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Infinitely many positive solutions for an iterative system of singular BVP on time scales

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ABSTRACT

In this paper, we consider an iterative system of singular two-point boundary value problems on time scales. By applying Hölder's inequality and Krasnoselskii's cone fixed point theorem in a Banach space, we derive sufficient conditions for the existence of infinitely many positive solutions. Finally, we provide an example to check the validity of our obtained results.

RESUMEN

En este artículo, consideramos un sistema iterativo de problemas de valor en la frontera singulares de dos puntos en escalas de tiempo. Aplicando la desigualdad de Hölder y el teorema de punto fijo cónico de Krasnoselskii en un espacio de Banach, derivamos condiciones suficientes para la existencia de una cantidad infinita de soluciones positivas. Finalmente, entregamos un ejemplo para verificar la validez de nuestros resultados.

Keywords and Phrases: Iterative system, time scales, singularity, cone, Krasnoselskii's fixed point theorem, positive solutions.

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1 Introduction

The theory of time scales was created to unify continuous and discrete analysis. Difference and differential equations can be studied simultaneously by studying dynamic equations on time scales. A time scale is any closed and nonempty subset of the real numbers. So, by this theory, we can extend known results from continuous and discrete analysis to a more general setting. As a matter of fact, this theory allows us to consider time scales which possess hybrid behaviours (both continuous and discrete). These types of time scales play an important role for applications, since most of the phenomena in the environment are neither only discrete nor only continuous, but they possess both behaviours. Moreover, basic results on this issue have been well documented in the articles [1, 2] and the monographs of Bohner and Peterson [6, 7]. There is a great deal of research activity devoted to existence of solutions to the dynamic equations on time scales, see for example [8, 9, 13, 16–19] and references therein.

In [14], Liang and Zhang studied countably many positive solutions for nonlinear singular m–point boundary value problems on time scales,

$$\begin{aligned}\big(\varphi(\upsilon^{\Delta}(t))\big)^{\nabla}+a(t)f\big(\upsilon(t)\big)=0,\ t\in[0,\mathfrak{T}]_{\mathbb{T}},\\ \upsilon(0)=\sum_{i=1}^{m-2}a_i\upsilon(\xi_i),\ \upsilon^{\Delta}(\mathfrak{T})=0,\end{aligned}$$

by using the fixed-point index theory and a new fixed-point theorem in cones.

In [12], Khuddush, Prasad and Vidyasagar considered second order n-point boundary value problem on time scales,

$$\begin{aligned}\upsilon_i^{\Delta\nabla}(t)+\uplambda(t)\mathrm{g}_\ell\big(\upsilon_{i+1}(t)\big)=0,\ 1\le i\le n,\ t\in(0,\upsigma(a)]_{\mathbb{T}},\\ \upsilon_{n+1}(t)=\upsilon_1(t),\ t\in(0,\upsigma(a)]_{\mathbb{T}},\\ \upsilon_i^{\Delta}(0)=0,\ \upsilon_i(\upsigma(a))=\sum_{k=1}^{n-2}c_k\upsilon_i(\zeta_k),\ 1\le i\le n,\end{aligned}$$

and established existence of positive solutions by applying Krasnoselskii's fixed point theorem.

Inspired by the aforementioned works, in this paper by applying Hölder's inequality and Krasnoselskii's cone fixed point theorem in a Banach space, we establish the existence of infinitely many positive solutions for the iterative system of two-point boundary value problems with n–singularities on time scales,

$$\left.\begin{aligned}\upsilon_\ell^{\Delta\Delta}(t)+\uplambda(t)\mathrm{g}_\ell\big(\upsilon_{\ell+1}(t)\big)=0,\ 1\le\ell\le m,\ t\in(0,\mathfrak{T})_{\mathbb{T}},\\ \upsilon_{m+1}(t)=\upsilon_1(t),\ t\in(0,\mathfrak{T})_{\mathbb{T}},\end{aligned}\right\}\tag{1.1}$$

$$\left.\begin{aligned}\upsilon_\ell(0)=\upsilon_\ell^{\Delta}(0),\ 1\le\ell\le m,\\ \upsilon_\ell(\mathfrak{T})=-\upsilon_\ell^{\Delta}(\mathfrak{T}),\ 1\le\ell\le m,\end{aligned}\right\}\tag{1.2}$$

where $m \in \mathbb{N}$, $\lambda(t) = \prod_{i=1}^{k} \lambda_i(t)$ and each $\lambda_i(t) \in L_{\Delta}^{p_i}([0, \mathfrak{T}]_{\mathbb{T}})$ $(p_i \geq 1)$ has n–singularities in the interval $(0, \mathfrak{T})_{\mathbb{T}}$.

We assume the following conditions are true throughout the paper:

(H_1) $\mathrm{g}_{\ell} : [0, +\infty) \to [0, +\infty)$ is continuous.

(H_2) $\lim\limits_{t \to t_i} \lambda_i(t) = \infty$, where $0 < t_n < t_{n-1} < \cdots < t_1 < \mathfrak{T}$.

2 Preliminaries

In this section, we introduce some basic definitions and lemmas which are useful for our later discussions.

Definition 2.1 ([6]). *A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$. $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology. It follows that the jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$, and the graininess $\mu : \mathbb{T} \to [0, +\infty)$ are defined by*

$$\sigma(t) = \inf\{\tau \in \mathbb{T} : \tau > t\},$$

$$\rho(t) = \sup\{\tau \in \mathbb{T} : \tau < t\},$$

and

$$\mu(t) = \sigma(t) - t,$$

respectively.

- *The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively.*
- *If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_{\kappa} = \mathbb{T} \backslash \{m\}$; otherwise $\mathbb{T}_{\kappa} = \mathbb{T}$.*
- *If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^{\kappa} = \mathbb{T} \backslash \{m\}$; otherwise $\mathbb{T}^{\kappa} = \mathbb{T}$.*
- *A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of all rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$.*
- *A function $f : \mathbb{T} \to \mathbb{R}$ is called ld-continuous provided it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at right-dense points in $\mathbb{T}$. The set of all ld-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{ld} = C_{ld}(\mathbb{T}) = C_{ld}(\mathbb{T}, \mathbb{R})$.*
- *By an interval time scale, we mean the intersection of a real interval with a given time scale, i.e., $[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T}$. Other intervals can be defined similarly.*

Definition 2.2 ([5,11]). *Let μ_Δ and μ_∇ be the Lebesgue $\Delta-$measure and the Lebesgue $\nabla-$measure on $\mathbb{T}$, respectively. If $A \subset \mathbb{T}$ satisfies $\mu_\Delta(A) = \mu_\nabla(A)$, then we call A measurable on $\mathbb{T}$, denoted $\mu(A)$ and this value is called the Lebesgue measure of A. Let P denote a proposition with respect to $t \in \mathbb{T}$.*

(i) If there exists $\Gamma_1 \subset A$ with $\mu_\Delta(\Gamma_1) = 0$ such that P holds on $A \backslash \Gamma_1$, then P is said to hold Δ–a.e. on A.

(ii) If there exists $\Gamma_2 \subset A$ with $\mu_\nabla(\Gamma_2) = 0$ such that P holds on $A \backslash \Gamma_2$, then P is said to hold ∇–a.e. on A.

Definition 2.3 ([4,5]). *Let $E \subset \mathbb{T}$ be a $\Delta-$measurable set and $p \in \bar{\mathbb{R}} \equiv \mathbb{R} \cup \{-\infty, +\infty\}$ be such that $p \geq 1$ and let $f : E \to \bar{\mathbb{R}}$ be a $\Delta-$measurable function. We say that f belongs to $L^p_\Delta(E)$ provided that either*

$$\int_E |f|^p(s)\Delta s < \infty \quad \textit{if} \quad p \in [1, +\infty),$$

or there exists a constant $M \in \mathbb{R}$ such that

$$|f| \leq M, \quad \Delta - a.e. \text{ on } E \quad \textit{if} \quad p = +\infty.$$

Lemma 2.4 ([20]). *Let $E \subset \mathbb{T}$ be a $\Delta-$measurable set. If $f : \mathbb{T} \to \mathbb{R}$ is $\Delta-$integrable on E, then*

$$\int_E f(s)\Delta s = \int_E f(s)ds + \sum_{i \in I_E} \big(\sigma(t_i) - t_i\big) f(t_i) + r(f, E),$$

where

$$r(f, E) = \begin{cases} \mu_{\mathtt{N}}(E) f(M), & \textit{if } \mathtt{N} \in \mathbb{T}, \\ \\ 0, & \textit{if } \mathtt{N} \notin \mathbb{T}, \end{cases}$$

$I_E := \{i \in I : t_i \in E\}$ and $\{t_i\}_{i \in I}$, $I \subset \mathbb{N}$, is the set of all right-scattered points of $\mathbb{T}$.

Lemma 2.5. *For any $\mathtt{y}(t) \in \mathcal{C}_{rd}([0, \mathfrak{T}]_\mathbb{T})$, the boundary value problem,*

$$\upsilon_1^{\Delta\Delta}(t) + \mathtt{y}(t) = 0, \ t \in (0, \mathfrak{T})_\mathbb{T}, \tag{2.1}$$

$$\upsilon_1(0) = \upsilon_1^\Delta(0), \ \upsilon_1(\mathfrak{T}) = -\upsilon_1^\Delta(\mathfrak{T}), \tag{2.2}$$

has a unique solution

$$\upsilon_1(t) = \int_0^\mathfrak{T} \aleph(t, \tau)\mathtt{y}(\tau)\Delta\tau, \tag{2.3}$$

where

$$\aleph(t, \tau) = \frac{1}{2 + \mathfrak{T}} \begin{cases} (\mathfrak{T} - t + 1)(\sigma(\tau) + 1), & \textit{if} \ \ \sigma(\tau) < t, \\ \\ (\mathfrak{T} - \sigma(\tau) + 1)(t + 1), & \textit{if} \ \ t < \tau. \end{cases} \tag{2.4}$$

Proof. Suppose υ_1 is a solution of (2.1), then

$$\begin{aligned}\upsilon_1(t) &= -\int_0^t \int_0^\tau \mathrm{y}(\tau_1)\Delta\tau_1\Delta\tau + A_1 t + A_2\\ &= -\int_0^t (t-\sigma(\tau))\mathrm{y}(\tau)\Delta\tau + A_1 t + A_2,\end{aligned}$$

where $A_1 = \upsilon_1^\Delta(0)$ and $A_2 = \upsilon_1(0)$. By the conditions (2.2), we get

$$A_1 = A_2 = \frac{1}{2+\mathfrak{T}}\int_0^{\mathfrak{T}} (\mathfrak{T}-\sigma(\tau)+1)\mathrm{y}(\tau)\Delta\tau.$$

So, we have

$$\begin{aligned}\upsilon_1(t) &= \int_0^t (t-\sigma(\tau))\mathrm{y}(\tau)\Delta\tau + \frac{1}{2+\mathfrak{T}}\int_0^{\mathfrak{T}} (\mathfrak{T}-\sigma(\tau)+1)(1+t)\mathrm{y}(\tau)\Delta\tau\\ &= \int_0^{\mathfrak{T}} \aleph(t,\tau)\mathrm{y}(\tau)\Delta\tau.\end{aligned}$$

This completes the proof. □

Lemma 2.6. *Suppose (H_1)–(H_2) hold. For $\varepsilon \in (0, \frac{\mathfrak{T}}{2})_{\mathbb{T}}$, let $\mathcal{G}(\varepsilon) = \dfrac{\varepsilon+1}{\mathfrak{T}+1} < 1$. Then $\aleph(t,\tau)$ has the following properties:*

(i) $0 \le \aleph(t,\tau) \le \aleph(\tau,\tau)$ *for all* $t,\tau \in [0,1]_{\mathbb{T}}$,

(ii) $\mathcal{G}(\varepsilon)\aleph(\tau,\tau) \le \aleph(t,\tau)$ *for all* $t \in [\varepsilon, \mathfrak{T}-\varepsilon]_{\mathbb{T}}$ *and* $\tau \in [0,1]_{\mathbb{T}}$.

Proof. *(i)* is evident. To prove *(ii)*, let $t \in [\varepsilon, \mathfrak{T}-\varepsilon]_{\mathbb{T}}$ and $t \le \tau$. Then

$$\frac{\aleph(t,\tau)}{\aleph(\tau,\tau)} = \frac{t+1}{\tau+1} \ge \frac{\varepsilon+1}{\mathfrak{T}+1} = \mathcal{G}(\varepsilon).$$

For $\tau \le t$,

$$\frac{\aleph(t,\tau)}{\aleph(\tau,\tau)} = \frac{\mathfrak{T}-t+1}{\mathfrak{T}-\tau+1} \ge \frac{\varepsilon+1}{\mathfrak{T}+1} = \mathcal{G}(\varepsilon).$$

This completes the proof. □

Notice that an $m-$tuple $(\upsilon_1(t), \upsilon_2(t), \upsilon_3(t), \ldots, \upsilon_m(t))$ is a solution of the iterative boundary value problem (1.1)–(1.2) if and only if

$$\begin{aligned}\upsilon_\ell(t) &= \int_0^1 \aleph(t,\tau)\lambda(\tau)\mathrm{g}_\ell(\upsilon_{\ell+1}(\tau))\Delta\tau,\ t \in (0,\mathfrak{T})_{\mathbb{T}},\ 1 \le \ell \le m,\\ \upsilon_{m+1}(t) &= \upsilon_1(t),\ t \in (0,\mathfrak{T})_{\mathbb{T}},\end{aligned}$$

i.e.,

$$\begin{aligned}\upsilon_1(t) = \int_0^1 \aleph(t,\tau_1)\lambda(\tau_1)\mathrm{g}_1\bigg(\int_0^1 \aleph(\tau_1,\tau_2)\lambda(\tau_2)\mathrm{g}_2\bigg(\int_0^1 \aleph(\tau_2,\tau_3)\cdots\\ \times \mathrm{g}_{m-1}\bigg(\int_0^1 \aleph(\tau_{m-1},\tau_m)\lambda(\tau_m)\mathrm{g}_m(\upsilon_1(\tau_m))\Delta\tau_m\bigg)\cdots\Delta\tau_3\bigg)\Delta\tau_2\bigg)\Delta\tau_1.\end{aligned}$$

Let $\mathtt{B}$ be the Banach space $C_{rd}((0,\mathfrak{T})_{\mathbb{T}},\mathbb{R})$ with the norm $\|\upsilon\| = \max_{t\in(0,\mathfrak{T})_{\mathbb{T}}} |\upsilon(t)|$. For $\varepsilon \in \big(0,\frac{\mathfrak{T}}{2}\big)_{\mathbb{T}}$, we define the cone $\mathtt{K}_\varepsilon \subset \mathtt{B}$ as

$$\mathtt{K}_\varepsilon = \left\{\upsilon \in \mathtt{B} : \upsilon(t) \text{ is nonnegative and } \min_{t\in[\varepsilon,\,\mathfrak{T}-\varepsilon]_{\mathbb{T}}} \upsilon(t) \geq \mathcal{G}(\varepsilon)\|\upsilon(t)\|\right\}.$$

For any $\upsilon_1 \in \mathtt{K}_\varepsilon$, define an operator $\Omega : \mathtt{K}_\varepsilon \to \mathtt{B}$ by

$$\begin{aligned}(\Omega\upsilon_1)(t) = \int_0^1 \aleph(t,\tau_1)\lambda(\tau_1)\mathtt{g}_1\bigg(\int_0^1 \aleph(\tau_1,\tau_2)\lambda(\tau_2)\mathtt{g}_2\bigg(\int_0^1 \aleph(\tau_2,\tau_3)\cdots\\ \times \mathtt{g}_{m-1}\bigg(\int_0^1 \aleph(\tau_{m-1},\tau_m)\lambda(\tau_m)\mathtt{g}_m(\upsilon_1(\tau_m))\Delta\tau_m\bigg)\cdots\Delta\tau_3\bigg)\Delta\tau_2\bigg)\Delta\tau_1.\end{aligned}$$

Lemma 2.7. *Assume that (H_1)–(H_2) hold. Then for each $\varepsilon \in \big(0,\frac{\mathfrak{T}}{2}\big)_{\mathbb{T}}$, $\Omega(\mathtt{K}_\varepsilon) \subset \mathtt{K}_\varepsilon$ and $\Omega : \mathtt{K}_\varepsilon \to \mathtt{K}_\varepsilon$ are completely continuous.*

Proof. From Lemma 2.6, $\aleph(t,\tau) \geq 0$ for all $t,\tau \in (0,\mathfrak{T})_{\mathbb{T}}$. So, $(\Omega\upsilon_1)(t) \geq 0$. Also, for $\upsilon_1 \in \mathtt{K}_\varepsilon$, we have

$$\begin{aligned}\|\Omega\upsilon_1\| &= \max_{t\in(0,\mathfrak{T})_{\mathbb{T}}} \int_0^1 \aleph(t,\tau_1)\lambda(\tau_1)\mathtt{g}_1\bigg(\int_0^1 \aleph(\tau_1,\tau_2)\lambda(\tau_2)\mathtt{g}_2\bigg(\int_0^1 \aleph(\tau_2,\tau_3)\cdots\\ &\qquad \times \mathtt{g}_{m-1}\bigg(\int_0^1 \aleph(\tau_{m-1},\tau_m)\lambda(\tau_m)\mathtt{g}_m(\upsilon_1(\tau_m))\Delta\tau_m\bigg)\cdots\Delta\tau_3\bigg)\Delta\tau_2\bigg)\Delta\tau_1\\ &\leq \int_0^1 \aleph(\tau_1,\tau_1)\lambda(\tau_1)\mathtt{g}_1\bigg(\int_0^1 \aleph(\tau_1,\tau_2)\lambda(\tau_2)\mathtt{g}_2\bigg(\int_0^1 \aleph(\tau_2,\tau_3)\cdots\\ &\qquad \times \mathtt{g}_{m-1}\bigg(\int_0^1 \aleph(\tau_{m-1},\tau_m)\lambda(\tau_m)\mathtt{g}_m(\upsilon_1(\tau_m))\Delta\tau_m\bigg)\cdots\Delta\tau_3\bigg)\Delta\tau_2\bigg)\Delta\tau_1.\end{aligned}$$

Again from Lemma 2.6, we get

$$\begin{aligned}\min_{t\in[\varepsilon,\mathfrak{T}-\varepsilon]_{\mathbb{T}}} \big\{(\Omega\upsilon_1)(t)\big\} \geq \mathcal{G}(\varepsilon)\int_0^1 \aleph(\tau_1,\tau_1)\lambda(\tau_1)\mathtt{g}_1\bigg(\int_0^1 \aleph(\tau_1,\tau_2)\lambda(\tau_2)\mathtt{g}_2\bigg(\int_0^1 \aleph(\tau_2,\tau_3)\cdots\\ \times \mathtt{g}_{m-1}\bigg(\int_0^1 \aleph(\tau_{m-1},\tau_m)\lambda(\tau_m)\mathtt{g}_m(\upsilon_1(\tau_m))\Delta\tau_m\bigg)\cdots\Delta\tau_3\bigg)\Delta\tau_2\bigg)\Delta\tau_1.\end{aligned}$$

It follows from the above two inequalities that

$$\min_{t\in[\varepsilon,\mathfrak{T}-\varepsilon]_{\mathbb{T}}} \big\{(\Omega\upsilon_1)(t)\big\} \geq \mathcal{G}(\varepsilon)\|\Omega\upsilon_1\|.$$

So, $\Omega\upsilon_1 \in \mathtt{K}_\varepsilon$ and thus $\Omega(\mathtt{K}_\varepsilon) \subset \mathtt{K}_\varepsilon$. Next, by standard methods and the Arzela-Ascoli theorem, it can be proved easily that the operator Ω is completely continuous. The proof is complete. □

3 Infinitely many positive solutions

For the existence of infinitely many positive solutions for iterative system of boundary value problem (1.1)–(1.2), we apply following theorems.

Theorem 3.1 ([10]). *Let $\mathcal{E}$ be a cone in a Banach space $\mathcal{X}$ and let $\mathtt{M}_1, \mathtt{M}_2$ be open sets with $0 \in \mathtt{M}_1, \overline{\mathtt{M}}_1 \subset \mathtt{M}_2$. Let $\mathcal{A} : \mathcal{E} \cap (\overline{\mathtt{M}}_2 \backslash \mathtt{M}_1) \to \mathcal{E}$ be a completely continuous operator such that* *(a)* $\|\mathcal{A}v\| \le \|v\|$, $v \in \mathcal{E} \cap \partial \mathtt{M}_1$, *and* $\|\mathcal{A}v\| \ge \|v\|$, $v \in \mathcal{E} \cap \partial \mathtt{M}_2$, *or*

(b) $\|\mathcal{A}v\| \ge \|v\|$, $v \in \mathcal{E} \cap \partial \mathtt{M}_1$, *and* $\|\mathcal{A}v\| \le \|v\|$, $v \in \mathcal{E} \cap \partial \mathtt{M}_2$.

Then $\mathcal{A}$ *has a fixed point in* $\mathcal{E} \cap (\overline{\mathtt{M}}_2 \backslash \mathtt{M}_1)$.

Theorem 3.2 ([7,15])**.** *Let* $f \in L^p_\nabla(J)$ *with* $p > 1$, $g \in L^q_\Delta(J)$ *with* $q > 1$, *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then* $fg \in L^1_\Delta(J)$ *and* $\|fg\|_{L^1_\Delta} \le \|f\|_{L^p_\Delta} \|g\|_{L^q_\Delta}$, *where*

$$\|f\|_{L^p_\Delta} := \begin{cases} \left[\int_J |f|^p(s)\Delta s \right]^{\frac{1}{p}}, & p \in \mathbb{R}, \\ \inf \left\{ M \in \mathbb{R} \,/\, |f| \le M \ \Delta - a.e. \ on \ J \right\}, & p = \infty, \end{cases}$$

and $J = [a,b)_{\mathbb{T}}$.

Theorem 3.3 (Hölder's inequality [3,4,15])**.** *Let* $f \in L^{p_i}_\Delta(J)$ *with* $p_i > 1$, *for* $i = 1,2,\ldots,n$ *and* $\sum_{i=1}^n \frac{1}{p_i} = 1$. *Then* $\prod_{i=1}^k \mathtt{g}_i \in L^1_\Delta(J)$ *and* $\left\| \prod_{i=1}^k \mathtt{g}_i \right\|_1 \le \prod_{i=1}^k \|\mathtt{g}_i\|_{p_i}$. *Further, if* $f \in L^1_\Delta(J)$ *and* $g \in L^\infty_\Delta(J)$, *then* $fg \in L^1_\Delta(J)$ *and* $\|fg\|_1 \le \|f\|_1 \|g\|_\infty$.

We need the following condition in the sequel:

(H_3) There exists $\delta_i > 0$ such that $\lambda_i(t) > \delta_i$ $(i = 1,2,\ldots,n)$ for $t \in [0,\mathfrak{T}]_{\mathbb{T}}$.

Consider the following three possible cases for $\lambda_i \in L^{p_i}_\Delta(0,\mathfrak{T})_{\mathbb{T}}$:

$$\sum_{i=1}^n \frac{1}{p_i} < 1, \quad \sum_{i=1}^n \frac{1}{p_i} = 1, \quad \sum_{i=1}^n \frac{1}{p_i} > 1.$$

Firstly, we seek infinitely many positive solutions for the case $\sum_{i=1}^n \frac{1}{p_i} < 1$.

Theorem 3.4. *Suppose* (H_1)–(H_3) *hold, let* $\{\varepsilon_r\}_{r=1}^\infty$ *be such that* $0 < \varepsilon_1 < \mathfrak{T}/2$, $\varepsilon \downarrow t^*$ *and* $0 < t^* < t_n$. *Let* $\{\Gamma_r\}_{r=1}^\infty$ *and* $\{\Lambda_r\}_{r=1}^\infty$ *be such that*

$$\Gamma_{r+1} < \mathcal{G}(\varepsilon_r)\Lambda_r < \Lambda_r < \theta\Lambda_r < \Gamma_r, \ r \in \mathbb{N},$$

where

$$\theta = \max\left\{ \left[\mathcal{G}(\varepsilon_1) \prod_{i=1}^k \delta_i \int_{\varepsilon_1}^{\mathfrak{T}-\varepsilon_1} \aleph(\tau,\tau)\Delta\tau \right]^{-1}, \ 1 \right\}.$$

Assume that $\mathtt{g}_\ell$ *satisfies*

(C_1) $\mathtt{g}_\ell(\upsilon) \le \mathfrak{N}_1\Gamma_r$ $\forall$ $t \in (0,\mathfrak{T})_{\mathbb{T}}$, $0 \le \upsilon \le \Gamma_r$, *where*

$$\mathfrak{N}_1 < \left[\|\aleph\|_{L^q_\Delta} \prod_{i=1}^k \|\lambda_i\|_{L^{p_i}_\Delta} \right]^{-1},$$

(C_2) $\mathtt{g}_\ell(\upsilon) \ge \theta\Lambda_r$ $\forall$ $t \in [\varepsilon_r, \mathfrak{T}-\varepsilon_r]_{\mathbb{T}}$, $\mathcal{G}(\varepsilon_r)\Lambda_r \le \upsilon \le \Lambda_r$.

Then the iterative boundary value problem (1.1)–(1.2) *has infinitely many solutions* $\{(\upsilon_1^{[r]}, \upsilon_2^{[r]}, \dots, \upsilon_m^{[r]})\}_{r=1}^{\infty}$ *such that* $\upsilon_\ell^{[r]}(t) \geq 0$ *on* $(0, \mathfrak{T})_{\mathbb{T}}$, $\ell = 1, 2, \dots, m$ *and* $r \in \mathbb{N}$.

Proof. Let

$$\mathtt{M}_{1,r} = \{\upsilon \in \mathtt{B} : \|\upsilon\| < \Gamma_r\}, \quad \mathtt{M}_{2,r} = \{\upsilon \in \mathtt{B} : \|\upsilon\| < \Lambda_r\},$$

be open subsets of $\mathtt{B}$. Let $\{\varepsilon_r\}_{r=1}^{\infty}$ be given in the hypothesis and we note that

$$t^* < t_{r+1} < \varepsilon_r < t_r < \frac{\mathfrak{T}}{2},$$

for all $r \in \mathbb{N}$. For each $r \in \mathbb{N}$, we define the cone $\mathtt{K}_{\varepsilon_r}$ by

$$\mathtt{K}_{\varepsilon_r} = \Big\{\upsilon \in \mathtt{B} : \upsilon(t) \geq 0, \min_{t \in [\varepsilon_r, \mathfrak{T} - \varepsilon_r]_{\mathbb{T}}} \upsilon(t) \geq \mathcal{G}(\varepsilon_r)\|\upsilon(t)\|\Big\}.$$

Let $\upsilon_1 \in \mathtt{K}_{\varepsilon_r} \cap \partial\mathtt{M}_{1,r}$. Then, $\upsilon_1(\tau) \leq \Gamma_r = \|\upsilon_1\|$ for all $\tau \in (0, \mathfrak{T})_{\mathbb{T}}$. By (C_1) and for $\tau_{m-1} \in (0, \mathfrak{T})_{\mathbb{T}}$, we have

$$\begin{aligned}\int_0^{\mathfrak{T}} \aleph(\tau_{m-1}, \tau_m)\lambda(\tau_m)\mathfrak{g}_m(\upsilon_1(\tau_m))\Delta\tau_m &\leq \int_0^{\mathfrak{T}} \aleph(\tau_m, \tau_m)\lambda(\tau_m)\mathfrak{g}_m(\upsilon_1(\tau_m))\Delta\tau_m \\ &\leq \mathfrak{N}_1\Gamma_r \int_0^{\mathfrak{T}} \aleph(\tau_m, \tau_m)\prod_{i=1}^{k}\lambda_i(\tau_m)\Delta\tau_m.\end{aligned}$$

There exists a $q > 1$ such that $\dfrac{1}{q} + \displaystyle\sum_{i=1}^{n}\dfrac{1}{p_i} = 1$. So,

$$\begin{aligned}\int_0^{\mathfrak{T}} \aleph(\tau_{m-1}, \tau_m)\lambda(\tau_m)\mathfrak{g}_m(\upsilon_1(\tau_m))\Delta\tau_m &\leq \mathfrak{N}_1\Gamma_r\big\|\aleph\big\|_{L^q_\Delta}\left\|\prod_{i=1}^{k}\lambda_i\right\|_{L^{p_i}_\Delta} \\ &\leq \mathfrak{N}_1\Gamma_r\|\aleph\|_{L^q_\Delta}\prod_{i=1}^{k}\|\lambda_i\|_{L^{p_i}_\Delta} \leq \Gamma_r.\end{aligned}$$

It follows in similar manner (for $\tau_{m-2} \in (0, \mathfrak{T})_{\mathbb{T}}$), that

$$\begin{aligned}&\int_0^{\mathfrak{T}} \aleph(\tau_{m-2}, \tau_{m-1})\lambda(\tau_{m-1})\mathfrak{g}_{m-1}\left(\int_0^{\mathfrak{T}} \aleph(\tau_{m-1}, \tau_m)\lambda(\tau_m)\mathfrak{g}_m(\upsilon_1(\tau_m))\Delta\tau_m\right)\Delta\tau_{m-1} \\ &\quad\leq \int_0^{\mathfrak{T}} \aleph(\tau_{m-2}, \tau_{m-1})\lambda(\tau_{m-1})\mathfrak{g}_{m-1}(\Gamma_r)\Delta\tau_{m-1} \\ &\quad\leq \int_0^{\mathfrak{T}} \aleph(\tau_{m-1}, \tau_{m-1})\lambda(\tau_{m-1})\mathfrak{g}_{m-1}(\Gamma_r)\Delta\tau_{m-1} \\ &\quad\leq \mathfrak{N}_1\Gamma_r\int_0^{\mathfrak{T}} \aleph(\tau_{m-1}, \tau_{m-1})\prod_{i=1}^{k}\lambda_i(\tau_{m-1})\Delta\tau_{m-1} \\ &\quad\leq \mathfrak{N}_1\Gamma_r\|\aleph\|_{L^q_\Delta}\prod_{i=1}^{k}\|\lambda_i\|_{L^{p_i}_\Delta} \leq \Gamma_r.\end{aligned}$$

Continuing with this bootstrapping argument, we get

$$\begin{aligned}(\Omega\upsilon_1)(t) &= \int_0^{\mathfrak{T}} \aleph(t,\tau_1)\lambda(\tau_1)\mathsf{g}_1\bigg(\int_0^{\mathfrak{T}} \aleph(\tau_1,\tau_2)\lambda(\tau_2)\mathsf{g}_2\bigg(\int_0^{\mathfrak{T}} \aleph(\tau_2,\tau_3)\cdots \\ &\quad \times \mathsf{g}_{m-1}\bigg(\int_0^{\mathfrak{T}} \aleph(\tau_{m-1},\tau_m)\lambda(\tau_m)\mathsf{g}_m(\upsilon_1(\tau_m))\Delta\tau_m\bigg)\cdots\Delta\tau_3\bigg)\Delta\tau_2\bigg)\Delta\tau_1 \\ &\leq \Gamma_r.\end{aligned}$$

Since $\Gamma_r = \|\upsilon_1\|$ for $\upsilon_1 \in \mathtt{K}_{\varepsilon_r} \cap \partial\mathtt{M}_{1,r}$, we get

$$\|\Omega\upsilon_1\| \leq \|\upsilon_1\|. \tag{3.1}$$

Let $t \in [\varepsilon_r, \mathfrak{T} - \varepsilon_r]_{\mathbb{T}}$. Then,

$$\Lambda_r = \|\upsilon_1\| \geq \upsilon_1(t) \geq \min_{t\in[\varepsilon_r,\mathfrak{T}-\varepsilon_r]_{\mathbb{T}}} \upsilon_1(t) \geq \mathcal{G}(\varepsilon_r)\,\|\upsilon_1\| \geq \mathcal{G}(\varepsilon_r)\Lambda_r.$$

By (C_2) and for $\tau_{m-1} \in [\varepsilon_r, \mathfrak{T} - \varepsilon_r]_{\mathbb{T}}$, we have

$$\begin{aligned}\int_0^{\mathfrak{T}} \aleph(\tau_{m-1},\tau_m)\lambda(\tau_m)\mathsf{g}_m(\upsilon_1(\tau_m))\Delta\tau_m &\geq \int_{\varepsilon_r}^{\mathfrak{T}-\varepsilon_r} \aleph(\tau_{m-1},\tau_m)\lambda(\tau_m)\mathsf{g}_m(\upsilon_1(\tau_m))\Delta\tau_m \\ &\geq \mathcal{G}(\varepsilon_r)\theta\Lambda_r \int_{\varepsilon_r}^{\mathfrak{T}-\varepsilon_r} \aleph(\tau_m,\tau_m)\lambda(\tau_m)\Delta\tau_m \\ &\geq \mathcal{G}(\varepsilon_r)\theta\Lambda_r \int_{\varepsilon_r}^{\mathfrak{T}-\varepsilon_r} \aleph(\tau_m,\tau_m)\prod_{i=1}^{k}\lambda_i(\tau_m)\Delta\tau_m \\ &\geq \mathcal{G}(\varepsilon_1)\theta\Lambda_r \prod_{i=1}^{k}\delta_i \int_{\varepsilon_1}^{\mathfrak{T}-\varepsilon_1} \aleph(\tau_m,\tau_m)\Delta\tau_m \\ &\geq \Lambda_r.\end{aligned}$$

Continuing with the bootstrapping argument, we get

$$\begin{aligned}(\Omega\upsilon_1)(t) &= \int_0^{\mathfrak{T}} \aleph(t,\tau_1)\lambda(\tau_1)\mathsf{g}_1\bigg(\int_0^{\mathfrak{T}} \aleph(\tau_1,\tau_2)\lambda(\tau_2)\mathsf{g}_2\bigg(\int_0^{\mathfrak{T}} \aleph(\tau_2,\tau_3)\cdots \\ &\quad \times \mathsf{g}_{m-1}\bigg(\int_0^{\mathfrak{T}} \aleph(\tau_{m-1},\tau_m)\lambda(\tau_m)\mathsf{g}_m(\upsilon_1(\tau_m))\Delta\tau_m\bigg)\cdots\Delta\tau_3\bigg)\Delta\tau_2\bigg)\Delta\tau_1 \\ &\geq \Lambda_r.\end{aligned}$$

Thus, if $\upsilon_1 \in \mathtt{K}_{\varepsilon_r} \cap \partial\mathtt{K}_{2,r}$, then

$$\|\Omega\upsilon_1\| \geq \|\upsilon_1\|. \tag{3.2}$$

It is evident that $0 \in \mathtt{M}_{2,k} \subset \overline{\mathtt{M}}_{2,k} \subset \mathtt{M}_{1,k}$. From (3.1)–(3.2), it follows from Theorem 3.1 that the operator Ω has a fixed point $\upsilon_1^{[r]} \in \mathtt{K}_{\varepsilon_r} \cap \big(\overline{\mathtt{M}}_{1,r}\backslash\mathtt{M}_{2,r}\big)$ such that $\upsilon_1^{[r]}(t) \geq 0$ on $(0,\mathfrak{T})_{\mathbb{T}}$, and $r \in \mathbb{N}$. Next setting $\upsilon_{m+1} = \upsilon_1$, we obtain infinitely many positive solutions $\{(\upsilon_1^{[r]}, \upsilon_2^{[r]}, \ldots, \upsilon_m^{[r]})\}_{r=1}^{\infty}$ of (1.1)–(1.2) given iteratively by

$$\upsilon_\ell(t) = \int_0^{\mathfrak{T}} \aleph(t,\tau)\lambda(\tau)\mathsf{g}_\ell(\upsilon_{\ell+1}(\tau))\Delta\tau,\ \ t \in (0,\mathfrak{T})_{\mathbb{T}},\ \ \ell = m, m-1, \ldots, 1.$$

The proof is completed. □

For $\displaystyle\sum_{i=1}^{n}\frac{1}{p_i}=1$, we have the following theorem.

Theorem 3.5. *Suppose* (H_1)–(H_3) *hold, let* $\{\varepsilon_r\}_{r=1}^{\infty}$ *be such that* $0<\varepsilon_1<\mathfrak{T}/2$, $\varepsilon\downarrow t^*$ *and* $0<t^*<t_n$. *Let* $\{\Gamma_r\}_{r=1}^{\infty}$ *and* $\{\Lambda_r\}_{r=1}^{\infty}$ *be such that*

$$\Gamma_{r+1}<\mathcal{G}(\varepsilon_r)\Lambda_r<\Lambda_r<\theta\Lambda_r<\Gamma_r,\ r\in\mathbb{N},$$

where

$$\theta=\max\left\{\left[\mathcal{G}(\varepsilon_1)\prod_{i=1}^{k}\delta_i\int_{\varepsilon_1}^{\mathfrak{T}-\varepsilon_1}\aleph(\tau,\tau)\Delta\tau\right]^{-1},\ 1\right\}.$$

Assume that $\mathtt{g}_\ell$ *satisfies* (C_2) *and*

(C_3) $\mathtt{g}_j(\upsilon)\leq\mathfrak{N}_2\Gamma_r\ \forall\ t\in(0,\mathfrak{T})_{\mathbb{T}}$, $0\leq\upsilon\leq\Gamma_r$, *where*

$$\mathfrak{N}_2<\min\left\{\left[\|\aleph\|_{L_\Delta^\infty}\prod_{i=1}^{k}\|\lambda_i\|_{L_\Delta^{p_i}}\right]^{-1},\theta\right\}.$$

Then the iterative boundary value problem (1.1)–(1.2) *has infinitely many solutions* $\{(\upsilon_1^{[r]},\upsilon_2^{[r]},\ldots,\upsilon_m^{[r]})\}_{r=1}^{\infty}$ *such that* $\upsilon_\ell^{[r]}(t)\geq 0$ *on* $(0,\mathfrak{T})_{\mathbb{T}}$, $\ell=1,2,\ldots,m$ *and* $r\in\mathbb{N}$.

Proof. For a fixed r, let $\mathtt{M}_{1,r}$ be as in the proof of Theorem 3.4 and let $\upsilon_1\in\mathtt{K}_{\varepsilon_r}\cap\partial\mathtt{M}_{2,r}$. Again

$$\upsilon_1(\tau)\leq\Gamma_r=\|\upsilon_1\|,$$

for all $\tau\in(0,\mathfrak{T})_{\mathbb{T}}$. By (C_3) and for $\tau_{\ell-1}\in(0,\mathfrak{T})_{\mathbb{T}}$, we have

$$\begin{aligned}\int_0^{\mathfrak{T}}\aleph(\tau_{m-1},\tau_m)\lambda(\tau_m)\mathtt{g}_m(\upsilon_1(\tau_m))\Delta\tau_m&\leq\int_0^{\mathfrak{T}}\aleph(\tau_m,\tau_m)\lambda(\tau_m)\mathtt{g}_m(\upsilon_1(\tau_m))\Delta\tau_m\\&\leq\mathfrak{N}_1\Gamma_r\int_0^{\mathfrak{T}}\aleph(\tau_m,\tau_m)\prod_{i=1}^{k}\lambda_i(\tau_m)\Delta\tau_m\\&\leq\mathfrak{N}_1\Gamma_r\|\aleph\|_{L_\Delta^\infty}\left\|\prod_{i=1}^{k}\lambda_i\right\|_{L_\Delta^{p_i}}\\&\leq\mathfrak{N}_1\Gamma_r\|\aleph\|_{L_\Delta^\infty}\prod_{i=1}^{k}\|\lambda_i\|_{L_\Delta^{p_i}}\leq\Gamma_r.\end{aligned}$$

It follows in similar manner (for $\tau_{m-2} \in [0,1]_{\mathbb{T}}$), that

$$\begin{aligned}
\int_0^{\mathfrak{T}} \aleph(\tau_{m-2},\tau_{m-1})\lambda(\tau_{m-1})\mathsf{g}_{m-1}\left(\int_0^{\mathfrak{T}} \aleph(\tau_{m-1},\tau_m)\lambda(\tau_m)\mathsf{g}_m(\upsilon_1(\tau_m))\Delta\tau_m\right)\Delta\tau_{m-1}&\\
\leq \int_0^{\mathfrak{T}} \aleph(\tau_{m-2},\tau_{m-1})\lambda(\tau_{m-1})\mathsf{g}_{m-1}(\Gamma_r)\Delta\tau_{m-1}&\\
\leq \int_0^{\mathfrak{T}} \aleph(\tau_{m-1},\tau_{m-1})\lambda(\tau_{m-1})\mathsf{g}_{m-1}(\Gamma_r)\Delta\tau_{m-1}&\\
\leq \mathfrak{N}_1\Gamma_r \int_0^{\mathfrak{T}} \aleph(\tau_{m-1},\tau_{m-1})\prod_{i=1}^{k}\lambda_i(\tau_{m-1})\Delta\tau_{m-1}&\\
\leq \mathfrak{N}_1\Gamma_r\|\aleph\|_{L^\infty_\Delta}\prod_{i=1}^{k}\|\lambda_i\|_{L^{p_i}_\Delta} \leq \Gamma_r.&
\end{aligned}$$

Continuing with this bootstrapping argument, we get

$$\begin{aligned}
(\Omega\upsilon_1)(t) &= \int_0^{\mathfrak{T}} \aleph(t,\tau_1)\lambda(\tau_1)\mathsf{g}_1\left(\int_0^{\mathfrak{T}} \aleph(\tau_1,\tau_2)\lambda(\tau_2)\mathsf{g}_2\left(\int_0^{\mathfrak{T}} \aleph(\tau_2,\tau_3)\cdots\right.\right.\\
&\quad\left.\left.\times\, \mathsf{g}_{m-1}\left(\int_0^{\mathfrak{T}} \aleph(\tau_{m-1},\tau_m)\lambda(\tau_m)\mathsf{g}_m(\upsilon_1(\tau_m))\Delta\tau_m\right)\cdots\Delta\tau_3\right)\Delta\tau_2\right)\Delta\tau_1\\
&\leq \Gamma_r.
\end{aligned}$$

Since $\Gamma_r = \|\upsilon_1\|$ for $\upsilon_1 \in \mathtt{K}_{\varepsilon_r} \cap \partial\mathtt{M}_{1,r}$, we get

$$\|\Omega\upsilon_1\| \leq \|\upsilon_1\|. \tag{3.3}$$

Now define $\mathtt{M}_{2,r} = \{\upsilon_1 \in \mathtt{B} : \|\upsilon_1\| < \Lambda_r\}$. Let $\upsilon_1 \in \mathtt{K}_{\varepsilon_r} \cap \partial\mathtt{M}_{2,r}$ and let $\tau \in [\varepsilon_r, \mathfrak{T} - \varepsilon_r]_{\mathbb{T}}$. Then, the argument leading to (3.2) can be done to the present case. Hence, the theorem. $\square$

Lastly, the case $\displaystyle\sum_{i=1}^{n}\frac{1}{p_i} > 1$.

Theorem 3.6. *Suppose (H_1)–(H_3) hold, let $\{\varepsilon_r\}_{r=1}^\infty$ be such that $0 < \varepsilon_1 < \mathfrak{T}/2$, $\varepsilon \downarrow t^*$ and $0 < t^* < t_n$. Let $\{\Gamma_r\}_{r=1}^\infty$ and $\{\Lambda_r\}_{r=1}^\infty$ be such that*

$$\Gamma_{r+1} < \mathcal{G}(\varepsilon_r)\Lambda_r < \Lambda_r < \theta\Lambda_r < \Gamma_r,\ r \in \mathbb{N},$$

where

$$\theta = \max\left\{\left[\mathcal{G}(\varepsilon_1)\prod_{i=1}^{k}\delta_i\int_{\varepsilon_1}^{\mathfrak{T}-\varepsilon_1}\aleph(\tau,\tau)\Delta\tau\right]^{-1},\ 1\right\}.$$

Assume that g_ℓ satisfies (C_2) and

(C_4) $\mathsf{g}_j(\upsilon) \leq \mathfrak{N}_3\Gamma_r\ \forall\ t \in (0,\mathfrak{T})_{\mathbb{T}},\ 0 \leq \upsilon \leq \Gamma_r,$ *where* $\mathfrak{N}_3 < \min\left\{\left[\|\aleph\|_{L^\infty_\Delta}\prod_{i=1}^{k}\|\lambda_i\|_{L^1_\Delta}\right]^{-1}, \theta\right\}$.

Then the iterative boundary value problem (1.1)–(1.2) has infinitely many solutions $\{(\upsilon_1^{[r]}, \upsilon_2^{[r]}, \ldots, \upsilon_m^{[r]})\}_{r=1}^\infty$ such that $\upsilon_\ell^{[r]}(t) \geq 0$ on $(0,\mathfrak{T})_{\mathbb{T}}$, $\ell = 1,2,\ldots,m$ and $r \in \mathbb{N}$.

Proof. The proof is similar to the proof of Theorem 3.4. So, we omit the details here. $\square$

4 Example

In this section, we present an example to check validity of our main results.

Example 4.1. Consider the following boundary value problem on $\mathbb{T} = \mathbb{R}$.

$$\left.\begin{aligned} \upsilon_\ell''(t) + \lambda(t)\mathrm{g}_\ell(\upsilon_{\ell+1}(t)) = 0,\ \ell = 1, 2, \\ \upsilon_3(t) = \upsilon_1(t), \end{aligned}\right\} \tag{4.1}$$

$$\left.\begin{aligned} \upsilon_\ell(0) = \upsilon_\ell'(0), \\ \upsilon_\ell(1) = -\upsilon_\ell'(1), \end{aligned}\right\} \tag{4.2}$$

where

$$\lambda(t) = \lambda_1(t)\lambda_2(t)$$

in which

$$\lambda_1(t) = \frac{1}{|t - \frac{1}{4}|^{\frac{1}{2}}} \quad \text{and} \quad \lambda_2(t) = \frac{1}{|t - \frac{3}{4}|^{\frac{1}{2}}},$$

$$\mathrm{g}_1(\upsilon) = \mathrm{g}_2(\upsilon) = \begin{cases} \frac{1}{5} \times 10^{-4}, & \upsilon \in (10^{-4}, +\infty), \\ \frac{25\times 10^{-(4r+3)} - \frac{1}{5}\times 10^{-4r}}{10^{-(4r+3)} - 10^{-4r}}(\upsilon - 10^{-4r}) + & \\ \frac{1}{5} \times 10^{-8r}, & \upsilon \in \left[10^{-(4r+3)}, 10^{-4r}\right], \\ 25 \times 10^{-(4r+3)}, & \upsilon \in \left(\frac{1}{5} \times 10^{-(4r+3)}, 10^{-(4r+3)}\right), \\ \frac{25\times 10^{-(4r+3)} - \frac{1}{5}\times 10^{-8r}}{\frac{1}{5}\times 10^{-(4r+3)} - 10^{-(4r+4)}}(\upsilon - 10^{-(4r+4)}) + & \\ \frac{1}{5} \times 10^{-8r}, & \upsilon \in \left(10^{-(4r+4)}, \frac{1}{5} \times 10^{-(4r+3)}\right], \\ 0, & \upsilon = 0. \end{cases}$$

Let

$$t_r = \frac{31}{64} - \sum_{k=1}^{r} \frac{1}{4(k+1)^4}, \quad \varepsilon_r = \frac{1}{2}(t_r + t_{r+1}), \quad r = 1, 2, 3, \ldots,$$

then

$$\varepsilon_1 = \frac{15}{32} - \frac{1}{648} < \frac{15}{32},$$

and

$$t_{r+1} < \varepsilon_r < t_r, \quad \varepsilon_r > \frac{1}{5}.$$

Therefore,

$$\mathcal{G}(\varepsilon_r) = \frac{\varepsilon_r + 1}{\mathfrak{T} + 1} = \frac{\varepsilon_r + 1}{2} > \frac{1}{5}, \quad r = 1, 2, 3, \ldots$$

It is clear that

$$t_1 = \frac{15}{32} < \frac{1}{2}, \quad t_r - t_{r+1} = \frac{1}{4(r+2)^4}, \quad r = 1, 2, 3, \ldots$$

Since $\sum_{x=1}^{\infty} \frac{1}{x^4} = \frac{\pi^4}{90}$ and $\sum_{x=1}^{\infty} \frac{1}{x^2} = \frac{\pi^2}{6}$, it follows that

$$t^* = \lim_{r\to\infty} t_r = \frac{31}{64} - \sum_{k=1}^{\infty} \frac{1}{4(r+1)^4} = \frac{47}{64} - \frac{\pi^4}{360} = 0.4637941914,$$

$$\lambda_1, \lambda_2 \in L^p[0,1] \quad \text{for all} \quad 0<p<2, \quad \text{and} \quad \delta_1=\delta_2=(4/3)^{1/4},$$

$$\mathcal{G}(\varepsilon_1)=0.7336033951.$$

$$\int_{\varepsilon_1}^{\mathfrak{T}-\varepsilon_1} \aleph(\tau,\tau)\Delta\tau = \int_{\frac{15}{32}-\frac{1}{648}}^{1-\frac{15}{32}+\frac{1}{648}} \frac{(2-\tau)(1+\tau)}{3} d\tau = 0.04918197801.$$

Thus, we get

$$\begin{aligned}\theta &= \max\left\{\left[\mathcal{G}(\varepsilon_1)\prod_{i=1}^{k}\delta_i\int_{\varepsilon_1}^{\mathfrak{T}-\varepsilon_1}\aleph(\tau,\tau)\nabla\tau\right]^{-1}, \ 1\right\}\\ &= \max\left\{\frac{1}{0.04166167167}, 1\right\}\\ &= 24.00287746.\end{aligned}$$

Next, let $0<\mathfrak{a}<1$ be fixed. Then $\lambda_1,\lambda_2 \in L^{1+\mathfrak{a}}[0,1]$ and $\frac{2}{1+\mathfrak{a}}>1$ for $0<\mathfrak{a}<1$. It follows that

$$\prod_{i=1}^{k}\|\lambda_i\|_{L_{\Delta}^{p_i}} \approx \pi - \ln(7-4\sqrt{3}),$$

and also $\|\aleph\|_\infty = \frac{2}{3}$. So, for $0<\mathfrak{a}<1$, we have

$$\mathfrak{N}_1 < \left[\|\aleph\|_\infty \prod_{i=1}^{k}\|\lambda_i\|_{L_{\Delta}^{p_i}}\right]^{-1} \approx 0.2597173925.$$

Taking $\mathfrak{N}_1 = \frac{1}{4}$. In addition if we take

$$\Gamma_r = 10^{-4r}, \quad \Lambda_r = 10^{-(4r+3)},$$

then

$$\Gamma_{r+1} = 10^{-(4r+4)} < \frac{1}{5}\times 10^{-(4r+3)} < \mathcal{G}(\varepsilon_r)\Lambda_r < \Lambda_r = 10^{-(4r+3)} < \Gamma_r = 10^{-4r},$$

$\theta\Lambda_r = 24.00287746\times 10^{-(4r+3)} < \frac{1}{4}\times 10^{-4r} = \mathfrak{N}_1\Gamma_r$, $r=1,2,3,\ldots$, and $\mathsf{g}_1, \mathsf{g}_2$ satisfy the following growth conditions:

$$\begin{aligned}&\mathsf{g}_1(\upsilon)=\mathsf{g}_2(\upsilon) \le \mathfrak{N}_1\Gamma_r = \frac{1}{4}\times 10^{-4r}, \quad \upsilon \in \left[0, 10^{-4r}\right],\\ &\mathsf{g}_1(\upsilon)=\mathsf{g}_2(\upsilon) \ge \theta\Lambda_r = 24.00287746\times 10^{-(4r+3)}, \quad \upsilon \in \left[\frac{1}{5}\times 10^{-(4r+3)}, 10^{-(4r+3)}\right].\end{aligned}$$

Then all the conditions of Theorem 3.4 are satisfied. Therefore, by Theorem 3.4, the iterative boundary value problem (1.1) has infinitely many solutions $\{(\upsilon_1^{[r]}, \upsilon_2^{[r]})\}_{r=1}^{\infty}$ such that $\upsilon_\ell^{[r]}(t)\ge 0$ on $[0,1]$, $\ell=1,2$ and $r\in\mathbb{N}$.

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