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Uniqueness of entire functions whose difference polynomials share a polynomial with finite weight

Goutam Haldar[1]

[1] *Department of Mathematics, Malda College, Rabindra Avenue, Malda, West Bengal 732101, India.*
goutamiit1986@gmail.com
goutamiitm@gmail.com

ABSTRACT

In this paper, we use the concept of weighted sharing of values to investigate the uniqueness results when two difference polynomials of entire functions share a nonzero polynomial with finite weight. Our result improves and extends some recent results due to Sahoo-Karmakar [J. Cont. Math. Anal. 52(2) (2017), 102–110] and that of Li *et al.* [Bull. Malays. Math. Sci. Soc., 39 (2016), 499–515]. Some examples have been exhibited which are relevant to the content of the paper.

RESUMEN

En este artículo, usamos el concepto de intercambio pesado de valores para investigar los resultados de unicidad cuando dos polinomios de diferencia de funciones enteras comparten un polinomio no cero con peso finito. Nuestro resultado mejora y extiende algunos resultados de Sahoo-Karmakar [J. Cont. Math. Anal. 52(2) (2017), 102–110] y los de Li *et al.* [Bull. Malays. Math. Sci. Soc., 39 (2016), 499–515]. Se exhiben algunos ejemplos que son relevantes para el contenido del artículo.

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1 Introduction

Let f and g be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$, the zero of $f - a$ and $g - a$ have the same locations as well as same multiplicities, we say that f and g share the value a CM (counting multiplicities), and if we do not consider the multiplicities into account, then f and g are said to share the value a IM (ignoring multiplicities)(see [37]). We adopt the standard notations of the Nevanlinna theory of meromorphic functions (see [14, 22, 41]). For a non-constant meromorphic function f, we denote by $T(r, f)$ the Nevanlinna characteristic function of f and by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$ outside of an exceptional set of finite linear measure.

We define shift and difference operators of $f(z)$ by $f(z + c)$ and $\Delta_c f(z) = f(z + c) - f(z)$, respectively. Note that $\Delta_c^n f(z) = \Delta_c^{n-1}(\Delta_c f(z))$, where c is a nonzero complex number and $n \geq 2$ is a positive integer.

For further generalization of $\Delta_c f$, we now define the linear difference operator of an entire (meromorphic) function f as $L_c(f) = f(z + c) + c_0 f(z)$, where c_0 is a finite complex constant. Clearly, for the particular choice of the constant $c_0 = -1$, we get $L_c(f) = \Delta_c f$.

In 1959, Hayman [13] proved the following result.

Theorem A ([13]). *Let f be a transcendental entire function and let n be an integer such that $n \geq 1$. Then $f^n f' = 1$ has infinitely many solutions.*

A number of authors have shown their interest to find the uniqueness of entire and meromorphic functions whose differential polynomials share certain values or fixed points, and obtained some remarkable results (see [3, 9, 10, 26, 33, 34, 36, 37, 39, 42]).

In recent years, the difference variant of the Nevanlinna theory has been established in [8, 11, 12]. Using these theories, some mathematicians in the world began to study the uniqueness questions of meromorphic functions sharing values with their shifts, and study the value distribution of the nonlinear difference polynomials, and produced many fine works, for example, see [1, 5, 6, 7, 11, 15, 16, 23, 27, 29, 30, 31, 40, 44]. We recall the following result from Laine-Yang [23].

Theorem B ([23]). *Let f be a transcendental entire function of finite order, and c be a non-zero complex constant. Then, for $n \geq 2$, $f(z)^n f(z + c)$ assumes every non-zero value $a \in \mathbb{C}$ infinitely often.*

Later on, Liu-Yang [28] extended Theorem B, and proved the following result:

Theorem C ([28]). *Let f be a transcendental entire function of finite order, and let η be a nonzero complex constant. Then for $n \geq 2$ the function $f(z)^n f(z + \eta) - P_0(z)$ has infinitely many zeros, where P_0 is any given polynomial such that $P_0 \not\equiv 0$.*

Regarding uniqueness corresponding to Theorem C, Li *et al.* [24] obtained the following result.

Theorem D ([24]). *Let f and g be two distinct transcendental entire functions of finite order, and let $P_0 \not\equiv 0$ be a polynomial. Let η is a nonzero complex constant and $n \geq 4$ is an integer such that $2\deg(P_0) < n+1$. Also, suppose that $f(z)^n f(z+\eta) - P_0(z)$ and $g(z)^n g(z+\eta) - P_0(z)$ share 0 CM. Then one of the following assertions holds.*

(I) If $n \geq 4$ and $f(z)^n f(z+\eta)/P_0(z)$ is a Mobius transformation of $g(z)^n g(z+\eta)/P_0(z)$, then either

(i) $f = tg$, where t is a constant satisfying $t^{n+1} = 1$

(ii) $f = e^Q$ and $g = te^{-Q}$, where P_0 reduces to a nonzero constant c, t is a constant such that $t^{n+1} = c^2$, and Q is a non-constant polynomial.

(II) If $n \geq 6$, then (I)(i) or (I)(ii) holds.

In 2016, Li-Li [25] obtained the IM analogues of the above Theorem D as follows.

Theorem E ([25]). *Let f and g be two distinct transcendental entire functions of finite order, and let $P_0 \not\equiv 0$ be a polynomial. Let η is a nonzero complex constant and $n \geq 4$ is an integer such that $2\deg(P_0) < n+1$. Also, suppose that $f(z)^n f(z+\eta) - P_0(z)$ and $g(z)^n g(z+\eta) - P_0(z)$ share 0 IM. Then one of the following assertions holds.*

(I) If $n \geq 4$ and $f(z)^n f(z+\eta)/P_0(z)$ is a Mobius transformation of $g(z)^n g(z+\eta)/P_0(z)$, then either

(i) $f = tg$, where t is a constant satisfying $t^{n+1} = 1$,

(ii) $f = e^Q$ and $g = te^{-Q}$, where P_0 reduces to a nonzero constant c, t is a constant such that $t^{n+1} = c^2$, and Q is a non-constant polynomial.

(II) If $n \geq 12$, then (I)(i) or (I)(ii) holds.

In 2001, the notion of weighted sharing was originally defined in the literature ([18, 19]), which is the gradual change of shared values from CM to IM. Below we recall the definition.

Definition 1.1 ([18, 19]). *Let k be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.*

Clearly, if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞), respectively.

Using the notion of weighted sharing, Sahoo-Karmakar [35] further improved Theorem D as follows.

Theorem F ([35])**.** *Let f, g, P_0 and n be defined as in Theorem D. Suppose that $f(z)^n f(z+\eta) - P_0(z)$ and $g(z)^n g(z+\eta) - P_0(z)$ share $(0, 2)$.*

(I) If $n \geq 4$ and $f(z)^n f(z+\eta)/P_0(z)$ is a Mobius transformation of $g(z)^n g(z+\eta)/P_0(z)$, then either

(i) $f = tg$, where t is a constant satisfying $t^{n+1} = 1$

(ii) $f = e^Q$ and $g = te^{-Q}$, where P_0 reduces to a nonzero constant c, t is a constant such that $t^{n+1} = c^2$, and Q is a non-constant polynomial.

(II) If $n \geq 6$, then (I)(i) or (I)(ii) holds.

Observing the above results, it is natural to ask the following questions.

Question 1.2. *What can be said about the relationship of two finite order non-constant meromorphic functions f and g if their more general nonlinear difference polynomials $f(z)^n L_c(f)$ and $g(z)^n L_c(g)$ share a polynomial $P(z) \not\equiv 0$, where $L_c(f) = f(z+c) + c_0 f(z)$ with c and c_0 being finite nonzero complex constants, and $n \geq 2$ being a positive integer?*

Question 1.3. *Is it possible to further reduce the nature of sharing from* $(0, 2)$ *to* $(0, 1)$ *in Theorem* F*?*

Question 1.4. *Can the lower bound of n be further reduced in Theorems* E *and* F*?*

Question 1.5. *What can be said about the uniqueness of f and g if we consider the difference polynomial of the form $f(z)^n \Delta_c f$ and $g(z)^n \Delta_c g$ in Theorems* E *and* F*?*

The purpose of this paper is to answer all the questions raised above. In fact we have been successfully able to reduce the nature of sharing of $f(z)^n f(z+\eta) - P_0(z)$ and $g(z)^n g(z+\eta) - P_0(z)$ in Theorem F. We have also reduced the lower bound of n in Theorems E and F successfully.

2 Main results

Now we state our main result.

Theorem 2.1. *Let f and g be two transcendental entire functions of finite order, $P \not\equiv 0$ be a polynomial. Let c be a non-zero complex constant, and n be a positive integer such that $2\deg(P) < n+1$. Let l be a non-negative integer such that $f(z)^n L_c(f) - P(z)$ and $g(z)^n L_c(g) - P(z)$ share $(0, l)$ and $g(z)$, $g(z+c)$ share* 0 *CM. If $n \geq 4$ and $f(z)^n L_c(f)/P(z)$ is a Mobius transformation of $g(z)^n L_c(g)/P(z)$, or one of the following conditions holds:*

(i) $l \geq 2$ and $n \geq 5$;

(ii) $l = 1$ *and* $n \geq 6$;

(iii) $l = 0$ *and* $n \geq 11$, *then one of the following conclusions can be realized:*

(a) $f = tg$, *where* t *is a constant satisfying* $t^{n+1} = 1$;

(b) *When* $c_0 = 0$, $f = e^U$ *and* $g = te^{-U}$, *where* $P(z)$ *reduces to a nonzero constant* d, t *is a constant such that* $t^{n+1} = d^2$ *and* U *is a non-constant polynomial;*

(c) *When* $c_0 \neq 0$, $f = c_1 e^{az}$, $g(z) = c_2 e^{-az}$, *where* a, c_1, c_2 *and* d *are non-zero constants satisfying* $(c_1c_2)^{n+1}(e^{ac} + c_0)(e^{-ac} + c_0) = d^2$.

If $L_c(f) = \Delta_c f$, then one can easily get the following corollary from Theorem 2.1 which answers Question 1.5.

Corollary 2.2. *Let* f *and* g *be two transcendental entire functions of finite order,* $P \not\equiv 0$ *be a polynomial. Let* c *be a non-zero complex constant, and* n *be a positive integer such that* $2\deg(P) < n+1$. *Let* l *be a non-negative integer such that* $f(z)^n\Delta_c f - P(z)$ *and* $g(z)^n\Delta_c g - P(z)$ *share* $(0, l)$ *and* $g(z)$, $g(z+c)$ *share* 0 *CM. If* $n \geq 4$ *and* $f(z)^n\Delta_c(f)/P(z)$ *is a Mobius transformation of* $g(z)^n\Delta_c(g)/P(z)$, *or one of the following conditions holds:*

(i) $l \geq 2$ *and* $n \geq 5$;

(ii) $l = 1$ *and* $n \geq 6$;

(iii) $l = 0$ *and* $n \geq 11$, *then one of the following conclusions can be realized:*

(a) $f = tg$, *where* t *is a constant satisfying* $t^{n+1} = 1$;

(b) $f = c_1e^{az}$, $g(z) = c_2e^{-az}$, *where* a, c_1, c_2 *and* d *are non-zero constants satisfying* $(c_1c_2)^{n+1}(e^{ac} + c_0)(e^{-ac} + c_0) = d^2$.

The following examples show that both the conclusions of Theorem 2.1 actually holds.

Example 2.3. *Let* $f(z) = e^z$ *and* $g = tf$, *where* t *is a constant such that* $t^{n+1} = 1$, *and* η *be any non-zero complex constant. Then for any given polynomial* p *such that* $p \not\equiv 0$ *with* $2\deg(p) < n+1$, $f(z)^n f(z+\eta) - p(z)$ *and* $g(z)^n g(z+\eta) - p(z)$ *share* $(0, \infty)$. *Also* $f(z)^n(f(z+\eta) - f(z)) - p(z)$ *and* $g(z)^n(g(z+\eta) - g(z)) - p(z)$*share* $(0, \infty)$. *Here* f *and* g *satisfy the conclusion* (a) *of Theorem 2.1.*

Example 2.4. *Let* $f(z) = e^{2\pi iz/\eta}$ *and* $g(z) = te^{-2\pi iz/\eta}$, *where* t *is a constant such that* $t^{n+1} = 1$, η *is a non-zero complex constant. Then* $f(z)^n f(z+\eta)$ *and* $g(z)^n g(z+\eta)$ *share* $(1, \infty)$. *Here* f *and* g *satisfy the conclusion* (b) *of Theorem 2.1.*

Example 2.5. *Let* $f(z) = e^z$, $g(z) = e^{-z}$, $\eta = -\log(-1)$ *and* $P(z) = 2$. *Then one can easily verify that* $f(z)^n(f(z+\eta) - f(z))$ *and* $g(z)^n(g(z+\eta) - g(z))$ *share* $(2, \infty)$. *Here* f *and* g *satisfy the conclusion* (b) *of Theorem 2.1.*

The following example shows that Theorem 2.1 is not true for infinite order entire functions.

Example 2.6. *Let* $f(z) = \dfrac{e^{2\pi iz/\eta}}{e^{e^{2\pi iz/\eta}}}$ *and* $g(z) = \dfrac{1}{e^{e^{2\pi iz/\eta}}}$*, where* η *is a non-zero constant. Then it is easy to verify that* $f(z)^n f(z+\eta)$ *and* $g(z)^n g(z+\eta)$ *share* $(1,\infty)$*. But there does not exist a non-zero constant* t *such that* $f = tg$ *or* $fg = t$*, where* $t^{n+1} = 1$*.*

3 Auxiliary definitions

Throughout the paper we have used the following definitions and notations.

Definition 3.1 ([17]). *Let* $a \in \mathbb{C} \cup \{\infty\}$*. We denote by* $N(r, a; f \mid= 1)$ *the counting function of simple a points of* f*. For* $p \in \mathbb{N}$ *we denote by* $N(r, a; f \mid\leq p)$ *the counting function of those a-points of* f *(counted with multiplicities) whose multiplicities are not greater than* p*. By* $\overline{N}(r, a; f \mid\leq p)$ *we denote the corresponding reduced counting function. In a similar manner we can define* $N(r, a; f \mid\geq p)$ *and* $\overline{N}(r, a; f \mid\geq p)$*.*

Definition 3.2 ([19]). *Let* $p \in \mathbb{N} \cup \{\infty\}$*. We denote by* $N_p(r, a; f)$ *the counting function of a-points of* f*, where an a-point of multiplicity* m *is counted* m *times if* $m \leq p$ *and* p *times if* $m > p$*. Then* $N_p(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2) + \cdots + \overline{N}(r, a; f \mid\geq p)$*. Clearly* $N_1(r, a; f) = \overline{N}(r, a; f)$*.*

Definition 3.3 ([43]). *Let* f *and* g *be two non-constant meromorphic functions such that* f *and* g *share* $(a, 0)$*. Let* z_0 *be an a-point of* f *with multiplicity* p*, an a-point of* g *with multiplicity* q*. We denote by* $\overline{N}_L(r, a; f)$ *the reduced counting function of those a-points of* f *and* g *where* $p > q$*, by* $N_E^{1)}(r, a; f)$ *the counting function of those a-points of* f *and* g *where* $p = q = 1$*, by* $\overline{N}_E^{(2}(r, a; f)$ *the reduced counting function of those a-points of* f *and* g *where* $p = q \geq 2$*. In the same way we can define* $\overline{N}_L(r, a; g)$*,* $\overline{N}_E^{1)}(r, a; g)$*,* $\overline{N}_E^{(2}(r, a; g)$*. In a similar manner we can define* $\overline{N}_L(r, a; f)$ *and* $\overline{N}_L(r, a; g)$ *for* $a \in \mathbb{C} \cup \{\infty\}$*.*

When f and g share (a, m), $m \geq 1$, then $N_E^{1)}(r, a; f) = N(r, a; f \mid= 1)$.

Definition 3.4 ([19]). *Let* f*,* g *share a value* $(a, 0)$*. We denote by* $\overline{N}_*(r, a; f, g)$ *the reduced counting function of those a-points of* f *whose multiplicities differ from the multiplicities of the corresponding a-points of* g*. Clearly* $\overline{N}_*(r, a; f, g) = \overline{N}_*(r, a; g, f)$ *and* $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$*.*

4 Some lemmas

We now prove several lemmas which will play key roles in proving the main results of the paper. Let F and G be two non-constant meromorphic functions. Henceforth we shall denote by H the following function

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{4.1}$$

Lemma 4.1 ([8])**.** *Let $f(z)$ be a meromorphic function of finite order ρ, and let c be a fixed non-zero complex constant. Then for each $\epsilon > 0$, we have*

$$T(r, f(z+c)) = T(r, f) + O(r^{\rho-1+\epsilon}) + O\{\log r\}.$$

Lemma 4.2 ([8])**.** *Let $f(z)$ be a meromorphic function of finite order ρ and let c be a non-zero complex number. Then for each $\epsilon > 0$, we have*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = O(r^{\rho-1+\epsilon}).$$

Lemma 4.3 ([32])**.** *Let f be a non-constant meromorphic function and let $\mathcal{R}(f) = \sum\limits_{i=0}^{n} a_i f^i / \sum\limits_{j=0}^{m} b_j f^j$ be an irreducible rational function in f with constant coefficients $\{a_i\}$ and $\{b_j\}$ where $a_n \neq 0$ and $b_m \neq 0$. Then*

$$T(r, \mathcal{R}(f)) = d\, T(r, f) + S(r, f), \text{ where } d = \max\{n, m\}.$$

Lemma 4.4 ([25])**.** *Let f and g be two transcendental entire functions of finite order, $c \neq 0$ be a complex constant, $\alpha(z)$ be a small function of f and g, $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be a nonzero polynomial, where $a_0, a_1, \ldots, a_n (\neq 0)$ are complex constants, and let $n > \Gamma_1$ be an integer. If $P(f)f(z+c)$ and $P(g)g(z+c)$ share $\alpha(z)$ IM, then $\rho(f) = \rho(g)$.*

Lemma 4.5. *Let f be a transcendental entire function of finite order, and $L_c(f) = f(z+c) + c_0 f(z)$, where $c,\ c_0 \in \mathbb{C} - \{0\}$. Then for $n \in \mathbb{N}$,*

$$nT(r, f) + S(r, f) \leq T(r, f(z)^n L_c(f)) \leq (n+1)T(r, f) + S(r, f).$$

Proof. This lemma can be proved in a similar manner as done in the proof of Lemma 2.4 and Remark 2.1 of [30]. □

Remark 4.6. *If $c_0 = 0$, then $L_c(f) = f(z+c)$ and therefore by Lemma 2.3 of [30], we can get*

$$T(r, f(z)^n L_c(f)) = (n+1)T(r, f) + S(r, f). \tag{4.2}$$

Remark 4.7. *If $c_0 \neq 1$, then the following example shows that one can not get equality just like* (4.2).

Example 4.8 ([30])**.** *If $f(z) = e^z$, $e^c = 2$, $c_0 = -1$, then $T(r, f(z)^n L_c(f)) = T(r, e^{(n+1)z}) = (n+1)T(r, f) + S(r, f)$. If $f(z) = e^z + z$, $c = 2\pi i$, $c_0 = -1$, then $T(r, f(z)^n L_c(f)) = T(r, 2\pi i(e^z + z)^n) = nT(r, f) + S(r, f)$.*

Remark 4.9. *From the above example, it can be easily seen that $f(z)$ and $f(z+c)$ share 0 CM for the first one, but for the second one $f(z)$ and $f(z+c)$ do not share 0 CM. Regarding this one may ask, in order to get equality just like* (4.2)*, is it sufficient to assume that $f(z)$ and $f(z+c)$ share 0 CM? In this direction, we prove the following lemma.*

Lemma 4.10. *Let $F = f(z)^n L_c(f)$, where $f(z)$ is an entire function of finite order, and $f(z)$, $f(z+c)$ share 0 CM. Then*

$$T(r,F) = (n+1)T(r,f) + S(r,f).$$

Proof. Keeping in view of Lemmas 4.1 and 4.3, we have

$$\begin{aligned} T(r,F) &= T(r, f(z)^n L_c(f)) = m(r, f^n L_c(f)) \\ &\leq m(r, f(z)^n) + m(r, L_c(f)) + S(r,f) \\ &\leq T(f(z)^n) + m\left(r, \frac{L_c(f)}{f(z)}\right) + m(r, f(z)) + S(r,f) \\ &\leq (n+1)T(r,f) + S(r,f). \end{aligned}$$

Since $f(z)$ and $f(z+c)$ share 0 CM, we must have $N\left(r,\infty; \frac{L_c(f)}{f(z)}\right) = S(r,f)$.

So, keeping in view of Lemmas 4.2 and 4.3, we obtain

$$\begin{aligned} (n+1)T(r,f) &= T(r, f(z)^{n+1}) = m(r, f(z)^{n+1} P(f(z))) \\ &= m\left(r, F\frac{f(z)}{L_c(f)}\right) \leq m(r,F) + m\left(r, \frac{f(z)}{L_c(f)}\right) + S(r,f) \\ &\leq T(r,F) + T\left(r, \frac{L_c(f)}{f(z)}\right) + S(r,f) = T(r,F) + N\left(r,\infty; \frac{L_c(f)}{f(z)}\right) \\ &\quad + m\left(r, \frac{L_c(f)}{f(z)}\right) + S(r,f) = T(r,F) + S(r,f). \end{aligned}$$

From the above two inequalities, we must have

$$T(r,F) = (n+1)T(r,f) + S(r,f). \quad \square$$

Lemma 4.11 ([37])**.** *Let F and G be non-constant meromorphic functions such that G is a Mobius transformation of F. Suppose that there exists a subset $I \subset \mathbb{R}^+$ with linear measure $mesI = +\infty$ such that for $r \in I$ and $r \longrightarrow \infty$*

$$\overline{N}(r,0;F) + \overline{N}(r,0;G) + \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) < (\lambda + o(1))T(r,G),$$

where $\lambda < 1$. If there exists a point $z_0 \in \mathbb{C}$ satisfying $F(z_0) = G(z_0) = 1$, then either $F = G$ or $FG = 1$.

Lemma 4.12 ([38])**.** *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. Then*

$$N\left(r,\infty;\frac{f}{g}\right) - N\left(r,\infty;\frac{g}{f}\right) = N(r,\infty;f) + N(r,0;g) - N(r,\infty;g) - N(r,0;f).$$

Lemma 4.13. *Let $f(z)$ be a transcendental entire function of finite order, $c \in \mathbb{C}-\{0\}$ be finite complex constant and $n \in \mathbb{N}$. Let $F(z) = f(z)^n L_c(f)$, where $L_c(f) \not\equiv 0$. Then*

$$nT(r,f) \leq T(r,F) - N(r,0;L_c(f)) + S(r,f).$$

Proof. Using Lemmas 4.2 and 4.12, and the first fundamental theorem of Nevanlinna, we obtain

$$\begin{aligned} m(r, f(z)^{n+1}) &= m\left(r, \frac{f(z)F}{L_c(f)}\right) \le m(r,F) + m\left(r, \frac{f(z)}{L_c(f)}\right) + S(r,f) \\ &\le m(r,F) + T\left(r, \frac{f(z)}{L_c(f)}\right) - N\left(r,\infty; \frac{f(z)}{L_c(f)}\right) + S(r,f) \\ &\le m(r,F) + T\left(r, \frac{L_c(f)}{f(z)}\right) - N\left(r,\infty; \frac{f(z)}{L_c(f)}\right) + S(r,f) \\ &\le m(r,F) + N\left(r,\infty; \frac{L_c(f)}{f(z)}\right) + m\left(r, \frac{L_c(f)}{f(z)}\right) - N\left(r,\infty; \frac{f(z)}{L_c(f)}\right) + S(r,f) \\ &\le m(r,F) + N(r,0;f) - N(r,0;L_c(f)) + S(r,f). \end{aligned}$$

i.e.,

$$m(f(z)^{n+1}) \le T(r,F) + T(r,f) - N(r,0;L_c(f)) + S(r,f).$$

By Lemma 4.3, we obtain

$$(n+1)T(r,f) = m(r, f^{n+1}) \le T(r,F) + T(r,f) - N(r,0;L_c(f)) + S(r,f),$$

i.e.,

$$nT(r,f) \le T(r,F) - N(r,0;L_c(f)) + S(r,f). \quad \square$$

Lemma 4.14 ([2])**.** *If f, g be two non-constant meromorphic functions sharing $(1,1)$, then*

$$2\overline{N}_L(r,1;f) + 2\overline{N}_L(r,1;g) + N_E^{(2}(r,1;f) - \overline{N}_{f>2}(r,1;g) \le N(r,1;g) - \overline{N}(r,1;g).$$

Lemma 4.15 ([4])**.** *If f, g be two non-constant meromorphic functions sharing $(1,1)$, then*

$$\overline{N}_{f>2}(r,1;g) \le \frac{1}{2}\overline{N}(r,0;f) + \frac{1}{2}\overline{N}(r,\infty;f) - \frac{1}{2}N_0(r,0;f') + S(r,f),$$

where $N_0(r,0;f')$ is the counting function of those zeros of f' which are not the zeros of $f(f-1)$.

Lemma 4.16 ([43])**.** *If f, g be two non-constant meromorphic functions sharing $(1,0)$ and $H \not\equiv 0$, then*

$$N_E^{1)}(r,1;f) \le N(r,0;H) + S(r,f) \le N(r,\infty;H) + S(r,f) + S(r,g).$$

Lemma 4.17 ([4])**.** *If f, g be two non-constant meromorphic functions such that they share $(1,0)$, then*

$$\overline{N}_L(r,1;f) + 2\overline{N}_L(r,1;g) + N_E^{(2}(r,1;f) - \overline{N}_{f>1}(r,1;g) - \overline{N}_{g>1}(r,1;f) \le N(r,1;g) - \overline{N}(r,1;g).$$

Lemma 4.18 ([4])**.** *If f, g be share $(1,0)$, then*

(i) $\overline{N}_L(r,1;f) \leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + S(r,f)$.

(ii) $\overline{N}_{f>1}(r,1;g) \leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) - N_0(r,0;f') + S(r,f)$.

(iii) $\overline{N}_{g>1}(r,1;f) \leq \overline{N}(r,0;g) + \overline{N}(r,\infty;g) - N_0(r,0;g') + S(r,g)$.

Lemma 4.19 ([20])**.** *If f, g be be two non-constant meromorphic functions that share $(1,0)$, $(\infty,0)$ and $H \not\equiv 0$, then*

$$\begin{aligned} N(r,\infty;H) \quad \leq \quad & \overline{N}(r,0;f \mid \geq 2) + \overline{N}(r,0;g \mid \geq 2) + \overline{N}_*(r,1;f,g) + \overline{N}_*(r,\infty;f,g) \\ & + \overline{N}_0(r,0;f') + \overline{N}_0(r,0;g') + S(r,f) + S(r,g), \end{aligned}$$

where $\overline{N}_0(r,0;f')$ is the reduced counting function of those zeros of f' which are not the zeros of $f(f-1)$ and $\overline{N}_0(r,0;g')$ is similarly defined.

Lemma 4.20 ([21])**.** *If $N(r,0;f^{(k)} \mid f \neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity, then*

$$N\left(r,0;f^{(k)} \mid f \neq 0\right) \leq k\overline{N}(r,\infty;f) + N\left(r,0;f \mid < k\right) + k\overline{N}\left(r,0 f \mid \geq k\right) + S(r,f).$$

5 Proofs of the theorems

Proof of Theorem 2.1. Let $F = f(z)^n L_c(f)/P(z)$ and $G = g(z)^n L_c(g)/P(z)$. Then F and G are two transcendental meromorphic functions that share $(1,l)$ except the zeros and poles of $P(z)$. Since $g(z)$ and $g(z+c)$ share 0 CM, from Lemma 4.10, we obtain

$$T(r,G) = (n+1)T(r,g) + O\{r^{\rho(f)-1+\epsilon}\} + O\{\log r\}. \tag{5.1}$$

Since f and g are of finite order, it follows from Lemma (4.5) and (5.1) that F and G are also of finite order. Moreover, from Lemma 4.4 we deduce that $\rho(f) = \rho(g) = \rho(F) = \rho(G)$.

We consider the following two cases separately.

Case 1: Suppose that F is a Mobius transformation of G, *i.e.*,

$$F = \frac{AG+B}{CG+D}, \tag{5.2}$$

where $A,\ B,\ C,\ D$ are complex constants satisfying $AD-BC \neq 0$. Let z_0 be a 1-point such that F. Since $F,\ G$ share $(1,2)$, z_0 is also a 1-point of G. Therefore, from (5.2), we obtain $A+B = C+D$, and hence (5.2) can be written as

$$F - 1 = \frac{G-1}{\alpha G + \beta},$$

where $\alpha = C/(A-C)$ and $\beta = D/(A-C)$. From this we can say that F, G share $(1,\infty)$.

Now using the standard Valiron-Mohon'ko Lemma 4.3, we obtain from (5.2) that

$$T(r,F) = T(r,G) + O(\log r).$$

Then using Lemmas 4.5 and 4.10 and the fact that f and g are transcendental entire functions of finite order, we deduce

$$T(r,f) \leq \frac{n+1}{n}T(r,g) + S(r,f) + S(r,g) \quad \text{and} \quad \frac{T(r,G)}{T(r,g)} \longrightarrow n+1 \tag{5.3}$$

as $r \longrightarrow \infty$, $r \in I$.

Now keeping in view of (5.3), Lemma 4.2 and the condition that f and g are transcendental entire functions, we obtain

$$\begin{aligned}
\overline{N}(r,0;F) + \overline{N}(r,\infty;F) &= \overline{N}(r,0;f(z)^n L_c(f)) + O(\log r) \\
&\leq \overline{N}(r,0;f(z)) + \overline{N}(r,0;L_c(f)) + O(\log r) \\
&\leq \overline{N}(r,0;f(z)) + T(r,L_c(f)) + O(\log r) \\
&\leq \overline{N}(r,0;f(z)) + m(r,L_c(f)) + O(\log r) \\
&\leq \overline{N}(r,0;f(z)) + m\left(r,\frac{L_c(f)}{f(z)}\right) + m(r,f(z)) + O(\log r) \\
&\leq 2T(r,f) + S(r,f) \leq \frac{2n+2}{n}T(r,g) + S(r,g).
\end{aligned}$$

Similarly, we obtain $\overline{N}(r,0;G) + \overline{N}(r,\infty;G) \leq 2T(r,g) + S(r,g)$. Thus using (5.3), we obtain

$$\overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,0;G) + \overline{N}(r,\infty;G) \leq \frac{2(2n+1)}{n(n+1)}T(r,G) + S(r,g). \tag{5.4}$$

Since, $g(z)$ and $g(z+c)$ share 0 CM, we get that $N(r,0;L_c(g)/g(z)) = S(r,g)$. Thus, keeping in view of this, Lemmas 4.2, 4.10 and applying the second fundamental theorem of Nevanlinna on G, we obtain

$$\begin{aligned}
(n+1)T(r,g) = T(r,G) &\leq \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,1;G) + S(r,g) \\
&\leq \overline{N}(r,0;g) + \overline{N}(r,0;L_c(g)) + \overline{N}(r,1;G) + S(r,g) \\
&\leq \overline{N}(r,0;g) + T(r,L_c(g)) + \overline{N}(r,1;G) + S(r,g) \\
&\leq \overline{N}(r,0;g) + T\left(r,\frac{L_c(g)}{g(z)}\right) + T(r,g) + S(r,g) \\
&\leq 2T(r,g) + \overline{N}(r,1;G) + S(r,g),
\end{aligned}$$

i.e.,

$$(n-1)T(r,g) \leq 2T(r,g) + \overline{N}(r,1;G) + S(r,g).$$

From this and the fact that F and G share $(1,2)$, we conclude that there exists a point $z_0 \in \mathbb{C}$ such that $F(z_0) = G(z_0) = 1$. Hence from (5.4), Lemma 4.11 and the condition $n \geq 4$, we conclude that either $FG = 1$ or $F = G$. Now we consider the following sub-cases.

Subcase 1.1: $F \equiv G$. Then we get

$$f(z)^n(f(z+c)+c_0f(z)) \equiv g(z)^n(g(z+c)+c_0g(z)).$$

Let $h(z) = f(z)/g(z)$. Then we deduce that

$$(h(z)^n h(z+c)-1)g(z+c) = -c_0(h^{n+1}(z)-1)g(z). \tag{5.5}$$

Suppose h is not constant. Then from (5.5), we obtain

$$\frac{g(z)}{g(z+c)} = \frac{h(z)^n h(z+c)-1}{c_0(h(z)^{n+1}-1)}.$$

As $g(z)$ and $g(z+c)$ share 0 CM, from the above equation we can say that $h(z)^{n+1}$ and $h(z)^n h(z+c)$ share $(1,\infty)$. Let z_0 be a zero of $h^{n+1}-1$. Then we must have $h(z)_0^{n+1} = 1$ and $h(z_0)^n h(z_0+c) = 1$. Hence $h(z_0+c) = h(z_0)$, and therefore by Lemma 4.1, we obtain

$$\overline{N}(r,1;h^{n+1}) \le \overline{N}(r,0;h(z+c)-h(z)) \le 2T(r,h)+S(r,h).$$

Keeping in mind the above inequality and Lemma 4.3 and applying the second fundamental theorem of Nevanlinna to h^{n+1}, we obtain

$$\begin{aligned}(n+1)T(r,h) &= T(r,h^{n+1}) \le \overline{N}(r,\infty;h^{n+1}) + \overline{N}(r,0;h^{n+1}) + \overline{N}(r,1;h^{n+1}) + S(r,h)\\ &\le 4T(r,h)+S(r,h),\end{aligned}$$

i.e.,

$$(n-3)T(r,h) \le S(r,h),$$

which is not possible since $n \ge 4$. Hence h is constant. Then (5.5) reduces to $(h^{n+1}-1)L_c(g) = 0$. As $L_c(g) \not\equiv 0$, we must have $h^{n+1} = 1$ and thus $f = tg$, for a constant t such that $t^{n+1} = 1$, which is the conclusion (a).

Subcase 1.2: Suppose $FG \equiv 1$. Then we have

$$f(z)^n L_c(f) g(z)^n L_c(g) = P_0(z)^2. \tag{5.6}$$

From (5.6) and the condition that f and g are transcendental entire functions, one can immediately say that both f and g have at most finitely many zeros. So, we may write

$$f(z) = P_1(z)e^{Q_1(z)},\ g(z) = P_1(z)e^{Q_2(z)}, \tag{5.7}$$

where P_1, P_2, Q_1, Q_2 are polynomials, and Q_1, Q_2 are non-constants. Substituting (5.7) in (5.6), we obtain

$$\begin{aligned}&(P_1P_2)^n e^{n(Q_1+Q_2)}[P_1(z+c)P_2(z+c)e^{Q_1(z+c)+Q_2(z+c)} + c_0^2P_1P_2e^{Q_1+Q_2}\\ &+c_0P_1P_2(z+c)e^{Q_1+Q_2(z+c)} + c_0P_1(z+c)P_2e^{Q_1(z+c)+Q_2}] = P(z)^2.\end{aligned} \tag{5.8}$$

Keeping in view of (5.7), we must have

$$n(Q_1(z)+Q_2(z))+Q_1(z+c)+Q_2(z+c)=A_1, \tag{5.9}$$

$$n(Q_1(z)+Q_2(z))+Q_1(z)+Q_2(z+c)=A_2, \tag{5.10}$$

$$n(Q_1(z)+Q_2(z))+Q_1(z+c)+Q_2(z)=A_3, \tag{5.11}$$

$$(n+1)(Q_1(z)+Q_2(z))=A_4, \tag{5.12}$$

where A_1, A_2, A_3, A_4 are constants. Let $Q_1(z)+Q_2(z)=W(z)$. Then (5.9) can be written as

$$nW(z)+W(z+c)=A_1, \tag{5.13}$$

for all $z \in \mathbb{C}$. Therefore, from (5.13), we must have $W=B$, where B is a constant, and therefore, we have

$$Q_2=B-Q_1. \tag{5.14}$$

Keeping in view of (5.14), (5.7) can be written as

$$f(z)=P_1e^{Q_1(z)},\ g(z)=P_2e^Be^{-Q_1(z)}. \tag{5.15}$$

Now (5.8) can be written as

$$\begin{aligned}&(P_1P_2)^n[P_1(z+c)P_2(z+c)e^{A_4}+c_0P_1(z+c)P_2e^{A_3}+c_0P_1P_2(z+c)e^{A_2}\\&+c_0^2P_1P_2e^{A_4}]=P(z)^2.\end{aligned} \tag{5.16}$$

If P_1P_2 is not a constant, then the degree of the left side of (5.16) is at least $n+1$. But the condition $2\deg(P)<n+1$ implies that the degree of the right side of (5.16) is less than $n+1$, which is a contradiction. Thus P_1P_2 and P reduce to non-zero constants.

Since P_1, P_2 are both polynomials and their product is constant, each of them must be constant. Therefore, (5.15) can be written as

$$f(z)=e^U,\ g(z)=e^Be^{-U}, \tag{5.17}$$

where U is a non-constant polynomial. Using the above forms of f and g and keeping in mind that P is a constant, say d, (5.6) reduces to

$$e^{(n+1)B}(e^{U(z+c)-U(z)}+c_0)(e^{-(U(z+c)-U(z))}+c_0)=d^2. \tag{5.18}$$

If $c_0=0$, (5.18) reduces to $e^{(n+1)B}=d^2$. Set $e^B=t$. Then (5.17) can be written as

$$f(z)=e^U,\ g(z)=te^{-U},\ \text{where } t \text{ is a constant such that }\ t^{n+1}=1,$$

which is the conclusion (b).

If $c_0 \neq 0$, then from (5.18), one can say that $e^{U(z+c)-U(z)} + c_0$ has no zeros. Then $\phi(z) = e^{U(z+c)-U(z)} \neq 0, -c_0, \infty$. By Picard's theorem, ϕ is constant and so $\deg(U(z)) = 1$. Therefore, from (5.17), one may obtain

$$f(z) = c_1 e^{az},\ g(z) = c_2 e^{-az},$$

where $a,\ c_1$ and c_2 are non-zero constants. Using these in (5.6), we obtain

$$(c_1c_2)^{n+1}(e^{ac} + c_0)(e^{-ac} + c_0) = d^2,$$

which is the conclusion (c).

Case 2: Suppose $n \geq 5$.

Since $f(z)^n L_c(f) - P(z)$ and $g(z)^n L_c(g) - P(z)$ share $(0, l)$, it follows that F and G share $(1, l)$. Let $H \not\equiv 0$. First suppose $l \geq 2$.

Using Lemmas 4.16 and 4.19, we obtain

$$\begin{aligned}\overline{N}(r,1;F) &= N(r,1;F \mid= 1) + \overline{N}(r,1;F \mid\geq 2) \leq N(r,\infty;H) + \overline{N}(r,1;F \mid\geq 2)\\ &\leq \overline{N}(r,0;F \mid\geq 2) + \overline{N}(r,0;G \mid\geq 2) + \overline{N}_*(r,1;F,G) + \overline{N}(r,1;F \mid\geq 2)\\ &\quad + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,F) + S(r,G). \end{aligned} \tag{5.19}$$

Keping in view of the above observation and Lemma 4.20, we see that

$$\begin{aligned}&\overline{N}_0(r,0;G') + \overline{N}(r,1;F \mid\geq 2) + \overline{N}_*(r,1;F,G)\\ \leq\ &\overline{N}_0(r,0;G') + \overline{N}(r,1;F \mid\geq 2) + \overline{N}(r,1;F \mid\geq 3) + S(r,F)\\ \leq\ &\overline{N}_0(r,0;G') + \overline{N}(r,1;G \mid\geq 2) + \overline{N}(r,1;G \mid\geq 3) + S(r,F) + S(r,G)\\ \leq\ &\overline{N}_0(r,0;G') + N(r,1;G) - \overline{N}(r,1;G) + S(r,F) + S(r,G)\\ \leq\ &N(r,0;G' \mid G \neq 0) \leq \overline{N}(r,0;G) + S(r,G). \end{aligned} \tag{5.20}$$

Since $g(z)$ and $g(z+c)$ share 0 CM, we must have $N(r,\infty,L_c(g)/g(z)) = 0$.

Hence using (5.19), (5.20), Lemmas 4.2, 4.13 and applying second fundamental theorem of Nevanlinna to F, we obtain

$$\begin{aligned}&nT(r,f) \leq T(r,F) - N(r,0;L_c(f)) + S(r,f)\\ \leq\ &\overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,1;F) - \overline{N}(r,0;F') - N(r,0;L_c(f)) + S(r,f)\\ \leq\ &N_2(r,0;F) + N_2(r,0;G) - N(r,0;L_c(f)) + S(r,f) + S(r,g)\\ \leq\ &N_2(r,0;f^nL_c(f)) + N_2\left(r,0;g^{n+1}\frac{L_c(g)}{g(z)}\right) - N(r,0;L_c(f)) + S(r,f) + S(r,g)\\ \leq\ &2\overline{N}(r,0;f) + 2\overline{N}(r,0;g) + N\left(r,0;\frac{L_c(g)}{g(z)}\right) + S(r,f) + S(r,g)\end{aligned}$$

$$
\begin{aligned}
&\leq 2(T(r,f)+T(r,g))+T\left(r,\frac{L_c(g)}{g(z)}\right)+S(r,f)+S(r,g)\\
&\leq 2(T(r,f)+T(r,g))+N\left(r,\infty;\frac{L_c(g)}{g(z)}\right)+m\left(r,\frac{L_c(g)}{g(z)}\right)+S(r,f)+S(r,g)\\
&\leq 2(T(r,f)+T(r,g))+S(r,f)+S(r,g). \qquad (5.21)
\end{aligned}
$$

Similarly, using Lemmas 4.2, 4.13 and applying second fundamental theorem of Nevanlinna to G, we obtain

$$
\begin{aligned}
&nT(r,g)\leq T(r,G)-N(r,0;L_c(g))+S(r,g)\\
\leq\ &\overline{N}(r,0;G)+\overline{N}(r,\infty;G)+\overline{N}(r,1;G)-\overline{N}(r,0;G')-N(r,0;L_c(g))+S(r,g)\\
\leq\ &N_2(r,0;F)+N_2(r,0;G)-N(r,0;L_c(g))+S(r,f)+S(r,g)\\
\leq\ &N_2(r,0;f(z)^nL_c(f))+N_2\left(r,0;g^nL_c(g)\right)-N(r,0;L_c(g))+S(r,f)+S(r,g)\\
\leq\ &2\overline{N}(r,0;f)+2\overline{N}(r,0;g)+N\left(r,0;L_c(f)\right)+S(r,f)+S(r,g)\\
\leq\ &2(T(r,f)+T(r,g))+T\left(r,L_c(f)\right)+S(r,f)+S(r,g)\\
\leq\ &2(T(r,f)+T(r,g))+m\left(r,\frac{L_c(f)}{f(z)}\right)+m(r,f(z))+S(r,f)+S(r,g)\\
\leq\ &2(T(r,f)+T(r,g))+T(r,f)+S(r,f)+S(r,g). \qquad (5.22)
\end{aligned}
$$

Combining (5.21) and (5.22), we get

$$(n-5)T(r,f)+(n-4)T(r,g)\leq S(r,f)+S(r,g),$$

which contradicts with $n\geq 5$.

When $l=1$, Keeping in view of Lemmas 4.14, 4.15, 4.16, 4.19 and 4.20, we obtain

$$
\begin{aligned}
\overline{N}(r,1;F) &= N(r,1;F\mid=1)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)\\
&\leq \overline{N}(r,0;F\mid\geq 2)+\overline{N}(r,0;G\mid\geq 2)+\overline{N}_*(r,1;F,G)+\overline{N}_L(r,1;F)\\
&\quad+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,F)+S(r,G)\\
&\leq \overline{N}(r,0;F\mid\geq 2)+\overline{N}(r,0;G\mid\geq 2)+2\overline{N}_L(r,1;F)+\overline{2}N_L(r,1;G)\\
&\quad+\overline{N}_E^{(2}(r,1;F)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,F)+S(r,G)\\
&\leq \overline{N}(r,0;F\mid\geq 2)+\overline{N}(r,0;G\mid\geq 2)+\overline{N}_{F>2}(r,1;G)+N(r,1;G)\\
&\quad-\overline{N}(r,1;G)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,F)+S(r,G)\\
&\leq \overline{N}(r,0;F\mid\geq 2)+\overline{N}(r,0;G\mid\geq 2)+N(r,0;G'\mid G\neq 0)\\
&\quad+\frac{1}{2}\overline{N}(r,0;F)+\overline{N}_0(r,0;F')+S(r,F)+S(r,G)\\
&\leq \overline{N}(r,0;F\mid\geq 2)+\frac{1}{2}\overline{N}(r,0;F)+N_2(r,0;G)+\overline{N}_0(r,0;F')\\
&\quad+S(r,F)+S(r,G). \qquad (5.23)
\end{aligned}
$$

Since $g(z)$, $g(z+c)$ share 0 CM, $N(r,\infty;g(z+c)/g(z))=0$, and therefore, using Lemma 4.2, we obtain $T(r,g(z+c)/g(z))=0$.

Hence using (5.23), Lemmas 4.2, 4.13 and applying second fundamental theorem of Nevanlinna to F, we obtain

$$\begin{aligned}
& nT(r,f) \le T(r,F) - N(r,0;L_c(f)) + S(r,f) \\
\le\ & \overline{N}(r,0;F) + \overline{N}(r,1;F) - \overline{N}(r,0;F') - N(r,0;L_c(f)) + S(r,f) \\
\le\ & N_2(r,0;F) + N_2(r,0;G) + \frac{1}{2}\overline{N}(r,0;F) - N(r,0;L_c(f)) + S(r,f) + S(r,g) \\
\le\ & 2\overline{N}(r,0;f) + N_2\left(r,0;g^{n+1}\frac{L_c(g)}{g}\right) + \frac{1}{2}\overline{N}(r,0;F) + S(r,f) + S(r,g) \\
\le\ & 2\overline{N}(r,0;f) + 2\overline{N}(r,0;g) + \frac{1}{2}(\overline{N}(r,0;f) + \overline{N}(r,0,L_c(f))) + N\left(r,0;\frac{L_c(g)}{g}\right) \\
& + S(r,f) + S(r,g) \\
\le\ & \frac{5}{2}T(r,f) + 2T(r,g) + \frac{1}{2}T(r,L_c(f)) + T\left(r,\frac{L_c(g)}{g}\right) + S(r,f) + S(r,g) \\
\le\ & \frac{5}{2}T(r,f) + 2T(r,g) + \frac{1}{2}m\left(r,\frac{L_c(f)}{f}\right) + \frac{1}{2}m(r,f(z)) + S(r,f) + S(r,g) \\
\le\ & 3T(r,f) + 2T(r,g) + S(r,f) + S(r,g). \qquad (5.24)
\end{aligned}$$

In a similar manner, we may obtain

$$nT(r,g) \le 3T(r,f) + \frac{5}{2}T(r,g) + S(r,f) + S(r,g). \qquad (5.25)$$

Combining (5.24) and (5.25), we obtain

$$(n-6)T(r,f) + \left(n - \frac{5}{2}\right)T(r,g) \le S(r,f) + S(r,g),$$

which is a contradiction since $n \ge 6$.

When $l = 0$, using Lemmas 4.16, 4.17, 4.18, 4.19 and 4.20, we obtain

$$\begin{aligned}
\overline{N}(r,1;F) \ =\ & N(r,1;F \mid= 1) + \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}_E^{(2}(r,1;F) \\
\le\ & \overline{N}(r,0;F \mid\ge 2) + \overline{N}(r,0;G \mid\ge 2) + \overline{N}_*(r,1;F,G) + \overline{N}_L(r,1;F) \\
& + \overline{N}_L(r,1;G) + \overline{N}_E^{(2}(r,1;F) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,F) + S(r,G) \\
\le\ & \overline{N}(r,0;F \mid\ge 2) + \overline{N}(r,0;G \mid\ge 2) + 2\overline{N}_L(r,1;F) + \overline{2}N_L(r,1;G) \\
& + \overline{N}_E^{(2}(r,1;F) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,F) + S(r,G) \\
\le\ & \overline{N}(r,0;F \mid\ge 2) + \overline{N}(r,0;G \mid\ge 2) + \overline{N}_L(r,1;F) + \overline{N}_{F>1}(r,1;G) \\
& + \overline{N}_{G>1}(r,1;F) + N(r,1;G) - \overline{N}(r,1;G) + \overline{N}_0(r,0;F') \\
& + \overline{N}_0(r,0;G') + S(r,F) + S(r,G) \\
\le\ & \overline{N}(r,0;F \mid\ge 2) + \overline{N}(r,0;G \mid\ge 2) + N(r,0;G' \mid G \ne 0) \\
& + 2\overline{N}(r,0;F) + \overline{N}(r,0;G) + \overline{N}_0(r,0;F') + S(r,F) + S(r,G) \\
\le\ & N_2(r,0;F) + \overline{N}(r,0;F) + N_2(r,0;G) + \overline{N}(r,0;G) + \overline{N}_0(r,0;F') \\
& + S(r,F) + S(r,G). \qquad (5.26)
\end{aligned}$$

Hence using (5.26), Lemmas 4.2, 4.13 and applying second fundamental theorem of Nevanlinna to F, we obtain

$$\begin{aligned}
& nT(r,f) \leq T(r,F) - N(r,0;L_c(f)) + S(r,f) \\
\leq \;& \overline{N}(r,0;F) + \overline{N}(r,1;F) - \overline{N}(r,0;F') - N(r,0;L_c(f)) + S(r,f) \\
\leq \;& N_2(r,0;F) + N_2(r,0;G) + 2\overline{N}(r,0;F) + \overline{N}(r,0;G) - N(r,0;L_c(f)) + S(r,f) + S(r,g) \\
\leq \;& 2\overline{N}(r,0;f) + N_2\left(r,0;g^{n+1}(z)\frac{L_c(g)}{g(z)}\right) + \overline{N}\left(r,0;g^{n+1}(z)\frac{L_c(g)}{g(z)}\right) \\
& + 2\overline{N}(r,0;f^n(z)L_c(f)) + S(r,f) + S(r,g) \\
\leq \;& 4\overline{N}(r,0;f) + 3\overline{N}(r,0;g) + \overline{N}\left(r,0;\frac{L_c(g)}{g(z)}\right) + N\left(r,0;\frac{L_c(g)}{g(z)}\right) \\
& + 2\overline{N}(r,0;L_c(f)) + S(r,f) + S(r,g) \\
\leq \;& 4T(r,f) + 3T(r,g) + 2T\left(r,\frac{L_c(g)}{g(z)}\right) + 2T(r,L_c(f)) + S(r,f) + S(r,g) \\
\leq \;& 4T(r,f) + 3T(r,g) + 2m(r,L_c(f)) + S(r,f) + S(r,g) \\
\leq \;& 4T(r,f) + 3T(r,g) + 2m\left(r,\frac{L_c(f)}{f(z)}\right) + 2m(r,f(z)) + S(r,f) + S(r,g) \\
\leq \;& 6T(r,f) + 3T(r,g) + S(r,f) + S(r,g). \qquad (5.27)
\end{aligned}$$

In a similar manner, we obtain

$$nT(r,g) \leq 5T(r,f) + 6T(r,g) + S(r,f) + S(r,g). \qquad (5.28)$$

Combining (5.27) and (5.28), we get

$$(n-11)T(r,f) + (n-9)T(r,g) \leq S(r,f) + S(r,g),$$

which is a contradiction since $n \geq 11$.

Thus $H \equiv 0$. Then by integration we obtain (5.2). Therefore, the results follows from Case 1. This completes the proof of the theorem. □

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