcal{P}_\omega^n \psi)_{n \geq 0}$ is a bounded sequence in $\mathcal{C}_0$. All conditions of Theorem 4.2 are satisfied and then $\mathcal{F}_{\mathcal{P}_\omega} \neq \varnothing$, which yields an ω-periodic solution of Equation (1.1). □

Corollary 4.4. *Under assumptions* $(\mathbf{H_0})$ *and* $(\mathbf{H_1})$. *If* $I - \mathcal{T}_0(\omega)$ *is semi-Fredholm in* $\overline{D(\mathcal{A})}$ *and if* $|L|$ *satisfies the following inequality*

$$|L| < \frac{\log\left(\dfrac{e^{-\omega_0^+ \omega}}{2M_0\left(k + m_\omega |S_M^{-1}|\right)(1+\sqrt{n})} + 1\right)}{M_0 \overline{M} \omega}.$$

Then, Equation (1.1) satisfies the property $(\mathcal{R})$.

To establish the proof, we need the following Lemma.

Lemma 4.5 ([14]). *Suppose that* $I - \mathcal{T}_0(\omega)$ *is semi-Fredholm on* $\overline{D(\mathcal{A})}$. *If there exists a constant* $\widetilde{c} > 0$ *such that*

$$\|\mathcal{G}\psi\|_\mathcal{C} \leq \widetilde{c}\|\psi\|_\mathcal{C} \quad \text{for all} \quad \psi \in D(\mathcal{G}).$$

Then,

$$|[\varphi]| \leq \widetilde{c}\|(I - \mathcal{X}(\omega))\varphi\|_\mathcal{C} \quad \text{for all} \quad \varphi \in \mathcal{C}_0.$$

Proof of Corollary 4.4: Since,

$$|L| < \frac{\log\left(\dfrac{e^{-\omega_0^+ \omega}}{2M_0\left(k + m_\omega |S_M^{-1}|\right)(1+\sqrt{n})} + 1\right)}{M_0 \overline{M} \omega}.$$

it follows that,

$$(k + m_\omega |S_M^{-1}|)(e^{M_0 \overline{M} \omega |L|} - 1) < \frac{e^{-\omega_0^+ \omega}}{2M_0 (1+\sqrt{n})}.$$

Lemma 4.5 and estimation (3.1) implies that

$$\widetilde{c} \leq k + m_\omega |S_M^{-1}|,$$

and then

$$\widetilde{c}(e^{M_0 \overline{M} \omega |L|} - 1) < \frac{e^{-\omega_0^+ \omega}}{2M_0 (1+\sqrt{n})}.$$

Finally

$$|L| < \frac{\log\left(\dfrac{e^{-\omega_0^+ \omega}}{2M_0 \widetilde{c}(1+\sqrt{n})} + 1\right)}{M_0 \overline{M} \omega}.$$

Now, Theorem 4.3 shows that Equation (1.1) satisfies the property $(\mathcal{R})$. □

In the particular case where the semigroup $(\mathcal{T}_0(t))_{t \geq 0}$ is exponentially stable, we have the following Theorem.

Theorem 4.6. *Under assumptions* $(\mathbf{H_0})$ *and* $(\mathbf{H_1})$. *If the semigroup* $(\mathcal{T}_0(t))_{t \geq 0}$ *is exponentially stable and if the operator* L *satisfies the following inequality*

$$|L| < \frac{\log \left(\dfrac{1}{2M_0 \left(k + m_\omega |S_M^{-1}| \right)} + 1 \right)}{M_0 \overline{M} \omega}.$$

Then, Equation (1.1) satisfies the property $(\mathcal{R})$.

Proof. From the exponential stability of $(\mathcal{T}_0(t))_{t \geq 0}$, we have

$$\omega_{ess}(\mathcal{T}_0) = \lim_{t \to +\infty} \frac{1}{t} \log |\mathcal{T}_0(t)|_\alpha \leq \lim_{t \to +\infty} \frac{1}{t} \log |\mathcal{T}_0(t)| = -\omega_0 < 0.$$

Consequently,

$$r_{ess}(\mathcal{T}_0(\omega)) = \exp \left(\omega \omega_{ess}(\mathcal{T}_0) \right) < 1.$$

Which implies that $Im(I - \mathcal{T}_0(\omega))$ is closed. On the other hand, one has

$$|\mathcal{T}_0(\omega)^n| = |\mathcal{T}_0(n\omega)| \leq M_0 e^{-\omega_0 n \omega}$$

and

$$|\mathcal{T}_0(n\omega)|^{\frac{1}{n}} \leq M_0^{\frac{1}{n}} e^{-\omega_0 \omega},$$

which implies that the spectral radius is estimated as

$$r(\mathcal{T}_0(\omega)) = \lim_{n \to +\infty} |\mathcal{T}_0(\omega)^n|^{\frac{1}{n}} \leq e^{-\omega_0 \omega} \lim_{n \to +\infty} M_0^{\frac{1}{n}} < \lim_{n \to +\infty} M_0^{\frac{1}{n}} < 1.$$

Consequently

$$1 \notin \sigma(\mathcal{T}_0(\omega)) \quad \text{and} \quad n = \dim \ker(I - \mathcal{T}_0(\omega)) = 0.$$

All conditions of Corollary 4.4 are satisfied with $n = 0$. Then, Equation (1.1) satisfies the property $(\mathcal{R})$. $\square$

5 Application

In order to apply our theoretical results, we consider the following delayed partial differential equation:

$$\begin{cases} \dfrac{\partial}{\partial t} y(t, \zeta) & = \dfrac{\partial^2}{\partial \zeta^2} y(t, \zeta) - a y(t, \zeta) + b y(t - r, \zeta) + g(t, \zeta) & \text{for } t \in \mathbb{R}^+ \text{ and } \zeta \in \mathbb{R}, \\ y(\theta, \zeta) & = \psi_0(\theta, \zeta) & \text{for } \theta \in [-r, 0] \text{ and } \zeta \in \mathbb{R}, \end{cases} \quad (5.1)$$

where a and b are positive constants, $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\phi : [-r, 0] \times \mathbb{R} \to \mathbb{R}$ are continuous functions where $\phi(\theta, \zeta)$ has a finite limit at $\pm \infty$.

Note that Equation (5.1) can be written in the form of Equation (1.1). In fact: we set $\overline{\overline{\mathbb{R}}} :=$ $[-\infty, +\infty]$ and we say that $z \in \mathbf{C}^k(\overline{\overline{\mathbb{R}}})$ if $z \in \mathbf{C}^k(\mathbb{R})$ and all derivatives of z up to the order k have finite limits at $\pm\infty$. Then, the space of continuous functions on $\overline{\overline{\mathbb{R}}}$, denoted by $\mathbb{E} = \mathbf{C}(\overline{\overline{\mathbb{R}}})$, endowed with the norm

$$\|z\|_\infty = \sup_{-\infty < \zeta < +\infty} |z(\zeta)|$$

becomes a Banach space. we consider the linear operator Δ defined from $D(\Delta) \subset \mathbb{E}$ to $\mathbb{E}$ by

$$\left\{ \begin{array}{rcl} D(\Delta) & = & \left\{ z \in \mathbf{C}^2\left(\overline{\overline{\mathbb{R}}}\right) : \lim\limits_{\zeta \to \pm\infty} z(\zeta) = 0 \right\}, \\ \Delta z & = & z''. \end{array} \right.$$

Then, we have

Lemma 5.1 ([7]).

$$(0, +\infty) \subset \rho(\Delta)$$

and for each $\lambda > 0$

$$\left| (\lambda I - \Delta)^{-1} \right| \le \frac{1}{\lambda}.$$

Clearly

$$\overline{D(\Delta)} = \left\{ z \in \mathbf{C}\left(\overline{\overline{\mathbb{R}}}\right) : \lim_{\zeta \to \pm\infty} z(\zeta) = 0 \right\}.$$

We write the part Δ_0 of Δ in $\overline{D(\Delta)}$ as

$$\left\{ \begin{array}{rcl} D(\Delta_0) & = & \left\{ z \in \mathbf{C}^2\left(\overline{\overline{\mathbb{R}}}\right) : \lim\limits_{\zeta \to \pm\infty} z(\zeta) = \lim\limits_{\zeta \to \pm\infty} z''(\zeta) = 0 \right\}, \\ \Delta_0 z & = & z''. \end{array} \right.$$

Lemma 5.2 ([7]). *Δ_0 is the infinitesimal generator of a strongly continuous semigroup $(\mathcal{T}_{\Delta_0}(t))_{t \ge 0}$ on $\overline{D(\Delta)}$. Furthermore,*

$$\left| \mathcal{T}_{\Delta_0}(t) \right| \le 1 \quad for \quad t \ge 0.$$

Let $\mathcal{A} : D(\mathcal{A}) \subset \mathbb{E} \to \mathbb{E}$ defined by:

$$\left\{ \begin{array}{l} D(\mathcal{A}) = \left\{ z \in \mathbf{C}^2\left(\overline{\overline{\mathbb{R}}}\right) : \lim\limits_{\zeta \to \pm\infty} z(\zeta) = 0 \right\}, \\ \\ \mathcal{A}z = z'' - az. \end{array} \right.$$

By Lemma 5.1, it is clear that

Lemma 5.3.

$$(-a, +\infty) \subset \rho(\mathcal{A})$$

and for each $\lambda > -a$

$$\left| (\lambda I - \mathcal{A})^{-1} \right| \le \frac{1}{\lambda + a}.$$

Lemma 5.3 guarantees that the assumption $(\mathbf{H_0})$ is satisfied with $\widehat{\omega} = -a$ and $\overline{M} = 1$. Moreover,

$$\overline{D(\mathcal{A})} = \left\{ z \in \mathbf{C}\left(\mathbb{R}\right) : \lim_{\zeta \to \pm \infty} z(\zeta) = 0 \right\} \subsetneq \mathbb{E}.$$

Moreover, we write the part $\mathcal{A}_0$ of the linear operator $\mathcal{A}$ in $\overline{D(\mathcal{A})}$ as:

$$\begin{cases} D(\mathcal{A}_0) = \left\{ z \in \mathbf{C}^2\left(\mathbb{R}\right) : \lim_{\zeta \to \pm \infty} z(\zeta) = 0 = \lim_{\zeta \to \pm \infty} z''(\zeta) = 0 \right\}, \\ \mathcal{A}_0 z = z'' - az. \end{cases}$$

Lemma 5.4. *$\mathcal{A}_0$ is the infinitesimal generator of an exponentially stable continuous semigroup $(\mathcal{T}_0(t))_{t \geq 0}$ on $\overline{D(\mathcal{A})}$. Moreover, for $t \geq 0$, we have*

$$|\mathcal{T}_0(t)| \leq e^{-at}.$$

Consider the following notations:

$$\begin{cases} y(t)(\zeta) &= y(t, \zeta) \quad \text{for} \quad t \in \mathbb{R}^+, \quad \zeta \in \mathbb{R}, \\ \psi(\theta)(\zeta) &= \psi_0(\theta, \zeta) \quad \text{for} \quad \theta \in [-r, 0], \quad \zeta \in \mathbb{R}, \end{cases}$$

and define the function $L : \mathcal{C} \to \mathbb{E}$ as follows

$$L(\phi)(\zeta) = b\phi(-r)(\zeta) \quad \text{for} \quad \zeta \in \mathbb{R} \quad \text{and} \quad \phi \in \mathcal{C}.$$

$h : \mathbb{R} \longrightarrow \mathbb{E}$ is defined by

$$h(t)(\zeta) = g(t, \zeta) \quad \text{for} \quad t \in \mathbb{R} \quad \text{and} \quad \zeta \in \mathbb{R}.$$

Clearly, L is a linear bounded operator from $\mathcal{C}$ to $\mathbb{E}$. Then, Equation (5.1) can be written in $\mathbb{E}$ as follows

$$\begin{cases} \dfrac{d}{dt} y(t) &= \mathcal{A}y(t) + L(y_t) + h(t) \quad \text{for} \quad t \geq 0, \\ y_0 &= \psi \in \mathcal{C}. \end{cases} \tag{5.2}$$

We suppose that $\lim\limits_{\zeta \to \pm \infty} \psi_0(0, \zeta) = 0$, then Equation (5.2) has a unique integral solution y on $[-r, +\infty)$.

To get the periodicity of solutions of Equation (5.2), we suppose that

$(\mathbf{H_2})$ $b < a$.

Let $\rho = 1 + \dfrac{|h|_\infty}{a - b}$ where $|h|_\infty = \sup\limits_{0 \leq t \leq \omega} |h(t)|$. Then, we have

Lemma 5.5. *Under assumption $(\mathbf{H_2})$. For every $\psi \in \mathcal{C}$ such that $\|\psi\|_\mathcal{C} < \rho$, the solution of Equation (5.2) is bounded by ρ on $\mathbb{R}^+$.*

Proof. We proceed by contradiction. Let

$$t_* = \inf\{t > 0 : |y(t, \psi)| > \rho\}.$$

From the continuity of y, one has

$$|y(t_*, \psi)| = \rho,$$

and there is $\alpha > 0$, with

$$|y(t, \psi)| > \rho \quad \text{for each} \quad t \in (t_*, t_* + \alpha).$$

Applying the variation-of-constants formula for Equation (5.2) with the initial value ψ,

$$y(t) = \mathcal{T}_0(t)\psi(0) + \lim_{\lambda \to +\infty} \lambda \int_0^t \mathcal{T}_0(t-s)R(\lambda, \mathcal{A})\left(L(y_s) + h(s)\right) \, ds.$$

Then, for $t \geq 0$

$$|y(t_*, \psi)| \leq |\mathcal{T}_0(t_*)| \, |\psi(0)| + \int_0^{t_*} |\mathcal{T}_0(t_* - s)| \, (|L(y_s)| + |h(s)|) \, ds.$$

Since for $0 < s < t_*$, it follows that $-r \leq s - r \leq t_* - r < t_*$ and then

$$|L(y_s)| = b|y(s-r)| \leq b\,\rho,$$

hence

$$
\begin{aligned}
|y(t_*, \psi)| &\leq \rho e^{-at_*} + (b\,\rho + |h|_\infty) \int_0^{t_*} e^{-a(t_*-s)} \, ds \\
&\leq \rho e^{-at_*} + \frac{(1 - e^{-at_*})}{a}\,(b\,\rho + |h|_\infty).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
|y(t_*, \psi)| &\leq \rho e^{-at_*} + (b\rho + (a-b)(\rho-1))\frac{(1-e^{-at_*})}{a} \\
&\leq \rho e^{-at_*} + \left(\rho - 1 + \frac{b}{a}\right)(1 - e^{-at_*}) \\
&\leq \rho e^{-at_*} + \rho(1 - e^{-at_*}) \\
&\leq \rho,
\end{aligned}
$$

which contradicts the definition of t_*, and we deduce that

$$|y(t, \psi)| \leq \rho \qquad \text{for} \quad t \geq 0. \qquad\qquad \square$$

To discuss the periodicity of solutions of Equation (5.2), we assume that:

($\mathbf{H_3}$) h is an ω-periodic function in t.

Theorem 5.6. *Suppose that* ($\mathbf{H_2}$) *and* ($\mathbf{H_3}$) *hold true. If*

$$|L| < \omega^{-1} \log\left(\frac{(1 - e^{-a\omega})(1 + 2k) + 2}{2 + 2k(1 - e^{-a\omega})}\right),$$

then, Equation (5.2) has an ω-periodic solution.

Proof. Let m_ω be the constant defined by

$$m_\omega = \sup_{0 \leq t \leq \omega} |\mathcal{T}_0(t)|.$$

Then

$$m_\omega \leq \sup_{0 \leq t \leq \omega} e^{-at} = 1.$$

Moreover, since $|\mathcal{T}_0(\omega)| < 1$, one has

$$
\begin{aligned}
|S_M^{-1}| = |(I - \mathcal{T}_0(\omega))^{-1}| &\leq \frac{1}{I - |\mathcal{T}_0(\omega)|} \\
&\leq \frac{1}{1 - e^{-a\omega}}.
\end{aligned}
$$

Thus,

$$k + m_\omega |S_M^{-1}| \leq k + \frac{1}{1 - e^{-a\omega}},$$

and

$$
\begin{aligned}
|L| &< \omega^{-1} \log\left(\frac{(1 - e^{-a\omega})(1 + 2k) + 2}{2 + 2k(1 - e^{-a\omega})}\right) \\
&< \omega^{-1} \log\left(\frac{1}{2\left(k + m_\omega |S_M^{-1}|\right)} + 1\right).
\end{aligned}
$$

All condition of Theorem 4.6 are satisfied. Then, Lemma 5.5 implies that Equation (5.2) has an $\omega-$periodic solution. $\qquad\square$

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