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Quasi bi-slant submersions in contact geometry

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ABSTRACT

The aim of the paper is to introduce the concept of quasi bi-slant submersions from almost contact metric manifolds onto Riemannian manifolds as a generalization of semi-slant and hemi-slant submersions. We mainly focus on quasi bi-slant submersions from cosymplectic manifolds. We give some non-trivial examples and study the geometry of leaves of distributions which are involved in the definition of the submersion. Moreover, we find some conditions for such submersions to be integrable and totally geodesic.

RESUMEN

El objetivo de este artículo es introducir el concepto de submersiones cuasi bi-inclinadas desde variedades casi contacto métricas hacia variedades Riemannianas, como una generalización de submersiones semi-inclinadas y hemi-inclinadas. Principalmente nos enfocamos en submersiones cuasi bi-inclinadas desde variedades cosimplécticas. Damos algunos ejemplos no triviales y estudiamos la geometría de hojas de distribuciones que están involucradas en la definición de la submersión. Más aún, encontramos algunas condiciones para que estas submersiones sean integrables y totalmente geodésicas.

Keywords and Phrases: Riemannian submersion, semi-invariant submersion, bi-slant submersion, quasi bi-slant submersion, horizontal distribution.

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1 Introductions

In differential geometry, there are so many important applications of immersions and submersions *both in mathematics and in physics.* The properties of slant submersions became an interesting subject in differential geometry, both in complex geometry and in contact geometry.

In 1966 and 1967, the theory of Riemannian submersions was initiated by O'Neill [17] and Gray [11] respectively. Nowadays, *Riemannian submersions are of great interest not only in mathematics, but also in theoretical pyhsics, owing to their applications in the Yang-Mills theory, Kaluza-Klein theory, supergravity and superstring theories* (see [7, 8, 10, 13, 14]). In 1976, the almost complex type of Riemannian submersions was studied by Watson [29]. He also introduced almost Hermitian submersions between almost Hermitian manifolds requiring that such Riemannian submersions are almost complex maps. In 1985, D. Chinea [9] extended the notion of almost Hermitian submersion to several kinds of sub-classes of almost contact manifolds. In [4] and [5], there are so many important and interesting results about Riemannian and almost Hermitian submersions. In 2010, B. Şahin introduced anti invariant submersions from almost Hermitian manifolds onto Riemannian manifolds [25]. Inspired by B. Şahin's article, many geometers introduced several new types of Riemannian submersions in different ambient spaces such as semi-invariant submersion [21, 23], generic submersion [27], slant submersion [12, 22], hemi-slant submersion [28], semi-slant submersion [18], bi-slant submersion [26], quasi hemi-slant submersion [16], quasi bi-slant submersion [19, 20], conformal anti-invariant submersion [1], conformal slant submersion [2] and conformal semi-slant submersion [3, 15]. Also, these kinds of submersions were considered in different kinds of structures such as cosymplectic, Sasakian, Kenmotsu, nearly Kaehler, almost product, paracontact, etc. Recent developments in the theory of submersions can be found in the book [24]. Inspired from the good and interesting results of above studies, we introduce the notion of quasi bi-slant submersions from cosymplectic manifolds onto Riemannian manifolds.

The paper is organized as follows: In the second section, we gather some basic definitions related to quasi bi-slant Riemannian submersion. In the third section, we obtain some results on quasi bi-slant Riemannian submersions from a cosymplectic manifold onto a Riemannian manifold. We also study the geometry of the leaves of the distributions involved in the considered submersions and discuss their totally geodesicity. We obtain conditions for the fibres or the horizontal distribution to be totally geodesic. In the last section, we provide some examples for such submersions.

2 Preliminaries

An $n-$dimensional smooth manifold M is said to have an almost contact structure, if there exist on M, a tensor field ϕ of type $(1,1)$, a vector field ξ and $1-$form η such that:

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \; \eta \circ \phi = 0, \tag{2.1}$$

$$\eta(\xi) = 1. \tag{2.2}$$

There exists a Riemannian metric g on an almost contact manifold M satisfying the next conditions:

$$g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V), \tag{2.3}$$

$$g(U, \xi) = \eta(U), \tag{2.4}$$

where U, V are vector fields on M.

An almost contact structure (ϕ, ξ, η) is said to be normal if the almost complex structure J on the product manifold $M \times \mathbb{R}$ is given by

$$J\left(U, \; \alpha\frac{d}{dt}\right) = \left(\phi U - \alpha\xi, \; \eta(U)\frac{d}{dt}\right) \tag{2.5}$$

and α is the differentiable function on $M \times \mathbb{R}$ has no torsion, *i.e.*, J is integrable. The condition for normality in terms of ϕ, ξ, and η is $[\phi, \phi] + 2d\eta \otimes \xi = 0$ on M, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ. Finally, the fundamental $2-$form Φ is defined by $\Phi(U, V) = g(U, \phi V)$.

An almost contact metric manifold with almost contact structure (ϕ, ξ, η, g) is said to be cosymplectic if

$$(\nabla_U \phi)V = 0, \tag{2.6}$$

for any U, V on M.

It is both normal and closed and the structure equation of a cosymplectic manifold is given by

$$\nabla_U \xi = 0, \tag{2.7}$$

for any U on M, where ∇ denotes the Riemannian connection of the metric g on M.

Example 2.1 ([6]). $\mathbb{R}^{2n+1}$ *with Cartesian coordinates* $(x_i, y_i, z)(i = 1, \ldots, n)$ *and its usual contact form*

$$\eta = dz.$$

The characteristic vector field ξ *is given by* $\frac{\partial}{\partial z}$ *and its Riemannian metric* g *and tensor field* ϕ *are given by*

$$g = \sum_{i=1}^{n}((dx_i)^2 + (dy_i)^2) + (dz)^2, \qquad \phi = \begin{pmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad i = 1, \ldots, n.$$

This gives a cosymplectic manifold on $\mathbb{R}^{2n+1}$. The vector fields $e_i = \frac{\partial}{\partial y_i}$, $e_{n+i} = \frac{\partial}{\partial x_i}$, ξ form a ϕ-basis for the cosymplectic structure.

Before giving our definition, we recall the following definition:

Definition 2.2 ([28])**.** *Let M be an almost Hermitian manifold with Hermitian metric g_M and almost complex structure J, and let N be a Riemannian manifold with Riemannian metric g_N. A Riemannian submersion $f : (M, g_M, J) \to (N, g_N)$ is called a hemi-slant submersion if the vertical distribution $\ker f_*$ of f admits two orthogonal complementary distributions D^θ and $D^\perp$ such that D^θ is slant with angle θ and $D^\perp$ is anti-invariant, i.e, we have*

$$\ker f_* = D^\theta \oplus D^\perp.$$

In this case, the angle θ is called the hemi-slant angle of the submersion.

Definition 2.3. *Let $(M, \phi, \xi, \eta, g_M)$ be an almost contact metric manifold and (N, g_N) a Riemannian manifold. A Riemannian submersion*

$$f : (M, \phi, \xi, \eta, g_M) \to (N, g_N),$$

is called a quasi bi-slant submersion if there exist four mutually orthogonal distributions D, D_1, D_2 and $< \xi >$ such that

(i) $\ker f_* = D \oplus D_1 \oplus D_2 \oplus < \xi >$,

(ii) $\phi(D) = D$ *i.e., D is invariant,*

(iii) $\phi(D_1) \perp D_2$ *and* $\phi(D_2) \perp D_1$,

(iv) for any non-zero vector field $U \in (D_1)_p$, $p \in M$, the angle θ_1 between ϕU and $(D_1)_p$ is constant and independent of the choice of the point p and U in $(D_1)_p$,

(v) for any non-zero vector field $U \in (D_2)_q$, $q \in M$, the angle θ_2 between ϕU and $(D_2)_q$ is constant and independent of the choice of point q and U in $(D_2)_q$,

These angles θ_1 and θ_2 are called the slant angles of the submersion.

We easily observe that

(a) If $\dim D \neq 0$, $\dim D_1 = 0$ and $\dim D_2 = 0$, then f is an invariant submersion.

(b) If $\dim D \neq 0$, $\dim D_1 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$ and $\dim D_2 = 0$, then f is proper semi-slant submersion.

(c) If $\dim D = 0$, $\dim D_1 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$ and $\dim D_2 = 0$, then f is slant submersion with slant angle θ_1.

(d) If $\dim D = 0$, $\dim D_1 = 0$ and $\dim D_2 \neq 0$, $0 < \theta_2 < \frac{\pi}{2}$, then f is slant submersion with slant angle θ_2.

(e) If $\dim D = 0$, $\dim D_1 \neq 0$, $\theta_1 = \frac{\pi}{2}$ and $\dim D_2 = 0$, then f is an anti-invariant submersion.

(f) If $\dim D \neq 0$, $\dim D_1 \neq 0$, $\theta_1 = \frac{\pi}{2}$ and $\dim D_2 = 0$, then f is an semi-invariant submersion.

(g) If $\dim D = 0$, $\dim D_1 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$ and $\dim D_2 \neq 0$, $\theta_2 = \frac{\pi}{2}$, then f is a hemi-slant submersion.

(h) If $\dim D = 0$, $\dim D_1 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$ and $\dim D_2 \neq 0$, $0 < \theta_2 < \frac{\pi}{2}$, then f is a bi-slant submersion.

(i) If $\dim D \neq 0$, $\dim D_1 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$ and $\dim D_2 \neq 0$, $\theta_2 = \frac{\pi}{2}$, then we may call f is an quasi-hemi-slant submersion.

(j) If $\dim D \neq 0$, $\dim D_1 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$ and $\dim D_2 \neq 0$, $0 < \theta_2 < \frac{\pi}{2}$, then f is proper quasi bi-slant submersion.

Define O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$ by

$$\mathcal{A}_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F, \tag{2.8}$$

$$\mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F, \tag{2.9}$$

for any vector fields E, F on M, where ∇ is the Levi-Civita connection of g_M. It is easy to see that $\mathcal{T}_E$ and $\mathcal{A}_E$ are skew-symmetric operators on the tangent bundle of M reversing the vertical and the horizontal distributions.

From equations (2.8) and (2.9) we have

$$\nabla_U V = \mathcal{T}_U V + \mathcal{V}\nabla_U V, \tag{2.10}$$

$$\nabla_U X = \mathcal{T}_U X + \mathcal{H}\nabla_U X, \tag{2.11}$$

$$\nabla_X U = \mathcal{A}_X U + \mathcal{V}\nabla_X U, \tag{2.12}$$

$$\nabla_X Y = \mathcal{H}\nabla_X Y + \mathcal{A}_X Y, \tag{2.13}$$

for $U, V \in \Gamma(\ker f_*)$ and $X, Y \in \Gamma(\ker f_*)^{\perp}$, where $\mathcal{H}\nabla_U Y = \mathcal{A}_Y U$, if Y is basic. It is not difficult to observe that $\mathcal{T}$ acts on the fibers as the second fundamental form, while $\mathcal{A}$ acts on the horizontal distribution and measures the obstruction to the integrability of this distribution.

It is seen that for $q \in M$, $U \in \mathcal{V}_q$ and $X \in \mathcal{H}_q$ the linear operators

$$\mathcal{A}_X,\ \mathcal{T}_U : T_q M \to T_q M$$

are skew-symmetric, that is

$$g_M(\mathcal{A}_X E, F) = -g_M(E, \mathcal{A}_X F) \text{ and } g_M(\mathcal{T}_U E, F) = -g_M(E, \mathcal{T}_U F) \tag{2.14}$$

for each $E, F \in T_q M$. Since $\mathcal{T}_U$ is skew-symmetric, we observe that f has totally geodesic fibres if and only if $\mathcal{T} \equiv 0$.

Let $(M, \phi, \xi, \eta, g_M)$ be a cosymplectic manifold, (N, g_N) be a Riemannian manifold and $f : M \to N$ a smooth map. Then the second fundamental form of f is given by

$$(\nabla f_*)(Y, Z) = \nabla^f_Y f_* Z - f_*(\nabla_Y Z), \quad \text{for } Y, Z \in \Gamma(T_p M), \tag{2.15}$$

where we denote conveniently by ∇ the Levi-Civita connections of the metrics g_M and g_N and ∇^f is the pullback connection.

We recall that a differentiable map f between two Riemannian manifolds is totally geodesic if

$$(\nabla f_*)(Y, Z) = 0, \text{ for all } Y, Z \in \Gamma(TM).$$

A totally geodesic map is that it maps every geodesic in the total space into a geodesic in the base space in proportion to arc lengths.

3 Quasi bi-slant submersions

Let f be quasi bi-slant submersion from an almost contact metric manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then, we have

$$TM = \ker f_* \oplus (\ker f_*)^\perp. \tag{3.1}$$

Now, for any vector field $U \in \Gamma(\ker f_*)$, we put

$$U = PU + QU + RU + \eta(U)\xi, \tag{3.2}$$

where P, Q and R are projection morphisms of $\ker f_*$ onto D, D_1 and D_2, respectively. For any $U \in \Gamma(\ker f_*)$, we set

$$\phi U = \psi U + \omega U, \tag{3.3}$$

where $\psi U \in \Gamma(\ker f_*)$ and $\omega U \in \Gamma(\ker f_*)^\perp$.

Now, let U_1, U_2 and U_3 be vector fields in D, D_1 and D_2 respectively. Since D is invariant, *i.e.* $\phi D = D$, we get $\omega U_1 = 0$. For any $U_2 \in \Gamma(D_1)$ we get $\omega U_2 \in \Gamma(\omega D_1)$ and for any $U_3 \in \Gamma(D_2)$ we get $\omega U_3 \in \Gamma(\omega D_2)$, hence $\omega U_2 \oplus \omega U_3 \in \Gamma(\omega D_1 \oplus \omega D_2) \subseteq \Gamma(\ker f_*)^\perp$.

From equations (3.2) and (3.3), we have

$$\begin{aligned} \phi U &= \phi(PU) + \phi(QU) + \phi(RU), \\ &= \psi(PU) + \omega(PU) + \psi(QU) + \omega(QU) + \psi(RU) + \omega(RU). \end{aligned}$$

Since $\phi D = D$, we get $\omega PU = 0$.

Hence above equation reduces to

$$\phi U = \psi PU + \psi QU + \omega QU + \psi RU + \omega RU. \tag{3.4}$$

Thus we have the following decomposition according to equation (3.4)

$$\phi(\ker f_*) = (\psi D) \oplus (\psi D_1 \oplus \psi D_2) \oplus (\omega D_1 \oplus \omega D_2), \tag{3.5}$$

where $\oplus$ denotes orthogonal direct sum.

Further, let $U \in \Gamma(D_1)$ and $V \in \Gamma(D_2)$. Then

$$g_M(U, V) = 0.$$

From Definition 2.3 (iii), we have

$$g_M(\phi U, V) = g_M(U, \phi V) = 0.$$

Now, consider

$$g_M(\psi U, V) = g_M(\phi U - \omega U, V) = g_M(\phi U, V) = 0.$$

Similarly, we have

$$g_M(U, \psi V) = 0.$$

Let $W \in \Gamma(D)$ and $U \in \Gamma(D_1)$. Then we have

$$g_M(\psi U, W) = g_M(\phi U - \omega U, W) = g_M(\phi U, W) = -g(U, \phi W) = 0,$$

as D is invariant, *i.e.*, $\phi W \in \Gamma(D)$.

Similarly, for $W \in \Gamma(D)$ and $V \in \Gamma(D_2)$, we obtain

$$g_M(\psi V, W) = 0,$$

From above equations, we have

$$g_M(\psi U, \psi V) = 0,$$

and

$$g_M(\omega U, \omega V) = 0,$$

for all $U \in \Gamma(D_1)$ and $V \in \Gamma(D_2)$.

So, we can write

$$\psi D_1 \cap \psi D_2 = \{0\}, \quad \omega D_1 \cap \omega D_2 = \{0\}.$$

If $\theta_2 = \frac{\pi}{2}$, then $\psi R = 0$ and D_2 is anti-invariant, *i.e.*, $\phi(D_2) \subseteq (\ker f_*)^{\perp}$.

We also have

$$\phi(\ker f_*) = \psi D \oplus \psi D_1 \oplus \omega D_1 \oplus \omega D_2. \tag{3.6}$$

Since $\omega D_1 \subseteq (\ker f_*)^\perp$, $\omega D_2 \subseteq (\ker f_*)^\perp$. So we can write

$$(\ker f_*)^\perp = \omega D_1 \oplus \omega D_2 \oplus \mathcal{V},$$

where $\mathcal{V}$ is invariant and orthogonal complement of $(\omega D_1 \oplus \omega D_2)$ in $(\ker f_*)^\perp$.

Also for any non-zero vector field $W \in \Gamma(\ker f)^\perp$, we have

$$\phi W = BW + CW, \tag{3.7}$$

where $BW \in \Gamma(\ker f)$ and $CW \in \Gamma(\mathcal{V})$.

Lemma 3.1. *Let f be a quasi bi-slant submersion from an almost contact metric manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then, we have*

$$\psi^2 U + B\omega U = -U + \eta(U)\xi, \quad \omega\psi U + C\omega U = 0,$$

$$\omega BW + C^2 W = -W, \quad \psi BW + BCW = 0,$$

for all $U \in \Gamma(\ker f_*)$ and $W \in \Gamma(\ker f_*)^\perp$.

Lemma 3.2. *Let f be a quasi bi-slant submersion from an almost contact metric manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then, we have*

(i) $\psi^2 U = -(\cos^2\theta_1)U$,

(ii) $g_M(\psi U, \psi V) = \cos^2\theta_1 g_M(U, V)$,

(iii) $g_M(\omega U, \omega V) = \sin^2\theta_1 g_M(U, V)$,

for all $U, V \in \Gamma(D_1)$.

Lemma 3.3. *Let f be a quasi bi-slant submersion from an contact metric manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then, we have*

(i) $\psi^2 W = -(\cos^2\theta_2)W$,

(ii) $g_M(\psi W, \psi Z) = \cos^2\theta_2 g_M(W, Z)$,

(iii) $g_M(\omega W, \omega Z) = \sin^2\theta_2 g_M(W, Z)$,

for all $W, Z \in \Gamma(D_2)$.

Lemma 3.4. *Let f be a quasi bi-slant submersion from a cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then, we have*

$$\mathcal{V}\nabla_U \psi V + \mathcal{T}_U \omega V = \psi \mathcal{V}\nabla_U V + B\mathcal{T}_U V, \tag{3.8}$$

$$\mathcal{T}_U \psi V + \mathcal{H}\nabla_U \omega V = \omega \mathcal{V}\nabla_U V + C\mathcal{T}_U V, \tag{3.9}$$

$$\mathcal{V}\nabla_X BY + \mathcal{A}_X CY = \psi \mathcal{A}_X Y + B\mathcal{H}\nabla_X Y, \tag{3.10}$$

$$\mathcal{A}_X BY + \mathcal{H}\nabla_X CY = \omega \mathcal{A}_X Y + C\mathcal{H}\nabla_X Y, \tag{3.11}$$

$$\mathcal{V}\nabla_U BX + \mathcal{T}_U CX = \psi \mathcal{T}_U X + B\mathcal{H}\nabla_U X, \tag{3.12}$$

$$\mathcal{T}_U BX + \mathcal{H}\nabla_U CX = \omega \mathcal{T}_U X + C\mathcal{H}\nabla_U X, \tag{3.13}$$

$$\mathcal{V}\nabla_Y \psi U + \mathcal{A}_Y \omega U = B\mathcal{A}_Y U + \psi \mathcal{V}\nabla_Y U, \tag{3.14}$$

$$\mathcal{A}_Y \psi U + \mathcal{H}\nabla_Y \omega U = C\mathcal{A}_Y U + \omega \mathcal{V}\nabla_Y U, \tag{3.15}$$

for any $U, V \in \Gamma(\ker f_*)$ and $X, Y \in \Gamma(\ker f_*)^{\perp}$.

Now, we define

$$(\nabla_U \psi)V = \mathcal{V}\nabla_U \psi V - \psi \mathcal{V}\nabla_U V, \tag{3.16}$$

$$(\nabla_U \omega)V = \mathcal{H}\nabla_U \omega V - \omega \mathcal{V}\nabla_U V, \tag{3.17}$$

$$(\nabla_X C)Y = \mathcal{H}\nabla_X CY - C\mathcal{H}\nabla_X Y, \tag{3.18}$$

$$(\nabla_X B)Y = \mathcal{V}\nabla_X BY - B\mathcal{H}\nabla_X Y, \tag{3.19}$$

for any $U, V \in \Gamma(\ker f_*)$ and $X, Y \in \Gamma(\ker f_*)^{\perp}$.

Lemma 3.5. *Let f be a quasi bi-slant submersion from a cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then, we have*

$$(\nabla_U \psi)V = B\mathcal{T}_U V - \mathcal{T}_U \omega V,$$

$$(\nabla_U \omega)V = C\mathcal{T}_U V - \mathcal{T}_U \psi V,$$

$$(\nabla_X C)Y = \omega \mathcal{A}_X Y - \mathcal{A}_X BY,$$

$$(\nabla_X B)Y = \psi \mathcal{A}_X Y - \mathcal{A}_X CY,$$

for any vectors $U, V \in \Gamma(\ker f_*)$ and $X, Y \in \Gamma(\ker f_*)^{\perp}$.

The proofs of above Lemmas follow from straightforward computations, so we omit them.

If the tensors ψ and ω are parallel with respect to the linear connection ∇ on M respectively, then

$$B\mathcal{T}_U V = \mathcal{T}_U \omega V,$$

and

$$C\mathcal{T}_U V = \mathcal{T}_U \psi V,$$

for any $U, V \in \Gamma(TM)$.

Lemma 3.6. *Let f be a quasi bi-slant submersion from a cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then, we have*

(i) $g_M(\nabla_X Y, \xi) = 0$ *for all* $X, Y \in \Gamma(D \oplus D_1 \oplus D_2)$,

(ii) $g_M([X, Y], \xi) = 0$ *for all* $X, Y \in \Gamma(D \oplus D_1 \oplus D_2)$.

Proof. Let $X, Y \in \Gamma(D \oplus D_1 \oplus D_2)$, consider

$$\nabla_X\{g_M(Y, \xi)\} = (\nabla_X g_M)(Y, \xi) + g_M(\nabla_X Y, \xi) + g_M(Y, \nabla_X \xi).$$

Since X and Y are orthogonal to ξ ie.

$$g_M(\nabla_X Y, \xi) = -g_M(Y, \nabla_X \xi),$$

using equation (2.7) and the property that metric tensor is $\nabla-$parallel, we have both results of this lemma. ☐

Theorem 3.7. *Let f be a proper quasi bi-slant submersion from a cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then, the invariant distribution D is integrable if and only if*

$$g_M(\mathcal{T}_V \psi U - \mathcal{T}_U \psi V, \omega QW + \omega RW) = g_M(\mathcal{V}\nabla_U \psi V - \mathcal{V}\nabla_V \psi U, \psi QW + \psi RW), \tag{3.20}$$

for $U, V \in \Gamma(D)$ and $W \in \Gamma(D_1 \oplus D_2)$.

Proof. For $U, V \in \Gamma(D)$, and $W \in \Gamma(D_1 \oplus D_2)$, using equations (2.1)–(2.4), (2.6), (2.7), (2.10), (3.2), (3.3) and Lemma 3.6 we have

$$\begin{aligned} g_M([U, V], W) &= g_M(\nabla_U \phi V, \phi W) + \eta(W)\eta(\nabla_U V) - g_M(\nabla_V \phi U, \phi W) - \eta(W)\eta(\nabla_V U), \\ &= g_M(\nabla_U \psi V, \phi W) - g_M(\nabla_V \psi U, \phi W), \\ &= g_M(\mathcal{T}_U \psi V - \mathcal{T}_V \psi U, \omega QW + \omega RW) - g_M(\mathcal{V}\nabla_U \psi V - \mathcal{V}\nabla_V \psi U, \psi QW + \psi RW), \end{aligned}$$

which completes the proof. ☐

Theorem 3.8. *Let f be a proper quasi bi-slant submersion from a cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then, the slant distribution D_1 is integrable if and only if*

$$\begin{aligned} g_M(\mathcal{T}_W \omega\psi Z - \mathcal{T}_Z \omega\psi W, X) &= g_M(\mathcal{T}_W \omega Z - \mathcal{T}_Z \omega W, \phi PX + \psi RX) \\ &\quad + g_M(\mathcal{H}\nabla_W \omega Z - \mathcal{H}\nabla_Z \omega W, \omega RX), \end{aligned} \tag{3.21}$$

for all $W, Z \in \Gamma(D_1)$ and $X \in \Gamma(D \oplus D_2)$.

Proof. For all $W, Z \in \Gamma(D_1)$ and $X \in \Gamma(D \oplus D_2)$, we have

$$g_M([W,Z],X) = g_M(\nabla_W Z, X) - g_M(\nabla_Z W, X).$$

Using equations (2.1)–(2.4), (2.6), (2.7), (2.11), (3.2), (3.3) and Lemma 3.2 we have

$$\begin{aligned}
g_M([W,Z],X) &= g_M(\nabla_W \phi Z, \phi X) - g_M(\nabla_Z \phi W, \phi X),\\
&= g_M(\nabla_W \psi Z, \phi X) + g_M(\nabla_W \omega Z, \phi X) - g_M(\nabla_Z \psi W, \phi X) - g_M(\nabla_W \omega Z, \phi X),\\
&= \cos^2\theta_1 g_M(\nabla_W Z, X) - \cos^2\theta_1 g_M(\nabla_Z W, X) - g_M(\mathcal{T}_W \omega\psi Z - \mathcal{T}_Z \omega\psi W, X)\\
&\quad + g_M(\mathcal{H}\nabla_W \omega Z + \mathcal{T}_W \omega Z, \phi PX + \psi RX + \omega RX)\\
&\quad - g_M(\mathcal{H}\nabla_Z \omega W + \mathcal{T}_Z \omega W, \phi PX + \psi RX + \omega RX).
\end{aligned}$$

Now, we obtain

$$\begin{aligned}
\sin^2\theta_1 g_M([W,Z],X) &= g_M(\mathcal{T}_W \omega Z - \mathcal{T}_Z \omega W, \phi PX + \psi RX) + g_M(\mathcal{H}\nabla_W \omega Z - \mathcal{H}\nabla_Z \omega W, \omega RX)\\
&\quad - g_M(\mathcal{T}_W \omega\psi Z - \mathcal{T}_Z \omega\psi W, X),
\end{aligned}$$

which completes the proof. □

Theorem 3.9. *Let f be a proper quasi bi-slant submersion from a cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then, the slant distribution D_2 is integrable if and only if*

$$\begin{aligned}
g_M(\mathcal{T}_U \omega\psi V - \mathcal{T}_V \omega\psi U, Y) &= g_M(\mathcal{H}\nabla_U \omega V - \mathcal{H}\nabla_V \omega U, \omega QY)\\
&\quad + g_M(\mathcal{T}_U \omega V - \mathcal{T}_V \omega U, \phi PY + \psi QY), \qquad (3.22)
\end{aligned}$$

for all $U, V \in \Gamma(D_2)$ and $Y \in \Gamma(D \oplus D_1)$.

Proof. For all $U, V \in \Gamma(D_2)$ and $Y \in \Gamma(D \oplus D_1)$, using equations (2.1)–(2.4), (2.6), (2.7), (3.3) and Lemma 3.6 we have

$$g_M([U,V],Y) = g_M(\nabla_U \psi V, \phi Y) + g_M(\nabla_U \omega V, \phi Y) - g_M(\nabla_V \psi U, \phi Y) - g_M(\nabla_V \omega U, \phi Y).$$

From equations (2.9), (3.2) and Lemma 3.3 we have

$$\begin{aligned}
g_M([U,V],Y) &= \cos^2\theta_2 g_M([U,V],Y) + g_M(\mathcal{H}\nabla_U \omega V - \mathcal{H}\nabla_V \omega U, \omega QY)\\
&\quad + g_M(\mathcal{T}_U \omega V - \mathcal{T}_V \omega U, \phi PY + \psi QY) - g_M(\mathcal{T}_U \omega\psi V - \mathcal{T}_V \omega\psi U, Y).
\end{aligned}$$

Now, we have

$$\begin{aligned}
\sin^2\theta_2 g_M([U,V],Y) &= g_M(\mathcal{T}_U \omega V - \mathcal{T}_V \omega U, \phi PY + \psi QY) - g_M(\mathcal{T}_U \omega\psi V - \mathcal{T}_V \omega\psi U, Y)\\
&\quad + g_M(\mathcal{H}\nabla_U \omega V - \mathcal{H}\nabla_V \omega U, \omega QY),
\end{aligned}$$

which the proof follows from the above equations. □

Theorem 3.10. *Let f be a proper quasi bi-slant submersion from a cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then the horizontal distribution $(\ker f_*)^\perp$ defines a totally geodesic foliation on M if and only if*

$$\begin{aligned} g_M(\mathcal{A}_U V, PW + \cos^2\theta_1 QW + \cos^2\theta_2 RW) &= g_M(\mathcal{H}\nabla_U V, \omega\psi PW + \omega\psi QW + \omega\psi RW) \\ &\quad - g_M(\mathcal{A}_U BV + \mathcal{H}\nabla_U CV, \omega W), \end{aligned} \tag{3.23}$$

for all $U, V \in \Gamma(\ker f_*)^\perp$ and $W \in \Gamma(\ker f_*)$.

Proof. For $U, V \in \Gamma(\ker f_*)^\perp$ and $W \in \Gamma(\ker f_*)$, we have

$$g_M(\nabla_U V, W) = g_M(\nabla_U V, PW + QW + RW + \eta(W)\xi).$$

Using equations (2.1)–(2.4), (2.6), (2.7), (2.12), (2.13), (3.2), (3.3), (3.7) and Lemmas 3.2 and 3.3 we have

$$\begin{aligned} g_M(\nabla_U V, W) &= g_M(\phi\nabla_U V, \phi PW) + g_M(\phi\nabla_U V, \phi QW) + g_M(\phi\nabla_U V, \phi RW), \\ &= g_M(\mathcal{A}_U V, PW + \cos^2\theta_1 QW + \cos^2\theta_2 RW) \\ &\quad - g_M(\mathcal{H}\nabla_U V, \omega\psi PW + \omega\psi QW + \omega\psi RW) \\ &\quad + g_M(\mathcal{A}_U BV + \mathcal{H}\nabla_U CV, \omega PW + \omega QW + \omega RW). \end{aligned}$$

Taking into account $\omega PW + \omega QW + \omega RW = \omega W$ and $\omega PW = 0$ in the above, one obtains

$$\begin{aligned} g_M(\nabla_U V, W) &= g_M(\mathcal{A}_U V, PW + \cos^2\theta_1 QW + \cos^2\theta_2 RW) \\ &\quad - g_M(\mathcal{H}\nabla_U V, \omega\psi PW + \omega\psi QW + \omega\psi RW) \\ &\quad + g_M(\mathcal{A}_U BV + \mathcal{H}\nabla_U CV, \omega W). \end{aligned}$$

□

Theorem 3.11. *Let f be a proper quasi bi-slant submersion from a cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then the vertical distribution $(\ker f_*)$ defines a totally geodesic foliation on M if and only if*

$$\begin{aligned} g_M(\mathcal{T}_X PY + \cos^2\theta_1 \mathcal{T}_X QY + \cos^2\theta_2 \mathcal{T}_X RY, U) &= g_M(\mathcal{H}\nabla_X \omega\psi PY + \mathcal{H}\nabla_X \omega\psi QY + \mathcal{H}\nabla_X \omega\psi RY, U) \\ &\quad + g_M(\mathcal{T}_X \omega Y, BU) + g_M(\mathcal{H}\nabla_X \omega Y, CU), \end{aligned} \tag{3.24}$$

for all $X, Y \in \Gamma(\ker f_*)$ and $U \in \Gamma(\ker f_*)^\perp$.

Proof. For all $X, Y \in \Gamma(\ker f_*)$ and $U \in \Gamma(\ker f_*)^\perp$, by using equations (2.1)–(2.4), (2.6) and (2.7) we have

$$g_M(\nabla_X Y, U) = g_M(\nabla_X \phi PY, \phi U) + g_M(\nabla_X \phi QY, \phi U) + g_M(\nabla_X \phi RY, \phi U).$$

Taking into account of (2.10), (2.11), (3.2), (3.3), (3.7) and Lemmas 3.2 and 3.3 we have

$$\begin{aligned} g_M(\nabla_X Y, U) &= g_M(\mathcal{T}_X PY, U) + \cos^2\theta_1 g_M(\mathcal{T}_X QY, U) + \cos^2\theta_2 g_M(\mathcal{T}_X RY, U) \\ &\quad - g_M(\mathcal{H}\nabla_X \omega\psi PY + \mathcal{H}\nabla_X \omega\psi QY + \mathcal{H}\nabla_X \omega\psi RY, U) \\ &\quad + g_M(\nabla_X \omega PY + \nabla_X \omega QY + \nabla_X \omega RY, \phi U). \end{aligned}$$

Since $\omega PY + \omega QY + \omega RY = \omega Y$ and $\omega PY = 0$, we derive

$$\begin{aligned} g_M(\nabla_X Y, U) &= g_M(\mathcal{T}_X PY + \cos^2\theta_1 \mathcal{T}_X QY + \cos^2\theta_2 \mathcal{T}_X RY, U) \\ &\quad - g_M(\mathcal{H}\nabla_X \omega\psi PY + \mathcal{H}\nabla_X \omega\psi QY + \mathcal{H}\nabla_X \omega\psi RY, U) \\ &\quad + g_M(\mathcal{T}_X \omega Y, BU) + g_M(\mathcal{H}\nabla_X \omega Y, CU), \end{aligned}$$

which completes the proof. □

From Theorems 3.10 and 3.11 we also have the following decomposition results.

Theorem 3.12. *Let f be a proper quasi bi-slant submersion from a cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then, the total space is locally a product manifold of the form $M_{\ker f_*} \times M_{(\ker f_*)^\perp}$, where $M_{\ker f_*}$ and $M_{(\ker f_*)^\perp}$ are leaves of $\ker f_*$ and $(\ker f_*)^\perp$ respectively if and only if*

$$\begin{aligned} g_M(\mathcal{A}_U V, PY + \cos^2\theta_1 QY + \cos^2\theta_2 RY) &= g_M(\mathcal{H}\nabla_U V, \omega\psi PY + \omega\psi QY + \omega\psi RY) \\ &\quad + g_M(\mathcal{A}_U BV + \mathcal{H}\nabla_U CV, \omega Y), \end{aligned}$$

and

$$\begin{aligned} g_M(\mathcal{T}_X Y + \cos^2\theta_1 \mathcal{T}_X QY + \cos^2\theta_2 \mathcal{T}_X RY, U) &= g_M(\mathcal{H}\nabla_X \omega\psi PY + \mathcal{H}\nabla_X \omega\psi QY + \mathcal{H}\nabla_X \omega\psi RY, U) \\ &\quad + g_M(\mathcal{T}_X \omega Y, BU) + g_M(\mathcal{H}\nabla_X \omega Y, CU), \end{aligned}$$

for all $X, Y \in \Gamma(\ker f_*)$ and $U, V \in \Gamma(\ker f_*)^\perp$.

Theorem 3.13. *Let f be a proper quasi bi-slant submersion from a cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then the distribution D defines a totally geodesic foliation if and only if*

$$g_M(\mathcal{T}_U \phi PV, \omega QW + \omega RW) = -g_M(\mathcal{V}\nabla_U \phi PV, \psi QW + \psi RW), \tag{3.25}$$

and

$$g_M(\mathcal{T}_U \phi PV, CY) = -g_M(\mathcal{V}\nabla_U \phi PV, BY), \tag{3.26}$$

for all $U, V \in \Gamma(D), W \in \Gamma(D_1 \oplus D_2)$ and $Y \in \Gamma(\ker f_*)^\perp$.

Proof. For all $U, V \in \Gamma(D)$, $W \in \Gamma(D_1 \oplus D_2)$ and $Y \in \Gamma(\ker f_*)^{\perp}$, using equations (2.1)–(2.4), (2.6), (2.7), (3.2), (3.3) and Lemma 3.6 we have

$$\begin{aligned} g_M(\nabla_U V, W) &= g_M(\nabla_U \phi V, \phi W), \\ &= g_M(\nabla_U \phi PV, \phi QW + \phi RW), \\ &= g_M(\mathcal{T}_U \phi PV, \omega QW + \omega RW) + g_M(\mathcal{V}\nabla_U \phi PV, \psi QW + \psi RW). \end{aligned}$$

Now, again using equations (2.10), (3.2), (3.3) and (3.7) we have

$$\begin{aligned} g_M(\nabla_U V, Y) &= g_M(\nabla_U \phi V, \phi Y), \\ &= g_M(\nabla_U \phi PV, BY + CY), \\ &= g_M(\mathcal{V}\nabla_U \phi PV, BY) + g_M(\mathcal{T}_U \phi PV, CY), \end{aligned}$$

which completes the proof. □

Theorem 3.14. *Let f be a proper quasi bi-slant submersion from a cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then the distribution D_1 defines a totally geodesic foliation if and only if*

$$g_M(\mathcal{T}_W \omega\psi Z, U) = g_M(\mathcal{T}_W \omega QZ, \phi PU + \psi RU) + g_M(\mathcal{H}\nabla_W \omega QZ, \omega RU), \tag{3.27}$$

and

$$g_M(\mathcal{H}\nabla_W \omega\psi Z, Y) = g_M(\mathcal{H}\nabla_W \omega Z, CY) + g_M(\mathcal{T}_W \omega Z, BY), \tag{3.28}$$

for all $W, Z \in \Gamma(D_1), U \in \Gamma(D \oplus D_2)$ and $Y \in \Gamma(\ker f_*)^{\perp}$.

Proof. For all $W, Z \in \Gamma(D_1)$, $U \in \Gamma(D \oplus D_2)$ and $Y \in \Gamma(\ker f_*)^{\perp}$, using equations (2.1)–(2.4), (2.6), (2.7), (2.11), (3.2), (3.3) and Lemma 3.2, we have

$$\begin{aligned} g_M(\nabla_W Z, U) &= g_M(\nabla_W \phi Z, \phi U) \\ &= g_M(\nabla_W \psi Z, \phi U) + g_M(\nabla_W \omega Z, \phi U), \\ &= \cos^2\theta_1 g_M(\nabla_W Z, U) - g_M(\mathcal{T}_W \omega\psi Z, U) \\ &\quad + g_M(\mathcal{T}_W \omega QZ, \phi PU + \psi RU) + g_M(\mathcal{H}\nabla_W \omega QZ, \omega RU). \end{aligned}$$

Now, we obtain

$$\sin^2\theta_1 g_M(\nabla_W Z, U) = -g_M(\mathcal{T}_W \omega\psi Z, U) + g_M(\mathcal{T}_W \omega QZ, \phi PU + \psi RU) + g_M(\mathcal{H}\nabla_W \omega QZ, \omega RZ)$$

Next, from equations (2.1)–(2.4), (2.6), (2.7), (2.12), (3.3), (3.7) and Lemma 3.2, we have

$$\begin{aligned} g_M(\nabla_W Z, Y) &= g_M(\nabla_W \phi Z, \phi Y), \\ &= g_M(\nabla_W \psi Z, \phi Y) + g_M(\nabla_W \omega Z, \phi Y), \\ &= \cos^2\theta_1 g_M(\nabla_W Z, Y) - g_M(\mathcal{H}\nabla_W \omega\psi Z, Y) \\ &\quad + g_M(\mathcal{H}\nabla_W \omega Z, CY) + g_M(\mathcal{T}_W \omega Z, BY). \end{aligned}$$

Now, we arrive

$$\sin^2\theta_1 g_M(\nabla_W Z, Y) = -g_M(\mathcal{H}\nabla_W\omega\psi Z, Y) + g_M(\mathcal{H}\nabla_W\omega Z, CY) + g_M(\mathcal{T}_W\omega Z, BY),$$

which completes the proof. □

Theorem 3.15. *Let f be a proper quasi bi-slant submersion from a cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then the distribution D_2 defines a totally geodesic foliation if and only if*

$$g_M(\mathcal{T}_U\omega\psi V, W) = g_M(\mathcal{T}_U\omega QV, \phi PW + \phi RW) + g_M(\mathcal{H}\nabla_U\omega QV, \omega RW), \tag{3.29}$$

and

$$g_M(\mathcal{H}\nabla_U\omega\psi V, Y) = g_M(\mathcal{H}\nabla_U\omega V, CY) + g_M(\mathcal{T}_U\omega V, BY), \tag{3.30}$$

for all $U, V \in \Gamma(D_2), W \in \Gamma(D \oplus D_1)$ and $Y \in \Gamma(\ker f_*)^\perp$.

Proof. For all $U, V \in \Gamma(D_2), W \in \Gamma(D \oplus D_1)$ and $Y \in \Gamma(\ker f_*)^\perp$, by using equations (2.1)–(2.4), (2.6), (2.7), (2.10), (3.3) and from Lemma 3.2 and Lemma 3.6, we have

$$\begin{aligned} g_M(\nabla_U V, W) &= g_M(\nabla_U\psi V, \phi W) + g_M(\nabla_U\omega V, \phi W), \\ &= \cos^2\theta_2 g_M(\nabla_U V, W) - g_M(\mathcal{T}_U\omega\psi V, W) \\ &\quad + g_M(\mathcal{T}_U\omega QV, \phi PW + \psi RW) + g_M(\mathcal{H}\nabla_U\omega QV, \omega RW). \end{aligned}$$

Now, we get

$$\sin^2\theta_2 g_M(\nabla_U V, W) = -g_M(\mathcal{T}_U\omega\psi V, W) + g_M(\mathcal{T}_U\omega QV, \phi PW + \psi RW) + g_M(\mathcal{H}\nabla_U\omega QV, \omega RW).$$

Next, from equations (2.1)–(2.4), (2.6), (2.7), (2.12), (3.2) (3.3), (3.7) and Lemma 3.2, we have

$$\begin{aligned} g_M(\nabla_U V, Y) &= g_M(\nabla_U\psi V, \phi Y) + g_M(\nabla_U\omega V, \phi Y), \\ &= \cos^2\theta_2 g_M(\nabla_U V, Y) - g_M(\mathcal{H}\nabla_U\omega\psi V, Y) \\ &\quad + g_M(\mathcal{H}\nabla_U\omega V, CY) + g_M(\mathcal{T}_U\omega V, BY). \end{aligned}$$

Now, we obtain

$$\sin^2\theta_1 g_M(\nabla_U V, Y) = -g_M(\mathcal{H}\nabla_U\omega\psi V, Y) + g_M(\mathcal{H}\nabla_U\omega V, CY) + g_M(\mathcal{T}_U\omega V, BY),$$

which completes the proof. □

We recall that a differentiable map f between two Riemannian manifolds is totally geodesic if

$$(\nabla f_*)(Y, Z) = 0, \text{ for all } Y, Z \in \Gamma(TM).$$

A totally geodesic map is that it maps every geodesic in the total space into a geodesic in the base space in proportion to arc lengths.

Theorem 3.16. *Let f be a proper quasi bi-slant submersion from a cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then the map f is totally geodesic if and only if*

$$g_M(\mathcal{H}\nabla_U \omega\psi QV + \mathcal{H}\nabla_U \omega\psi RV - \cos^2\theta_1 \mathcal{T}_U QV - \cos^2\theta_2 \mathcal{T}_U RV, W)$$
$$= g_M(\mathcal{V}\nabla_U \phi PV + \mathcal{T}_U \omega QV + \mathcal{T}_U \omega RV, BW) + g_M(\mathcal{T}_U \phi PV + \mathcal{H}\nabla_U \omega QV + \mathcal{H}\nabla_U \omega RV, CW),$$

and

$$g_M(\mathcal{H}\nabla_W \omega\psi QU + \mathcal{H}\nabla_W \omega\psi RU - \cos^2\theta_1 \mathcal{A}_W QU - \cos^2\theta_2 \mathcal{A}_W RU, Z)$$
$$= g_M(\mathcal{V}\nabla_W \phi PU + \mathcal{A}_W \omega QU + \mathcal{A}_W \omega RU, BZ) + g_M(\mathcal{A}_W \phi PU + \mathcal{H}\nabla_W \omega QU + \mathcal{H}\nabla_W \omega RU, CZ),$$

for all $U, V \in \Gamma(\ker f_*)$ and $W, Z \in \Gamma(\ker f_*)^\perp$.

Proof. For all $U, V \in \Gamma(\ker f_*)$ and $W, Z \in \Gamma(\ker f_*)^\perp$, making use of (2.1)–(2.4), (2.6), (2.7), (2.10), (2.11), (3.2), (3.3), (3.7) and from Lemma 3.2 and 3.3, we derive

$$\begin{aligned}
g_M(\nabla_U V, W) &= g_M(\nabla_U \phi V, \phi W) \\
&= g_M(\nabla_U \phi PV, \phi W) + g_M(\nabla_U \phi QV, \phi W) + g_M(\nabla_U \phi RV, \phi W), \\
&= g_M(\nabla_U \phi PV, \phi W) + g_M(\nabla_U \psi QV, \phi W) + g_M(\nabla_U \psi RV, \phi W) \\
&\quad + g_M(\nabla_U \omega QV, \phi W) + g_M(\nabla_U \omega RV, \phi W), \\
&= g_M(\mathcal{V}\nabla_U \phi PV + \mathcal{T}_U \omega QV + \mathcal{T}_U \omega RV, W) \\
&\quad + g_M(\mathcal{T}_U \phi PV + \mathcal{H}\nabla_U \omega QV + \mathcal{H}\nabla_U \omega RV, CW) \\
&\quad + g_M(\cos^2\theta_1 \mathcal{T}_U QV + \cos^2\theta_2 \mathcal{T}_U RV - \mathcal{H}\nabla_U \omega\psi QV - \mathcal{H}\nabla_U \omega\psi RV, W).
\end{aligned}$$

Next, taking account of (2.1)–(2.4), (2.6), (2.7), (2.10), (2.12), (2.13), (3.2), (3.3), (3.7) and Lemmas 3.2 and 3.3, we have

$$\begin{aligned}
g_M(\nabla_W U, Z) &= g_M(\phi\nabla_W U, \phi Z) \\
&= g_M(\nabla_W \phi U, \phi Z), \\
&= g_M(\nabla_W \phi PU, \phi Z) + g_M(\nabla_W \phi QU, \phi Z) + g_M(\nabla_W \phi RU, \phi Z), \\
&= g_M(\nabla_W \phi PU, \phi Z) + g_M(\nabla_W \psi QU, \phi Z) + g_M(\nabla_W \psi RU, \phi Z) \\
&\quad + g_M(\nabla_W \omega QU, \phi Z) + g_M(\nabla_W \omega RU, \phi Z), \\
&= g_M(\mathcal{V}\nabla_W \phi PU + \mathcal{A}_W \omega QU + \mathcal{A}_W \omega RU, BZ) \\
&\quad + g_M(\mathcal{A}_W \phi PU + \mathcal{H}\nabla_W \omega QU + \mathcal{H}\nabla_W \omega RU, CZ) \\
&\quad + g_M(\cos^2\theta_1 \mathcal{A}_W QU + \cos^2\theta_2 \mathcal{A}_W RU - \mathcal{H}\nabla_W \omega\psi QU - \mathcal{H}\nabla_W \omega\psi RU, Z),
\end{aligned}$$

which completes the proof. □

4 Examples

In this section, we are going to give some non-trivial examples. We will use the notation mentioned in Example 2.1.

Example 4.1. *Define a map*

$$\begin{aligned} \pi &: \mathbb{R}^{15} \to \mathbb{R}^6 \\ \pi(x_1, x_2, \ldots, x_7, y_1, y_2, \ldots, y_7, z) &= (x_2 \cos\theta_1 - y_3 \sin\theta_1, y_2, x_4 \sin\theta_2 - y_5 \cos\theta_2, x_5, x_7, y_7), \end{aligned}$$

which is a quasi bi-slant submersion such that

$$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = \frac{\partial}{\partial y_1}, \quad X_3 = \frac{\partial}{\partial x_2}\sin\theta_1 + \frac{\partial}{\partial y_3}\cos\theta_1, \quad X_4 = \frac{\partial}{\partial x_3},$$

$$X_5 = \frac{\partial}{\partial x_4}\cos\theta_2 + \frac{\partial}{\partial y_5}\sin\theta_2, \quad X_6 = \frac{\partial}{\partial y_4}, \quad X_7 = \frac{\partial}{\partial x_6}, \quad X_8 = \frac{\partial}{\partial y_6},$$

$$X_9 = \xi = \frac{\partial}{\partial z},$$

$$(\ker \pi_*) = (D \oplus D_1 \oplus D_2 \oplus \langle \xi \rangle),$$

where

$$D = \left\langle X_1 = \frac{\partial}{\partial x_1}, X_2 = \frac{\partial}{\partial y_1}, X_7 = \frac{\partial}{\partial x_6}, X_8 = \frac{\partial}{\partial y_6} \right\rangle,$$

$$D_1 = \left\langle X_3 = \frac{\partial}{\partial x_2}\sin\theta_1 + \frac{\partial}{\partial y_3}\cos\theta_1, X_4 = \frac{\partial}{\partial x_3} \right\rangle,$$

$$D_2 = \left\langle X_5 = \frac{\partial}{\partial x_4}\cos\theta_2 + \frac{\partial}{\partial y_5}\sin\theta_2, X_6 = \frac{\partial}{\partial y_4} \right\rangle,$$

$$\langle \xi \rangle = \left\langle X_9 = \frac{\partial}{\partial z} \right\rangle,$$

and

$$(\ker \pi_*)^{\perp} = \left\langle \frac{\partial}{\partial x_2}\cos\theta_1 - \frac{\partial}{\partial y_3}\sin\theta_1, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial x_4}\sin\theta_2 - \frac{\partial}{\partial y_5}\cos\theta_2, \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_7}, \frac{\partial}{\partial y_7} \right\rangle,$$

with bi-slant angles θ_1 and θ_2. Thus the above example verifies the Lemmas 3.1, 3.2, 3.3 and 3.6.

Example 4.2. *Define a map*

$$\begin{aligned} \pi &: \mathbb{R}^{13} \to \mathbb{R}^6 \\ \pi(x_1, x_2, \ldots, x_6, y_1, y_2, \ldots, y_6, z) &= \left(\frac{x_1 - x_2}{\sqrt{2}}, y_1, \frac{\sqrt{3}x_4 - x_5}{2}, y_5, x_6, y_6 \right), \end{aligned}$$

which is a quasi bi-slant submersion such that

$$X_1 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right), \quad X_2 = \frac{\partial}{\partial y_2}, \quad X_3 = \frac{\partial}{\partial x_3}, \quad X_4 = \frac{\partial}{\partial y_3},$$

$$X_5 = \frac{1}{2}\left(\frac{\partial}{\partial x_4} + \sqrt{3}\frac{\partial}{\partial x_5} \right), \quad X_6 = \frac{\partial}{\partial y_4},$$

$$X_7 = \xi = \frac{\partial}{\partial z},$$
$$(\ker \pi_*) = (D \oplus D_1 \oplus D_2 \oplus \langle \xi \rangle),$$

where

$$D = \left\langle X_3 = \frac{\partial}{\partial x_3}, X_4 = \frac{\partial}{\partial y_3} \right\rangle,$$
$$D_1 = \left\langle X_1 = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right), X_2 = \frac{\partial}{\partial y_2} \right\rangle,$$
$$D_2 = \left\langle X_5 = \frac{1}{2} \left(\frac{\partial}{\partial x_4} + \sqrt{3} \frac{\partial}{\partial x_5} \right), X_6 = \frac{\partial}{\partial y_4} \right\rangle,$$
$$\langle \xi \rangle = \left\langle X_7 = \frac{\partial}{\partial z} \right\rangle,$$

and

$$(\ker \pi_*)^\perp = \left\langle \frac{\partial}{\partial y_1}, \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right), \frac{1}{2} \left(\sqrt{3} \frac{\partial}{\partial x_4} - \frac{\partial}{\partial x_5} \right), \frac{\partial}{\partial y_5}, \frac{\partial}{\partial x_6}, \frac{\partial}{\partial y_6} \right\rangle,$$

with bi-slant angles $\theta_1 = \dfrac{\pi}{4}$ *and* $\theta_2 = \dfrac{\pi}{3}$. *Therefore, the above example verifies the Lemmas 3.1, 3.2, 3.3 and 3.6.*

References

[1] M. A. Akyol, "Conformal anti-invariant submersions from cosymplectic manifolds", Hacet. J. Math. Stat., vol. 46, no. 2, pp. 177–192, 2017.

[2] M. A. Akyol and B. Şahin, "Conformal slant submersions", Hacet. J. Math. Stat., vol 48, no.1, pp. 28–44, 2019.

[3] M. A. Akyol, "Conformal semi-slant submersions", Int. J. Geom. Methods Mod. Phys., vol. 14, no. 7, 1750114, 25 pages, 2017.

[4] P. Baird and J. C. Wood, *Harmonic morphism between Riemannian manifolds*, Oxford science publications, Oxford, 2003.

[5] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds*. Progress in Mathematics. 203, Birkhäuser Boston, Basel, Berlin, 2002.

[6] M. Cengizhan and I. K. Erken, "Anti-invariant Riemannian submersions from cosymplectic manifolds onto Riemannian submersions", Filomat, vol. 29, no. 7, pp. 1429–1444, 2015.

[7] J.-P. Bourguignon and H. B. Lawson, "Stability and isolation phenomena for Yang-Mills fields", Comm. Math. Phys., vol. 79, no. 2, pp.189–230, 1981.

[8] J.-P. Bourguignon, "A mathematician's visit to Kaluza-Klein theory", Rend. Sem. Mat. Univ. Politec. Torino, Special Issue, pp. 143–163, 1989.

[9] D. Chinea, "Almost contact metric submersions", Rend. Circ. Mat. Palermo, vol. 34, no. 1, pp. 89–104, 1985.

[10] M. Falcitelli, A. M. Pastore and S. Ianus, *Riemannian submersions and related topics*, World Scientific, River Edge, NJ, 2004.

[11] A. Gray, "Pseudo-Riemannian almost product manifolds and submersions", J. Math. Mech., vol. 16, pp. 715–738, 1967.

[12] Y. Gündüzalp and M. A. Akyol, "Conformal slant submersions from cosymplectic manifolds", Turkish J. Math., vol. 42, no. 5, pp. 2672–2689, 2018.

[13] S. Ianuş and M. Visinescu, "Kaluza-Klein theory with scalar fields and generalized Hopf manifolds", Classical Quantum Gravity, vol. 4, no. 5, pp. 1317–1325, 1987.

[14] S. Ianuş, A. M. Ionescu, R. Mocanu and G. E. Vîlcu, "Riemannian submersions from almost contact metric manifolds", Abh. Math. Semin. Univ. Hambg., vol. 81, no. 1, pp. 101–114, 2011.

[15] S. Kumar, R. Prasad and P. K. Singh, “Conformal semi-slant submersions from Lorentzian para Sasakian manifolds”, Commun. Korean Math. Soc., vol. 34, no. 2, pp. 637–655, 2019.

[16] S. Longwap, F. Massamba and N. E. Homti, “On quasi-hemi-slant Riemannian submersion”, Journal of Advances in Mathematics and Computer Science, vol. 34, no. 1, pp. 1–14, 2019.

[17] B. O’Neill, “The fundamental equations of a submersion”, Michigan Math. J., vol. 33, no. 13, pp. 459–469, 1966.

[18] K.-S. Park and R. Prasad, “Semi-slant submersions”, Bull. Korean Math. Soc, vol. 50, no. 3, pp. 951–962, 2013.

[19] R. Prasad, S. S. Shukla and S. Kumar, “On quasi-bi-slant submersions”, Mediterr. J. Math., vol. 16, no. 6, paper no. 155, 18 pages, 2019.

[20] R. Prasad, P. K. Singh and S. Kumar, “On quasi-bi-slant submersions from Sasakian manifolds onto Riemannian manifolds”, Afr. Mat., vol. 32, no. 3-4, pp. 403–417, 2020.

[21] B. Ṣahin, “Semi-invariant submersions from almost Hermitian manifolds”, Canad. Math. Bull., vol. 56, no. 1, pp. 173–182, 2013.

[22] B. Ṣahin, “Slant submersions from almost Hermitian manifolds”, Bull. Math. Soc. Sci. Math. Roumanie, vol. 54(102), no. 1, pp. 93–105, 2011.

[23] B. Şahin, “Riemannian submersion from almost Hermitian manifolds”, Taiwanese J. Math., vol. 17, no. 2, pp. 629–659, 2013.

[24] B. Ṣahin, *Riemannian submersions, Riemannian maps in Hermitian geometry and their applications*, Elsevier, Academic Press, London, 2017.

[25] B. Ṣahin, “Anti-invariant Riemannian submersions from almost Hermitian manifolds”, Cent. Eur. J. Math., vol. 8, no. 3, pp. 437–447, 2010.

[26] C. Sayar, M. A. Akyol and R. Prasad, “Bi-slant submersions in complex geometry”, Int. J. Geom. Methods Mod. Phys., vol. 17, no. 4, 17 pages, 2020.

[27] C. Sayar, H. M. Taştan, F. Özdemir and M. M. Tripathi, “Generic submersions from Kaehler manifolds”, Bull. Malays. Math. Sci. Soc., vol. 43, no. 1, pp. 809–831, 2020.

[28] H. M. Taştan, B. Şahin and Ş. Yanan, “Hemi-slant submersions”, Mediterr. J. Math., vol 13, no. 4, pp. 2171–2184, 2016.

[29] B. Watson, “Almost Hermitian submersions”, J. Differential Geometry, vol. 11, no. 1, pp. 147–165, 1976.