CUBO, A Mathematical Journal
Vol. 24, no. 02, pp. 239–261, August 2022
DOI: 10.56754/0719-0646.2402.0239

On an *a priori* L^∞ estimate for a class of Monge-Ampère type equations on compact almost Hermitian manifolds

Masaya Kawamura[1,] ✉

[1] *Department of General Education, National Institute of Technology, Kagawa College 355, Chokushi-cho, Takamatsu, Kagawa, Japan, 761-8058.*
kawamura-m@t.kagawa-nct.ac.jp ✉

ABSTRACT

We investigate Monge-Ampère type equations on almost Hermitian manifolds and show an *a priori* L^∞ estimate for a smooth solution of these equations.

RESUMEN

Investigamos ecuaciones de tipo Monge-Ampère en variedades casi Hermitianas y mostramos una estimación L^∞ *a priori* para una solución suave de estas ecuaciones.

Keywords and Phrases: Monge-Ampère type equation, almost Hermitian manifold, Chern connection.

2020 AMS Mathematics Subject Classification: 32Q60, 53C15, 53C55.

Accepted: 7 April, 2022
Received: 19 July, 2021

1 Introduction

Let (M^{2n}, J, ω) be a compact almost Hermitian manifold of real dimension $2n$ with $n \geq 2$. Let χ be a smooth real $(1,1)$-form on M. We define for a function $u \in C^2(M)$,

$$\chi_u := \chi + \sqrt{-1}\partial\bar{\partial}u$$

and

$$[\chi] := \{\chi_u | u \in C^2(M)\}, \quad [\chi]^+ := \{\chi' \in [\chi] | \chi' > 0\}, \quad \mathcal{H}(M,\chi) := \{u \in C^2(M) | \chi_u > 0\}$$

and

$$\mathcal{C}_\alpha(\psi) := \{[\chi] | \exists \chi' \in [\chi]^+, n\chi'^{n-1} > (n-\alpha)\psi\chi'^{n-\alpha-1} \wedge \omega^\alpha\}.$$

We consider the following fully nonlinear Monge-Ampère type equations, which are called the $(n, n-\alpha)$-quotient equations for $1 \leq \alpha \leq n$:

$$\chi_u^n = \psi\chi_u^{n-\alpha} \wedge \omega^\alpha \text{ with } \chi_u > 0, \tag{1.1}$$

where ψ is a smooth positive function. We will call a function $u \in C^2(M)$ admissible if it satisfies that $u \in \mathcal{H}(M,\chi)$. When solutions u are admissible, the equations (1.1) are elliptic. Since the equation (1.1) is invariant under the addition of constants to u, we may assume that u satisfies the normalized condition such that

$$\sup_M u = 0. \tag{1.2}$$

W. Sun has studied a class of fully nonlinear elliptic equations on closed Hermitian manifolds and derived some *a priori* estimates for these equations (cf. [5, 6]). In [5], W. Sun has proven a uniform *a priori* C^∞ estimates of a smooth solution of the equation (1.1) and shown the existence of a solution of (1.1) on a closed Hermitian manifold. In [12], J. Zhang has shown that on a compact almost Hermitian manifold (M^{2n}, J, ω), if there exists an admissible $\mathcal{C}$-subsolution and an admissible supersolution for the equation (1.1) for $\chi = \omega$, there exists a pair of (u, b) with $b \in \mathbb{R}$ such that $u \in \mathcal{H}(M,\omega)$, $\sup_M u = 0$, $\omega_u^n = e^b\psi\omega^{n-\alpha} \wedge \omega^\alpha$ for $1 \leq \alpha \leq n$ on M. L. Chen has studied a Hessian equation with its structure as a combination of elementary symmetric functions on a closed Kähler manifold and Chen has provided a sufficient and necessary condition for the solvability of this equation in [1]. Q. Tu and N. Xiang have investigated the Dirichlet problem for a class of Hessian type equation with its structure as a combination of elementary symmetric functions on a closed Hermitian manifold with smooth boundary and they have derived *a priori* estimates for the complex mixed Hessian equation in [9].

In this paper, we show that we have the *a priori* L^∞ estimate for a smooth solution of the equation (1.1) on general almost Hermitian manifolds.

Theorem 1.1. *Let* (M, J, ω) *be a compact almost Hermitian manifold of real dimension* $2n$ *with* $n \geq 2$ *and* u *be a smooth admissible solution to* (1.1). *Suppose that* $\chi \in \mathcal{C}_\alpha(\psi)$. *Then there is a uniform a priori* L^∞ *estimate for* u *depending only on* (M, J, ω), χ, ψ.

This paper is organized as follows: in section 2, we recall some basic definitions and computations on an almost Hermitian manifold (M, J, ω). In section 3, for an arbitrary chosen smooth function φ on M, we show the result that $\partial\partial\bar{\partial}\varphi$ and $\bar{\partial}\partial\bar{\partial}\varphi$ depend only on the first derivative of φ and some geometric quantities of (M, J, ω). In section 4, we give a proof for Theorem 1.1. Notice that we assume the Einstein convention omitting the symbol of sum over repeated indexes in all this paper.

2 Preliminaries

2.1 The Nijenhuis tensor of the almost complex structure

Let M be a $2n$-dimensional smooth differentiable manifold. An almost complex structure on M is an endomorphism J of TM, $J \in \Gamma(\mathrm{End}(TM))$, satisfying $J^2 = -Id_{TM}$, where TM is the real tangent vector bundle of M. The pair (M, J) is called an almost complex manifold. Let (M, J) be an almost complex manifold. We define a bilinear map on $C^\infty(M)$ for $X, Y \in \Gamma(TM)$ by

$$4N(X, Y) := [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y], \tag{2.1}$$

which is the Nijenhuis tensor of J. The Nijenhuis tensor N satisfies $N(X, Y) = -N(Y, X)$, $N(JX, Y) = -JN(X, Y)$, $N(X, JY) = -JN(X, Y)$, $N(JX, JY) = -N(X, Y)$. For any $(1,0)$-vector fields W and V, $N(V, W) = -[V, W]^{(0,1)}$, $N(V, \bar{W}) = N(\bar{V}, W) = 0$ and $N(\bar{V}, \bar{W}) = -[\bar{V}, \bar{W}]^{(1,0)}$ since we have $4N(V, W) = -2([V, W] + \sqrt{-1}J[V, W])$, $4N(\bar{V}, \bar{W}) = -2([\bar{V}, \bar{W}] - \sqrt{-1}J[\bar{V}, \bar{W}])$. An almost complex structure J is called integrable if $N = 0$ on M. Giving a complex structure to a differentiable manifold M is equivalent to giving an integrable almost complex structure to M (cf. [4]). A Riemannian metric g on M is called J-invariant if J is compatible with g, *i.e.*, for any $X, Y \in \Gamma(TM)$, $g(X, Y) = g(JX, JY)$. In this case, the pair (J, g) is called an almost Hermitian structure.

The complexified tangent vector bundle is given by $T^{\mathbb{C}}M = TM \otimes_{\mathbb{R}} \mathbb{C}$ for the real tangent vector bundle TM. By extending J $\mathbb{C}$-linearly and g $\mathbb{C}$-bilinearly to $T^{\mathbb{C}}M$, they are also defined on $T^{\mathbb{C}}M$ and we observe that the complexified tangent vector bundle $T^{\mathbb{C}}M$ can be decomposed as $T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$, where $T^{1,0}M$, $T^{0,1}M$ are the eigenspaces of J corresponding to eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively:

$$T^{1,0}M = \{X - \sqrt{-1}JX \big| X \in TM\}, \quad T^{0,1}M = \{X + \sqrt{-1}JX \big| X \in TM\}. \tag{2.2}$$

Let $\Lambda^r M = \bigoplus_{p+q=r} \Lambda^{p,q} M$ for $0 \leq r \leq 2n$ denote the decomposition of complex differential r-forms into (p,q)-forms, where $\Lambda^{p,q}M = \Lambda^p(\Lambda^{1,0}M) \otimes \Lambda^q(\Lambda^{0,1}M)$,

$$\Lambda^{1,0}M = \{\eta + \sqrt{-1}J\eta \big| \eta \in \Lambda^1 M\}, \quad \Lambda^{0,1}M = \{\eta - \sqrt{-1}J\eta \big| \eta \in \Lambda^1 M\} \tag{2.3}$$

and $\Lambda^1 M$ denotes the dual of $T^{\mathbb{C}}M$.

Let $\{Z_r\}$ be a local $(1,0)$-frame on (M,J) with an almost Hermitian metric g and let $\{\zeta^r\}$ be a local associated coframe with respect to $\{Z_r\}$, *i.e.*, $\zeta^i(Z_j) = \delta^i_j$ for $i, j = 1, \ldots, n$. Since g is almost Hermitian, its components satisfy $g_{ij} = g_{\bar{i}\bar{j}} = 0$ and $g_{i\bar{j}} = g_{\bar{j}i} = \bar{g}_{\bar{i}j}$. Using these local frame $\{Z_r\}$ and coframe $\{\zeta^r\}$, we have

$$N(Z_{\bar{i}}, Z_{\bar{j}}) = -[Z_{\bar{i}}, Z_{\bar{j}}]^{(1,0)} =: N^k_{\bar{i}\bar{j}} Z_k, \quad N(Z_i, Z_j) = -[Z_i, Z_j]^{(0,1)} = \overline{N^k_{\bar{i}\bar{j}}} Z_{\bar{k}},$$

and

$$N = \frac{1}{2}\overline{N^k_{\bar{i}\bar{j}}} Z_{\bar{k}} \otimes (\zeta^i \wedge \zeta^j) + \frac{1}{2} N^k_{\bar{i}\bar{j}} Z_k \otimes (\zeta^{\bar{i}} \wedge \zeta^{\bar{j}}). \tag{2.4}$$

Let (M,J,g) be an almost Hermitian manifold with $\dim_{\mathbb{R}} M = 2n$. An affine connection D on $T^{\mathbb{C}}M$ is called almost Hermitian connection if $Dg = DJ = 0$. For the almost Hermitian connection, we have the following Lemma (cf. [10, 13]).

Lemma 2.1. *Let (M,J,g) be an almost Hermitian manifold with $\dim_{\mathbb{R}} M = 2n$. Then for any given vector valued $(1,1)$-form $\Theta = (\Theta^i)_{1\leq i\leq n}$, there exists a unique almost Hermitian connection ∇ on (M,J,g) such that the $(1,1)$-part of the torsion is equal to the given Θ.*

If the $(1,1)$-part of the torsion of an almost Hermitian connection vanishes everywhere, then the connection is called the second canonical connection or the Chern connection. We will refer the connection as the Chern connection and denote it by ∇. Now let ∇ be the Chern connection on M. We denote the structure coefficients of Lie bracket by

$$[Z_i, Z_j] = B^r_{ij} Z_r + B^{\bar{r}}_{ij} Z_{\bar{r}}, \quad [Z_i, Z_{\bar{j}}] = B^r_{i\bar{j}} Z_r + B^{\bar{r}}_{i\bar{j}} Z_{\bar{r}}, \quad [Z_{\bar{i}}, Z_{\bar{j}}] = B^r_{\bar{i}\bar{j}} Z_r + B^{\bar{r}}_{\bar{i}\bar{j}} Z_{\bar{r}}.$$

We have $B^k_{ij} = -B^k_{ji}$ since $[Z_i, Z_j] = -[Z_j, Z_i]$. Notice that J is integrable if and only if the $B^{\bar{r}}_{ij}$'s vanish.

For any p-form ψ, there holds that

$$\begin{aligned} d\psi(X_1, \ldots, X_{p+1}) &= \sum_{i=1}^{p+1} (-1)^{i+1} X_i(\psi(X_1, \ldots, \widehat{X_i}, \ldots, X_{p+1})) \\ &\quad + \sum_{i<j} (-1)^{i+j} \psi([X_i, X_j], X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_{p+1}) \end{aligned} \tag{2.5}$$

for any vector fields $X_1, \ldots, X_{p+1}$ on M (cf. [13]). We directly compute that

$$d\zeta^s = -\frac{1}{2} B^s_{kl} \zeta^k \wedge \zeta^l - B^s_{k\bar{l}} \zeta^k \wedge \zeta^{\bar{l}} - \frac{1}{2} B^s_{\bar{k}\bar{l}} \zeta^{\bar{k}} \wedge \zeta^{\bar{l}}. \tag{2.6}$$

For any real $(1,1)$-form $\eta = \sqrt{-1}\eta_{i\bar{j}}\zeta^i \wedge \zeta^{\bar{j}}$, we have

$$\partial\eta = \frac{\sqrt{-1}}{2}\Big(Z_i(\eta_{j\bar{k}}) - Z_j(\eta_{i\bar{k}}) - B^s_{ij}\eta_{s\bar{k}} - B^{\bar{s}}_{i\bar{k}}\eta_{j\bar{s}} + B^{\bar{s}}_{j\bar{k}}\eta_{i\bar{s}}\Big)\zeta^i \wedge \zeta^j \wedge \zeta^{\bar{k}}, \tag{2.7}$$

$$\bar{\partial}\eta = \frac{\sqrt{-1}}{2}\Big(Z_{\bar{j}}(\eta_{k\bar{i}}) - Z_{\bar{i}}(\eta_{k\bar{j}}) - B^s_{k\bar{i}}\eta_{s\bar{j}} + B^s_{k\bar{j}}\eta_{s\bar{i}} + B^{\bar{s}}_{\bar{i}\bar{j}}\eta_{k\bar{s}}\Big)\zeta^k \wedge \zeta^{\bar{i}} \wedge \zeta^{\bar{j}}. \tag{2.8}$$

We can split the exterior differential operator $d : \Lambda^p M \otimes_{\mathbb{R}} \mathbb{C} \to \Lambda^{p+1} M \otimes_{\mathbb{R}} \mathbb{C}$, into four components

$$d = A + \partial + \bar{\partial} + \bar{A}$$

with

$$\partial : \Lambda^{p,q}M \to \Lambda^{p+1,q}M, \quad \bar{\partial} : \Lambda^{p,q}M \to \Lambda^{p,q+1}M,$$

$$A : \Lambda^{p,q}M \to \Lambda^{p+2,q-1}M, \quad \bar{A} : \Lambda^{p,q}M \to \Lambda^{p-1,q+2}M.$$

In terms of these components, the condition $d^2 = 0$ can be written as

$$A^2 = 0, \quad \partial A + A\partial = 0, \quad \bar{\partial}\bar{A} + \bar{A}\bar{\partial} = 0, \quad \bar{A}^2 = 0,$$

$$A\bar{\partial} + \partial^2 + \bar{\partial}A = 0, \quad A\bar{A} + \partial\bar{\partial} + \bar{\partial}\partial + \bar{A}A = 0, \quad \partial\bar{A} + \bar{\partial}^2 + \bar{A}\partial = 0. \tag{2.9}$$

A direct computation yields for any $\varphi \in C^\infty(M, \mathbb{R})$,

$$\sqrt{-1}\partial\bar{\partial}\varphi = \frac{1}{2}(dJd\varphi)^{(1,1)} = \sqrt{-1}(Z_i Z_{\bar{j}} - [Z_i, Z_{\bar{j}}]^{(0,1)})(\varphi)\zeta^i \wedge \zeta^{\bar{j}}, \tag{2.10}$$

so we write locally

$$\partial_i\partial_{\bar{j}}\varphi = (Z_i Z_{\bar{j}} - [Z_i, Z_{\bar{j}}]^{(0,1)})\varphi. \tag{2.11}$$

2.2 The torsion and the curvature on almost complex manifolds

Since the Chern connection ∇ preserves J, we have

$$\nabla_i Z_j := \nabla_{Z_i} Z_j = \Gamma^r_{ij} Z_r, \quad \nabla_i Z_{\bar{j}} := \nabla_{Z_i} Z_{\bar{j}} = \Gamma^{\bar{r}}_{i\bar{j}} Z_{\bar{r}},$$

where $\Gamma^r_{ij} = g^{r\bar{s}} Z_i(g_{j\bar{s}}) - g^{r\bar{s}} g_{j\bar{l}} B^{\bar{l}}_{i\bar{s}}$. We can obtain that $\Gamma^{\bar{r}}_{i\bar{j}} = B^{\bar{r}}_{i\bar{j}}$ since the $(1,1)$-part of the torsion of the Chern connection vanishes everywhere.

Note that the mixed derivatives $\nabla_i Z_{\bar{j}}$ do not depend on g (cf. [10]). Let $\{\gamma^i_j\}$ be the connection form, which is defined by $\gamma^i_j = \Gamma^i_{sj}\zeta^s + \Gamma^i_{\bar{s}j}\zeta^{\bar{s}}$. The torsion T of the Chern connection ∇ is given by $T^i = d\zeta^i - \zeta^p \wedge \gamma^i_p$, $T^{\bar{i}} = d\zeta^{\bar{i}} - \zeta^{\bar{p}} \wedge \gamma^{\bar{i}}_{\bar{p}}$, which has no $(1,1)$-part and the only non-vanishing components are as follows:

$$T^s_{ij} = \Gamma^s_{ij} - \Gamma^s_{ji} - B^s_{ij}, \quad T^{\bar{s}}_{ij} = -B^{\bar{s}}_{ij}.$$

These tell us that $T = (T^i)$ splits into $T = T' + T''$, where $T' \in \Gamma(\Lambda^{2,0}M \otimes T^{1,0}M)$, $T'' \in \Gamma(\Lambda^{0,2}M \otimes T^{1,0}M)$.

We denote by Ω the curvature of the Chern connection ∇. We can regard Ω as a section of $\Lambda^2 M \otimes \Lambda^{1,1}M$, $\Omega \in \Gamma(\Lambda^2 M \otimes \Lambda^{1,1}M)$ and Ω splits in $\Omega = \mathcal{H} + \mathcal{R} + \bar{\mathcal{H}}$, where $\mathcal{R} \in \Gamma(\Lambda^{1,1}M \otimes \Lambda^{1,1}M)$, $\mathcal{H} \in \Gamma(\Lambda^{2,0}M \otimes \Lambda^{1,1}M)$. The curvature form can be expressed by $\Omega^i_j = d\gamma^i_j + \gamma^i_s \wedge \gamma^s_j$.

In terms of Z_r's, we have

$$\mathcal{R}_{i\bar{j}k}{}^r = \Omega^r_k(Z_i, Z_{\bar{j}}) = Z_i(\Gamma^r_{\bar{j}k}) - Z_{\bar{j}}(\Gamma^r_{ik}) + \Gamma^r_{is}\Gamma^s_{\bar{j}k} - \Gamma^r_{\bar{j}s}\Gamma^s_{ik} - B^s_{i\bar{j}}\Gamma^r_{sk} + B^{\bar{s}}_{\bar{j}i}\Gamma^r_{\bar{s}k} = -\mathcal{R}_{\bar{j}ik}{}^r, \quad (2.12)$$

$$\mathcal{H}_{ijk}{}^r = \Omega^r_k(Z_i, Z_j) = Z_i(\Gamma^r_{jk}) - Z_j(\Gamma^r_{ik}) + \Gamma^r_{is}\Gamma^s_{jk} - \Gamma^r_{js}\Gamma^s_{ik} - B^s_{ij}\Gamma^r_{sk} - B^{\bar{s}}_{ij}\Gamma^r_{\bar{s}k} = -\mathcal{H}_{jik}{}^r, \quad (2.13)$$

$$\mathcal{H}_{\bar{i}\bar{j}k}{}^r = \Omega^r_k(Z_{\bar{i}}, Z_{\bar{j}}) = Z_{\bar{i}}(\Gamma^r_{\bar{j}k}) - Z_{\bar{j}}(\Gamma^r_{\bar{i}k}) + \Gamma^r_{\bar{i}s}\Gamma^s_{\bar{j}k} - \Gamma^r_{\bar{j}s}\Gamma^s_{\bar{i}k} - B^s_{\bar{i}\bar{j}}\Gamma^r_{sk} - B^{\bar{s}}_{\bar{i}\bar{j}}\Gamma^r_{\bar{s}k} = -\mathcal{H}_{\bar{j}\bar{i}k}{}^r. \quad (2.14)$$

Lemma 2.2 (The first Bianchi identity for the Chern curvature)**.** *For any $X, Y, Z \in T^{\mathbb{C}}M$,*

$$\sum \Omega(X,Y)Z = \sum \Big(T(T(X,Y),Z) + \nabla_X T(Y,Z)\Big),$$

where the sum is taken over all cyclic permutations.

This identity induces the following formulae:

$$\mathcal{R}_{i\bar{j}k}{}^l = \mathcal{R}_{k\bar{j}i}{}^l - T^{\bar{r}}_{ik}T^l_{\bar{r}\bar{j}} + \nabla_{\bar{j}}T^l_{ki} = \mathcal{R}_{k\bar{j}i}{}^l - B^{\bar{r}}_{ik}B^l_{\bar{r}\bar{j}} + \nabla_{\bar{j}}T^l_{ki}, \quad (2.15)$$

$$\mathcal{H}_{ijk}{}^l = T^{\bar{r}}_{ji}T^{\bar{k}}_{\bar{r}\bar{l}} + \nabla_{\bar{l}}T^{\bar{k}}_{ji} = -B^{\bar{r}}_{ji}T^{\bar{k}}_{\bar{r}\bar{l}} + \nabla_{\bar{l}}T^{\bar{k}}_{ji}, \quad (2.16)$$

where used that $\mathcal{R}_{ij\bar{k}\bar{l}} = \mathcal{R}_{\bar{i}\bar{j}kl} = \mathcal{H}_{j\bar{l}ik} = \mathcal{H}_{\bar{j}l\bar{i}\bar{k}} = \mathcal{H}_{\bar{l}ijk} = \mathcal{H}_{l\bar{i}\bar{j}\bar{k}} = 0$.

Let $\{Z_r\}$ be a local unitary $(1,0)$-frame with respect to g around a fixed point $p \in M$. Note that unitary frames always exist locally since we can take any frame and apply the Gram-Schmidt process. Then with respect to a local g-unitary frame, we have $g_{i\bar{j}} = \delta_{ij}$ for any $i, j, k = 1, \dots, n$, and the Christoffel symbols satisfy

$$\Gamma^k_{ij} = -\Gamma^{\bar{j}}_{i\bar{k}}, \quad \Gamma^{\bar{k}}_{\bar{i}\bar{j}} = -\Gamma^j_{\bar{i}k},$$

since we have

$$\Gamma^k_{ij} = g(\nabla_i Z_j, Z_{\bar{k}}) = Z_i(g_{j\bar{k}}) - g(Z_j, \nabla_i Z_{\bar{k}}) = -\Gamma^{\bar{j}}_{i\bar{k}},$$

$$\Gamma^{\bar{k}}_{\bar{i}\bar{j}} = g(Z_k, \nabla_{\bar{i}} Z_{\bar{j}}) = Z_{\bar{i}}(g_{k\bar{j}}) - g(\nabla_{\bar{i}} Z_k, Z_{\bar{j}}) = -\Gamma^j_{\bar{i}k}.$$

And also we have

$$\begin{aligned}
\mathcal{R}_{i\bar{j}k}{}^r &= Z_i(\Gamma^r_{\bar{j}k}) - Z_{\bar{j}}(\Gamma^r_{ik}) + \Gamma^r_{is}\Gamma^s_{\bar{j}k} - \Gamma^r_{\bar{j}s}\Gamma^s_{ik} - B^s_{i\bar{j}}\Gamma^r_{sk} + B^{\bar{s}}_{\bar{j}i}\Gamma^r_{\bar{s}k} \\
&= -Z_i(\Gamma^{\bar{k}}_{\bar{j}\bar{r}}) + Z_{\bar{j}}(\Gamma^{\bar{k}}_{i\bar{r}}) + \Gamma^{\bar{s}}_{i\bar{r}}\Gamma^{\bar{k}}_{\bar{j}\bar{s}} - \Gamma^{\bar{s}}_{\bar{j}\bar{r}}\Gamma^{\bar{k}}_{i\bar{s}} + B^s_{i\bar{j}}\Gamma^{\bar{k}}_{s\bar{r}} - B^{\bar{s}}_{\bar{j}i}\Gamma^{\bar{k}}_{\bar{s}\bar{r}} \\
&= -\mathcal{R}_{i\bar{j}\bar{r}}{}^{\bar{k}}, \qquad (2.17)
\end{aligned}$$

$$\begin{aligned}
{\mathcal{H}_{ijk}}^{r} &= Z_i(\Gamma^r_{jk}) - Z_j(\Gamma^r_{ik}) + \Gamma^r_{is}\Gamma^s_{jk} - \Gamma^r_{js}\Gamma^s_{ik} - B^s_{ij}\Gamma^r_{sk} - B^{\bar{s}}_{ij}\Gamma^r_{\bar{s}k} \\
&= -Z_i(\Gamma^{\bar{k}}_{j\bar{r}}) - Z_j(\Gamma^{\bar{k}}_{i\bar{r}}) + \Gamma^{\bar{s}}_{i\bar{r}}\Gamma^{\bar{k}}_{j\bar{s}} - \Gamma^{\bar{s}}_{j\bar{r}}\Gamma^{\bar{k}}_{i\bar{s}} + B^s_{ij}\Gamma^{\bar{k}}_{s\bar{r}} + B^{\bar{s}}_{ij}\Gamma^{\bar{k}}_{\bar{s}\bar{r}} \\
&= -{\mathcal{H}_{ij\bar{r}}}^{\bar{k}} \qquad (2.18)
\end{aligned}$$

and

$$\begin{aligned}
\overline{{\mathcal{R}_{i\bar{j}k}}^{r}} &= Z_{\bar{i}}(\Gamma^{\bar{r}}_{j\bar{k}}) - Z_j(\Gamma^{\bar{r}}_{\bar{i}\bar{k}}) + \Gamma^{\bar{r}}_{\bar{i}\bar{s}}\Gamma^{\bar{s}}_{j\bar{k}} - \Gamma^{\bar{r}}_{j\bar{s}}\Gamma^{\bar{s}}_{\bar{i}\bar{k}} - B^{\bar{s}}_{\bar{i}j}\Gamma^{\bar{r}}_{\bar{s}\bar{k}} + B^s_{j\bar{i}}\Gamma^{\bar{r}}_{s\bar{k}} \\
&= Z_j(\Gamma^k_{\bar{i}r}) - Z_{\bar{i}}(\Gamma^k_{jr}) + \Gamma^s_{\bar{i}r}\Gamma^k_{js} - \Gamma^s_{jr}\Gamma^k_{\bar{i}s} - B^s_{j\bar{i}}\Gamma^k_{sr} + B^{\bar{s}}_{\bar{i}j}\Gamma^k_{\bar{s}r} \\
&= {\mathcal{R}_{j\bar{i}r}}^{k}, \qquad (2.19)
\end{aligned}$$

$$\begin{aligned}
\overline{{\mathcal{H}_{ijk}}^{r}} &= Z_{\bar{i}}(\Gamma^{\bar{r}}_{\bar{j}\bar{k}}) - Z_{\bar{j}}(\Gamma^{\bar{r}}_{\bar{i}\bar{k}}) + \Gamma^{\bar{r}}_{\bar{i}\bar{s}}\Gamma^{\bar{s}}_{\bar{j}\bar{k}} - \Gamma^{\bar{r}}_{\bar{j}\bar{s}}\Gamma^{\bar{s}}_{\bar{i}\bar{k}} - B^{\bar{s}}_{\bar{i}\bar{j}}\Gamma^{\bar{r}}_{\bar{s}\bar{k}} - B^s_{\bar{i}\bar{j}}\Gamma^{\bar{r}}_{s\bar{k}} \\
&= -Z_{\bar{i}}(\Gamma^k_{\bar{j}r}) + Z_{\bar{j}}(\Gamma^k_{\bar{i}r}) + \Gamma^s_{\bar{i}r}\Gamma^k_{\bar{j}s} - \Gamma^s_{\bar{j}r}\Gamma^k_{\bar{i}s} - B^{\bar{s}}_{\bar{j}\bar{i}}\Gamma^k_{\bar{s}r} - B^s_{\bar{j}\bar{i}}\Gamma^k_{sr} \\
&= {\mathcal{H}_{\bar{j}\bar{i}r}}^{k}. \qquad (2.20)
\end{aligned}$$

Hence we obtain $\mathcal{R}_{i\bar{j}k\bar{r}} = -\mathcal{R}_{i\bar{j}\bar{r}k}$, $\mathcal{H}_{ijk\bar{r}} = -\mathcal{H}_{ij\bar{r}k}$ and $\overline{\mathcal{R}_{i\bar{j}k\bar{r}}} = \mathcal{R}_{j\bar{i}r\bar{k}}$, $\overline{\mathcal{H}_{ijk\bar{r}}} = \mathcal{H}_{\bar{j}\bar{i}r\bar{k}}$ by using a local unitary $(1,0)$-frame with respect to g.

3 Some results for a smooth function on almost Hermitian manifolds

Let (M, J, g) be an almost Hermitian manifold. Here note that $B^{\bar{q}}_{j\bar{b}}$, $B^q_{\bar{j}b}$'s do not depend on the metric g, which depend only on the almost complex structure J since the mixed derivatives $\nabla_j Z_{\bar{b}}$, $\nabla_{\bar{j}} Z_b$ do not depend on g. Since we have $B^q_{b\bar{j}} = -B^q_{\bar{j}b}$, we have that $B^q_{b\bar{j}}$, $B^{\bar{q}}_{\bar{b}j}$'s also do not depend on g (cf. [10]). Also note that $B^{\bar{s}}_{ri}$, $B^s_{\bar{r}\bar{i}}$ do not depend on g, depend only on J. We can choose a local unitary frame $\{Z_r\}$ around an arbitrary chosen point $p_0 \in M$ such that $g_{i\bar{j}}(p_0) = \delta_{ij}$ and $\nabla Z(p_0) = 0$ (cf. [11]). Then we have $\Gamma^k_{ij}(p_0) = 0$ since $\nabla_i Z_j(p_0) = \Gamma^k_{ij}(p_0)Z_k = 0$, also we obtain that

$$[Z_i, Z_{\bar{j}}](p_0) = \nabla_i Z_j(p_0) - \nabla_j Z_i(p_0) - T(Z_i, Z_{\bar{j}})(p_0) = 0 \quad \text{for all } i, j = 1, \ldots, n. \qquad (3.1)$$

Then we have that $0 = [Z_i, Z_{\bar{j}}](p_0) = B^k_{i\bar{j}}(p_0)Z_k + B^{\bar{k}}_{i\bar{j}}(p_0)Z_{\bar{k}}$, which gives that $B^k_{i\bar{j}}(p_0) = 0$ for all $i, j, k = 1, \ldots, n$ and that $B^{\bar{k}}_{i\bar{j}}(p_0) = 0$ for all $i, j, k = 1, \ldots, n$. By choosing such a local unitary frame around a point p_0, we have that the torsion tensor T' satisfies that $T^k_{ij}(p_0) = -B^k_{ij}(p_0)$ for all $i, j, k = 1, \ldots, n$, and for instance from the formula (2.11), we have that $\varphi_{i\bar{j}}(p_0) = \partial_i\partial_{\bar{j}}\varphi(p_0) = Z_i Z_{\bar{j}}(\varphi)(p_0) = Z_{\bar{j}} Z_i(\varphi)(p_0) = \varphi_{\bar{j}i}(p_0)$ for a smooth real-valued function φ. We show the following critical lemma for proving the main result. We choose and fix a local unitary frame $\{Z_r\}$ around an arbitrary chosen point $p_0 \in M$ such that $g_{i\bar{j}}(p_0) = \delta_{ij}$ and $\nabla Z(p_0) = 0$. Our computations will be done at the point p_0.

We introduce some results for a smooth function on almost Hermitian manifolds. We write that $\varphi_s := \nabla_s \varphi = \partial\varphi(Z_s) = Z_s(\varphi)$.

Lemma 3.1. *One has for a smooth real-valued function φ on M,*

$$\partial\partial\bar{\partial}\varphi(Z_k, Z_j, Z_{\bar{i}}) = \bar{\partial}(B^{\bar{s}}_{kj})(Z_{\bar{i}})\bar{\partial}\varphi(Z_{\bar{s}}). \tag{3.2}$$

Proof. We compute that from (2.7),

$$\begin{aligned}
\partial\partial\bar{\partial}\varphi(Z_k, Z_j, Z_{\bar{i}}) &= Z_k(\varphi_{j\bar{i}}) - Z_j(\varphi_{k\bar{i}}) - B^s_{kj}\varphi_{s\bar{i}} - B^{\bar{s}}_{k\bar{i}}\varphi_{j\bar{s}} + B^{\bar{s}}_{j\bar{i}}\varphi_{k\bar{s}} \\
&= Z_k(Z_j Z_{\bar{i}}(\varphi) - B^{\bar{s}}_{j\bar{i}}\varphi_{\bar{s}}) - Z_j(Z_k Z_{\bar{i}}(\varphi) - B^{\bar{s}}_{k\bar{i}}\varphi_{\bar{s}}) - B^s_{kj}(Z_s Z_{\bar{i}}(\varphi) - B^{\bar{r}}_{s\bar{i}}\varphi_{\bar{r}}) \\
&= Z_k Z_j Z_{\bar{i}}(\varphi) - Z_j Z_k Z_{\bar{i}}(\varphi) - B^s_{kj} Z_s Z_{\bar{i}}(\varphi) - Z_k(B^{\bar{s}}_{j\bar{i}})\varphi_{\bar{s}} + Z_j(B^{\bar{s}}_{k\bar{i}})\varphi_{\bar{s}} \\
&= [Z_k, Z_j]Z_{\bar{i}}(\varphi) - B^s_{kj} Z_s Z_{\bar{i}}(\varphi) - Z_k(B^{\bar{s}}_{j\bar{i}})\varphi_{\bar{s}} + Z_j(B^{\bar{r}}_{s\bar{i}})\varphi_{\bar{r}} \\
&= B^{\bar{s}}_{kj} Z_{\bar{s}} Z_{\bar{i}}(\varphi) - Z_k(B^{\bar{s}}_{j\bar{i}})\varphi_{\bar{s}} + Z_j(B^{\bar{s}}_{k\bar{i}})\varphi_{\bar{s}} \\
&= B^{\bar{s}}_{kj}[Z_{\bar{s}}, Z_{\bar{i}}](\varphi) + B^{\bar{s}}_{kj} Z_{\bar{i}} Z_{\bar{s}}(\varphi) - \left\{ Z_k(\Gamma^{\bar{s}}_{j\bar{i}}) - Z_j(\Gamma^{\bar{s}}_{k\bar{i}}) \right\}\varphi_{\bar{s}} \\
&= B^{\bar{s}}_{kj} B^r_{\bar{s}\bar{i}}\varphi_r + B^{\bar{s}}_{kj} B^{\bar{r}}_{\bar{s}\bar{i}}\varphi_{\bar{r}} + B^{\bar{s}}_{kj} Z_{\bar{i}} Z_{\bar{s}}(\varphi) - \overline{\mathcal{H}_{\bar{k}\bar{j}i}{}^s\varphi_s},
\end{aligned} \tag{3.3}$$

where we have used that $\Gamma^{\bar{k}}_{i\bar{j}}(p_0) = B^{\bar{k}}_{i\bar{j}}(p_0) = 0$, $\Gamma^k_{\bar{i}j}(p_0) = B^k_{\bar{i}j}(p_0) = 0$, $\Gamma^k_{ij}(p_0) = 0$ for all $i, j, k = 1, \ldots, n$, and that from (2.14),

$$\begin{aligned}
\mathcal{H}_{\bar{k}\bar{j}i}{}^s(p_0) &= \left\{ Z_{\bar{k}}(\Gamma^s_{\bar{j}i}) - Z_{\bar{j}}(\Gamma^s_{\bar{k}i}) + \Gamma^s_{\bar{k}r}\Gamma^r_{\bar{j}i} - \Gamma^s_{\bar{j}r}\Gamma^r_{\bar{k}i} - B^r_{\bar{k}\bar{j}}\Gamma^s_{ri} - B^{\bar{r}}_{\bar{k}\bar{j}}\Gamma^s_{\bar{r}i} \right\}(p_0) \\
&= Z_{\bar{k}}(\Gamma^s_{\bar{j}i})(p_0) - Z_{\bar{j}}(\Gamma^s_{\bar{k}i})(p_0).
\end{aligned}$$

We compute that

$$\begin{aligned}
B^{\bar{s}}_{kj} Z_{\bar{i}} Z_{\bar{s}}(\varphi) &= Z_{\bar{i}}(B^{\bar{s}}_{kj} Z_{\bar{s}}(\varphi)) - Z_{\bar{i}}(B^{\bar{s}}_{kj}) Z_{\bar{s}}(\varphi) \\
&= Z_{\bar{i}}(\partial^2\varphi(Z_k, Z_j)) - Z_{\bar{i}}(B^{\bar{s}}_{kj})\bar{\partial}\varphi(Z_{\bar{s}}) \\
&= \bar{\partial}\partial^2\varphi(Z_{\bar{i}}, Z_k, Z_j) - Z_{\bar{i}}(B^{\bar{s}}_{kj})\bar{\partial}\varphi(Z_{\bar{s}}),
\end{aligned} \tag{3.4}$$

where we used that

$$\begin{aligned}
\partial^2\varphi(Z_k, Z_j) &= Z_k Z_j(\varphi) - Z_j Z_k(\varphi) - B^s_{kj} Z_s(\varphi) \\
&= [Z_k, Z_j](\varphi) - B^s_{kj} Z_s(\varphi) \\
&= B^{\bar{s}}_{kj} Z_{\bar{s}}(\varphi),
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
\bar{\partial}\partial^2\varphi(Z_{\bar{i}}, Z_k, Z_j) &= Z_{\bar{i}}(\partial^2\varphi(Z_k, Z_j)) - \partial^2\varphi([Z_{\bar{i}}, Z_k], Z_j) + \partial^2\varphi([Z_{\bar{i}}, Z_j], Z_k) \\
&= Z_{\bar{i}}(\partial^2\varphi(Z_k, Z_j)) - B^{\bar{r}}_{sj} B^s_{\bar{i}k}\varphi_{\bar{r}} + B^{\bar{r}}_{sk} B^s_{\bar{i}j}\varphi_{\bar{r}} \\
&= Z_{\bar{i}}(\partial^2\varphi(Z_k, Z_j)).
\end{aligned}$$

By combining (3.3) with (3.4), we obtain

$$\begin{aligned}
\partial\partial\bar{\partial}\varphi(Z_k, Z_j, Z_{\bar{i}}) &= \bar{\partial}\partial^2\varphi(Z_{\bar{i}}, Z_k, Z_j) + B^{\bar{s}}_{kj} B^r_{\bar{s}\bar{i}}\partial\varphi(Z_r) + \left\{ B^{\bar{r}}_{kj} B^{\bar{s}}_{\bar{r}\bar{i}} - Z_{\bar{i}}(B^{\bar{s}}_{kj}) - \overline{\mathcal{H}_{\bar{k}\bar{j}i}{}^s} \right\}\bar{\partial}\varphi(Z_{\bar{s}}) \\
&= \bar{\partial}\partial^2\varphi(Z_{\bar{i}}, Z_k, Z_j) + B^{\bar{s}}_{kj} B^r_{\bar{s}\bar{i}}\partial\varphi(Z_r),
\end{aligned}$$

where we have used that from (2.16) and (2.20),

$$\begin{aligned}\overline{\mathcal{H}_{\bar{k}\bar{j}i}{}^{s}} &= \mathcal{H}_{jks}{}^{i}\\ &= -B_{kj}^{\bar{r}}T_{\bar{r}\bar{i}}^{\bar{s}} + \nabla_{\bar{i}}T_{kj}^{\bar{s}}\\ &= B_{kj}^{\bar{r}}B_{\bar{r}\bar{i}}^{\bar{s}} - Z_{\bar{i}}(B_{kj}^{\bar{s}}). \qquad (3.6)\end{aligned}$$

We compute by using (3.5),

$$\begin{aligned}\partial\partial\bar{\partial}\varphi(Z_k, Z_j, Z_{\bar{i}}) &= \bar{\partial}\partial^2\varphi(Z_{\bar{i}}, Z_k, Z_j) + B_{kj}^{\bar{s}}B_{\bar{s}\bar{i}}^{r}\partial\varphi(Z_r)\\ &= \bar{\partial}(B_{kj}^{\bar{s}}\bar{\partial}\varphi(Z_{\bar{s}}))(Z_{\bar{i}}) + T_{kj}^{\bar{s}}T_{\bar{s}\bar{i}}^{r}\partial\varphi(Z_r)\\ &= \bar{\partial}(B_{kj}^{\bar{s}})(Z_{\bar{i}})\bar{\partial}\varphi(Z_{\bar{s}}) + B_{kj}^{\bar{s}}\bar{\partial}^2\varphi(Z_{\bar{i}}, Z_{\bar{s}}) + T_{kj}^{\bar{s}}T_{\bar{s}\bar{i}}^{r}\partial\varphi(Z_r)\\ &= \bar{\partial}(B_{kj}^{\bar{s}})(Z_{\bar{i}})\bar{\partial}\varphi(Z_{\bar{s}}) - T_{kj}^{\bar{s}}B_{\bar{i}\bar{s}}^{r}\partial\varphi(Z_r) + T_{kj}^{\bar{s}}T_{\bar{s}\bar{i}}^{r}\partial\varphi(Z_r)\\ &= \bar{\partial}(B_{kj}^{\bar{s}})(Z_{\bar{i}})\bar{\partial}\varphi(Z_{\bar{s}}),\end{aligned}$$

where we have used that $B_{\bar{i}\bar{s}}^{r} = -T_{\bar{i}\bar{s}}^{r} = T_{\bar{s}\bar{i}}^{r}$. □

Lemma 3.2. *One has for a smooth real-valued function φ on M,*

$$\bar{\partial}\partial\bar{\partial}\varphi(Z_{\bar{k}}, Z_i, Z_{\bar{j}}) = \partial(B_{\bar{k}\bar{j}}^{s})(Z_i)\partial\varphi(Z_s). \qquad (3.7)$$

Proof. We compute that from (2.8), using $B_{i\bar{j}}^{s}(p_0) = 0$ for all $i, j, s = 1, \ldots, n$, $B_{i\bar{s}}^{\bar{r}}(p_0) = 0$ for all $i, r, s = 1, \ldots, n$ and $[Z_{\bar{k}}, Z_i](p_0) = 0$ for all $i, k = 1, \ldots, n$,

$$\begin{aligned}\bar{\partial}\partial\bar{\partial}\varphi(Z_{\bar{k}}, Z_i, Z_{\bar{j}}) &= Z_{\bar{k}}(\varphi_{i\bar{j}}) - Z_{\bar{j}}(\varphi_{i\bar{k}}) - B_{i\bar{j}}^{s}\varphi_{s\bar{k}} + B_{i\bar{k}}^{s}\varphi_{s\bar{j}} + B_{\bar{j}\bar{k}}^{\bar{s}}\varphi_{i\bar{s}}\\ &= Z_{\bar{k}}(Z_iZ_{\bar{j}}(\varphi) - B_{i\bar{j}}^{\bar{s}}\varphi_{\bar{s}}) - Z_{\bar{j}}(Z_iZ_{\bar{k}}(\varphi) - B_{i\bar{k}}^{\bar{s}}\varphi_{\bar{s}}) + B_{\bar{j}\bar{k}}^{\bar{s}}\varphi_{i\bar{s}}\\ &= Z_{\bar{k}}Z_iZ_{\bar{j}}(\varphi) - Z_{\bar{j}}Z_iZ_{\bar{k}}(\varphi) + B_{\bar{j}\bar{k}}^{\bar{s}}(Z_iZ_{\bar{s}}(\varphi) - B_{i\bar{s}}^{\bar{r}}\varphi_{\bar{r}})\\ &\quad -Z_{\bar{k}}(B_{i\bar{j}}^{\bar{s}})\varphi_{\bar{s}} + Z_{\bar{j}}(B_{i\bar{k}}^{\bar{s}})\varphi_{\bar{s}}\\ &= Z_iZ_{\bar{k}}Z_{\bar{j}}(\varphi) + [Z_{\bar{k}}, Z_i]Z_{\bar{j}}(\varphi) - Z_iZ_{\bar{j}}Z_{\bar{k}}(\varphi) - [Z_{\bar{j}}, Z_i]Z_{\bar{k}}(\varphi)\\ &\quad +B_{\bar{j}\bar{k}}^{\bar{s}}Z_iZ_{\bar{s}}(\varphi) - Z_{\bar{k}}(B_{i\bar{j}}^{\bar{s}})\varphi_{\bar{s}} + Z_{\bar{j}}(B_{i\bar{k}}^{\bar{s}})\varphi_{\bar{s}}\\ &= Z_i[Z_{\bar{k}}, Z_{\bar{j}}](\varphi) - B_{\bar{k}\bar{j}}^{\bar{s}}Z_iZ_{\bar{s}}(\varphi) - Z_{\bar{k}}(\Gamma_{i\bar{j}}^{\bar{s}})\varphi_{\bar{s}} + Z_{\bar{j}}(\Gamma_{i\bar{k}}^{\bar{s}})\varphi_{\bar{s}}\\ &= B_{\bar{k}\bar{j}}^{s}Z_iZ_s(\varphi) + Z_i(B_{\bar{k}\bar{j}}^{s})\varphi_s + Z_i(B_{\bar{k}\bar{j}}^{\bar{s}})\varphi_{\bar{s}} - Z_{\bar{k}}(\Gamma_{i\bar{j}}^{\bar{s}})\varphi_{\bar{s}} + Z_{\bar{j}}(\Gamma_{i\bar{k}}^{\bar{s}})\varphi_{\bar{s}}\\ &= B_{\bar{k}\bar{j}}^{s}Z_iZ_s(\varphi) - Z_i(T_{\bar{k}\bar{j}}^{s})\varphi_s - \Big\{Z_i(\Gamma_{\bar{k}\bar{j}}^{\bar{s}}) - Z_i(\Gamma_{\bar{j}\bar{k}}^{\bar{s}}) - Z_i(B_{\bar{k}\bar{j}}^{\bar{s}})\Big\}\varphi_{\bar{s}}\\ &\quad -\Big\{Z_{\bar{k}}(\Gamma_{i\bar{j}}^{\bar{s}}) - Z_i(\Gamma_{\bar{k}\bar{j}}^{\bar{s}})\Big\}\varphi_{\bar{s}} + \Big\{Z_{\bar{j}}(\Gamma_{i\bar{k}}^{\bar{s}}) - Z_i(\Gamma_{\bar{j}\bar{k}}^{\bar{s}})\Big\}\varphi_{\bar{s}}\\ &= B_{\bar{k}\bar{j}}^{s}Z_iZ_s(\varphi) - Z_i(T_{\bar{k}\bar{j}}^{s})\varphi_s - Z_i(T_{\bar{k}\bar{j}}^{\bar{s}})\varphi_{\bar{s}} - \overline{\mathcal{R}_{k\bar{i}j}{}^{s}}\varphi_{\bar{s}} + \overline{\mathcal{R}_{j\bar{i}k}{}^{s}}\varphi_{\bar{s}}, \qquad (3.8)\end{aligned}$$

where we have used that $B_{\bar{k}\bar{j}}^{\bar{s}} = -B_{\bar{j}\bar{k}}^{\bar{s}}$ and that

$$\begin{aligned}Z_iZ_{\bar{k}}Z_{\bar{j}}(\varphi) - Z_iZ_{\bar{j}}Z_{\bar{k}}(\varphi) &= Z_i[Z_{\bar{k}}, Z_{\bar{j}}](\varphi)\\ &= Z_i(B_{\bar{k}\bar{j}}^{s}Z_s + B_{\bar{k}\bar{j}}^{\bar{s}}Z_{\bar{s}})(\varphi)\\ &= Z_i(B_{\bar{k}\bar{j}}^{s})\varphi_s + B_{\bar{k}\bar{j}}^{s}Z_iZ_s(\varphi) + Z_i(B_{\bar{k}\bar{j}}^{\bar{s}})\varphi_{\bar{s}} + B_{\bar{k}\bar{j}}^{\bar{s}}Z_iZ_{\bar{s}}(\varphi),\end{aligned}$$

and from (2.12),

$$\begin{aligned}\mathcal{R}_{k\bar{i}j}{}^{s}(p_0) &= \Big\{Z_k(\Gamma^s_{\bar{i}j}) - Z_{\bar{i}}(\Gamma^s_{kj}) + \Gamma^s_{kr}\Gamma^r_{\bar{i}j} - \Gamma^s_{\bar{i}r}\Gamma^r_{kj} - B^r_{k\bar{i}}\Gamma^s_{rj} + B^{\bar{r}}_{\bar{i}k}\Gamma^s_{\bar{r}j}\Big\}(p_0)\\ &= Z_k(\Gamma^s_{\bar{i}j})(p_0) - Z_{\bar{i}}(\Gamma^s_{kj})(p_0).\end{aligned}$$

We compute that

$$\begin{aligned}B^s_{\bar{k}\bar{j}}Z_iZ_s(\varphi) &= Z_i(B^s_{\bar{k}\bar{j}}Z_s(\varphi)) - Z_i(B^s_{\bar{k}\bar{j}})Z_s(\varphi)\\ &= Z_i(\bar{\partial}^2\varphi(Z_{\bar{k}}, Z_{\bar{j}})) - Z_i(B^s_{\bar{k}\bar{j}})\partial\varphi(Z_s)\\ &= \partial\bar{\partial}^2\varphi(Z_i, Z_{\bar{k}}, Z_{\bar{j}}) + Z_i(T^s_{\bar{k}\bar{j}})\partial\varphi(Z_s), \qquad (3.9)\end{aligned}$$

where we used that

$$\begin{aligned}\bar{\partial}^2\varphi(Z_{\bar{k}}, Z_{\bar{j}}) &= Z_{\bar{k}}Z_{\bar{j}}(\varphi) - Z_{\bar{j}}Z_{\bar{k}}(\varphi) - B^{\bar{s}}_{\bar{k}\bar{j}}Z_{\bar{s}}(\varphi)\\ &= [Z_{\bar{k}}, Z_{\bar{j}}](\varphi) - B^{\bar{s}}_{\bar{k}\bar{j}}Z_{\bar{s}}(\varphi)\\ &= B^s_{\bar{k}\bar{j}}Z_s(\varphi), \qquad (3.10)\end{aligned}$$

$$\begin{aligned}\partial\bar{\partial}^2\varphi(Z_i, Z_{\bar{k}}, Z_{\bar{j}}) &= Z_i(\bar{\partial}^2\varphi(Z_{\bar{k}}, Z_{\bar{j}})) - \bar{\partial}^2\varphi([Z_i, Z_{\bar{k}}], Z_{\bar{j}}) + \bar{\partial}^2\varphi([Z_i, Z_{\bar{j}}], Z_{\bar{k}})\\ &= Z_i(\partial^2\varphi(Z_{\bar{k}}, Z_{\bar{j}})) - B^r_{\bar{s}\bar{j}}B^{\bar{s}}_{i\bar{k}}\varphi_r + B^r_{\bar{s}\bar{k}}B^{\bar{s}}_{i\bar{j}}\varphi_r\\ &= Z_i(\partial^2\varphi(Z_{\bar{k}}, Z_{\bar{j}})).\end{aligned}$$

Combining (3.8) with (3.9), we obtain that

$$\begin{aligned}\bar{\partial}\partial\bar{\partial}\varphi(Z_{\bar{k}}, Z_i, Z_{\bar{j}}) &= \partial\bar{\partial}^2\varphi(Z_i, Z_{\bar{k}}, Z_{\bar{j}}) + \Big\{\overline{\mathcal{R}_{j\bar{i}k}{}^{s}} - \overline{\mathcal{R}_{k\bar{i}j}{}^{s}} - Z_i(T^{\bar{s}}_{\bar{k}\bar{j}})\Big\}\bar{\partial}\varphi(Z_{\bar{s}})\\ &= \partial\bar{\partial}^2\varphi(Z_i, Z_{\bar{k}}, Z_{\bar{j}}) + T^r_{\bar{k}\bar{j}}T^{\bar{s}}_{ri}\bar{\partial}\varphi(Z_{\bar{s}}),\end{aligned}$$

where we have used that from (2.15),

$$\begin{aligned}\overline{\mathcal{R}_{j\bar{i}k}{}^{s} - \mathcal{R}_{k\bar{i}j}{}^{s}} &= \overline{-B^r_{j\bar{k}}B^{\bar{s}}_{ri} + \nabla_{\bar{i}}T^s_{kj}}\\ &= T^r_{\bar{k}\bar{j}}T^{\bar{s}}_{ri} + Z_i(T^{\bar{s}}_{\bar{k}\bar{j}}).\end{aligned}$$

We compute that by applying (3.5) and (3.10),

$$\begin{aligned}\bar{\partial}\partial\bar{\partial}\varphi(Z_{\bar{k}}, Z_i, Z_{\bar{j}}) &= \partial\bar{\partial}^2\varphi(Z_i, Z_{\bar{k}}, Z_{\bar{j}}) + T^r_{\bar{k}\bar{j}}T^{\bar{s}}_{ri}\bar{\partial}\varphi(Z_{\bar{s}})\\ &= \partial(B^s_{\bar{k}\bar{j}}\partial\varphi(Z_s))(Z_i) + T^r_{\bar{k}\bar{j}}T^{\bar{s}}_{ri}\bar{\partial}\varphi(Z_{\bar{s}})\\ &= \partial(B^s_{\bar{k}\bar{j}})(Z_i)\partial\varphi(Z_s) + B^r_{\bar{k}\bar{j}}\partial^2\varphi(Z_i, Z_r) + T^r_{\bar{k}\bar{j}}T^{\bar{s}}_{ri}\bar{\partial}\varphi(Z_{\bar{s}})\\ &= \partial(B^s_{\bar{k}\bar{j}})(Z_i)\partial\varphi(Z_s) - T^r_{\bar{k}\bar{j}}B^{\bar{s}}_{ir}\bar{\partial}\varphi(Z_{\bar{s}}) + T^r_{\bar{k}\bar{j}}T^{\bar{s}}_{ri}\bar{\partial}\varphi(Z_{\bar{s}})\\ &= \partial(B^s_{\bar{k}\bar{j}})(Z_i)\partial\varphi(Z_s),\end{aligned}$$

where we have used that $B^{\bar{s}}_{ir} = -T^{\bar{s}}_{ir} = T^{\bar{s}}_{ri}$. □

Lemma 3.3. *One has for a smooth real-valued function φ on M,*

$$\partial\bar{\partial}\partial\bar{\partial}\varphi(Z_l, Z_{\bar{k}}, Z_i, Z_{\bar{j}}) = -\partial^2(T^s_{\bar{k}\bar{j}})(Z_l, Z_i)\partial\varphi(Z_s) + \partial(T^s_{\bar{k}\bar{j}})(Z_i)T^{\bar{r}}_{ls}\bar{\partial}\varphi(Z_{\bar{r}}). \tag{3.11}$$

Proof. By applying (3.5) and (3.6), we have that

$$\begin{aligned}\partial\bar{\partial}\partial\bar{\partial}\varphi(Z_l, Z_{\bar{k}}, Z_i, Z_{\bar{j}}) &= \partial^2(B^s_{\bar{k}\bar{j}})(Z_l, Z_i)\partial\varphi(Z_s) + \partial(B^s_{\bar{k}\bar{j}})(Z_i)\partial^2\varphi(Z_l, Z_s)\\ &= -\partial^2(T^s_{\bar{k}\bar{j}})(Z_l, Z_i)\partial\varphi(Z_s) + \partial(T^s_{\bar{k}\bar{j}})(Z_i)T^{\bar{r}}_{ls}\bar{\partial}\varphi(Z_{\bar{r}}). \quad \square\end{aligned}$$

In order to avoid a notational quagmire, we adopt the following $*$-convention $\mathcal{C}_1 * \mathcal{C}_2$ between two geometric quantities $\mathcal{C}_1$ and $\mathcal{C}_2$ with respect to a metric g:

(1) Summation over pairs of maching upper and lower indices.

(2) Contraction on upper indices with respect to the metric.

(3) Contraction on lower indices with respect to the dual metrics.

Since the point p_0 was chosen arbitrary, the computations in Lemma 3.1–3.3 hold globally on an almost Hermitian manifold M for any real-valued smooth function φ, which implies that we can write (3.2), (3.7), and (3.11) globally on M as follows:

$$\partial\bar{\partial}\partial\bar{\partial}\varphi =: \mathcal{T}_1 * \partial\varphi + \mathcal{T}_2 * \bar{\partial}\varphi, \quad \partial^2\bar{\partial}\varphi =: \mathcal{T}_3 * \bar{\partial}\varphi, \quad \bar{\partial}\partial\bar{\partial}\varphi =: \mathcal{T}_4 * \partial\varphi. \tag{3.12}$$

4 Proof of Theorem 1.1

Let (M^{2n}, J, ω) be a compact almost Hermitian manifold of real dimension $2n$ with $n \geq 2$ in this whole section. Let u be a smooth solution of (1.1). As in [5], we let $S_k(\lambda)$ denote the k-th elementary symmetric polynomial of $\lambda \in \mathbb{R}^n$:

$$S_k(\lambda) := \sum_{1\leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}.$$

For a square matrix U, we define $S_\alpha(U) := S_\alpha(\lambda(U))$, where $\lambda(U)$ denote the eigenvalues of the matrix U. Locally, we can write the equation (1.1) in the following form:

$$\frac{S_n(\chi_u)}{S_{n-\alpha}(\chi_u)} = \frac{\psi}{C_n^\alpha}, \tag{4.1}$$

where $C_n^\alpha := \frac{n!}{(n-\alpha)!\alpha!}$. We need the following generalized Newton-MacLaurin inequality.

Lemma 4.1 (cf. [1, Proposition 3], [9, Proposition 2.1])**.** *For $\lambda \in \Gamma_k := \{\lambda \in \mathbb{R}^n : S_k(\lambda) > 0,$ $\forall\ 1 \leq i \leq k\}$ and $0 \leq l < k \leq n$, $0 \leq s < r$, $r \leq k$, $s \leq l$, we have*

$$\left[\frac{\frac{S_k(\lambda)}{C_n^k}}{\frac{S_l(\lambda)}{C_n^l}}\right]^{\frac{1}{k-l}} \leq \left[\frac{\frac{S_r(\lambda)}{C_n^r}}{\frac{S_s(\lambda)}{C_n^s}}\right]^{\frac{1}{r-s}}. \tag{4.2}$$

In this section, the positive constant C may be changed from line to line, but it depends on the allowed data.

Proof of Theorem 1.1. It suffices to show the following key inequality:

$$\int_M |\partial e^{-\frac{p}{2}u}|_g^2 \omega^n \leq Cp \int_M e^{-pu}\omega^n \tag{4.3}$$

for p large enough.

Lemma 4.2. *Let u be a smooth admissible solution to the Monge-Ampère type equation (1,1). Then, there are uniform constants C, p_0 such that for any $p \geq p_0$, we have the inequality* (4.3).

Proof. Without loss of generality, we may assume that

$$n\chi^{n-1} > (n-\alpha)\psi\chi^{n-\alpha-1} \wedge \omega^\alpha, \tag{4.4}$$

and there exist uniform positive constants $\lambda, \Lambda > 0$ such that

$$\lambda\omega \leq \chi \leq \Lambda\omega. \tag{4.5}$$

As the local expression (4.1):

$$\frac{\chi_u^n}{\chi_u^{n-\alpha} \wedge \omega^\alpha} = C_n^\alpha \frac{S_n(\chi_u)}{S_{n-\alpha}(\chi_u)} = \psi,$$

we locally have that

$$C_{n-1}^\alpha \frac{S_{n-1}(\chi_u)}{S_{n-\alpha-1}(\chi_u)} = \frac{\chi_u^{n-1}}{\chi_u^{n-\alpha-1} \wedge \omega^\alpha}$$

and which implies that the following inequality

$$n\chi_u^{n-1} > (n-\alpha)\psi\chi_u^{n-\alpha-1} \wedge \omega^\alpha \tag{4.6}$$

is equivalent to

$$\frac{S_{n-1}(\chi_u)}{S_{n-\alpha-1}(\chi_u)} > \frac{S_n(\chi_u)}{S_{n-\alpha}(\chi_u)} \tag{4.7}$$

since we have locally that

$$\frac{n-\alpha}{n} \cdot \psi = \frac{n-\alpha}{n} \cdot C_n^\alpha \frac{S_n(\chi_u)}{S_{n-\alpha}(\chi_u)} = C_{n-1}^\alpha \frac{S_n(\chi_u)}{S_{n-\alpha}(\chi_u)}.$$

Note that we may apply Lemma 4.1 to χ_u since $\chi_u > 0$. Applying the inequality (4.2), we have

$$\left[\frac{\frac{S_n(\chi_u)}{C_n^n}}{\frac{S_{n-\alpha}(\chi_u)}{C_n^{n-\alpha}}}\right]^{\frac{1}{\alpha}} \leq \left[\frac{\frac{S_{n-1}(\chi_u)}{C_n^{n-1}}}{\frac{S_{n-\alpha-1}(\chi_u)}{C_n^{n-\alpha-1}}}\right]^{\frac{1}{\alpha}},$$

which can be written by

$$\begin{aligned}
\frac{S_n(\chi_u)}{S_{n-\alpha}(\chi_u)} &\leq \frac{C_n^{n-\alpha-1}}{C_n^{n-1}C_n^{n-\alpha}} \frac{S_{n-1}(\chi_u)}{S_{n-\alpha-1}(\chi_u)} \\
&= \frac{n-\alpha}{n(\alpha+1)} \frac{S_{n-1}(\chi_u)}{S_{n-\alpha-1}(\chi_u)} \\
&< \frac{S_{n-1}(\chi_u)}{S_{n-\alpha-1}(\chi_u)},
\end{aligned}$$

where we used that $\frac{n-\alpha}{n(\alpha+1)} < 1$. Therefore, the inequality (4.7) holds and as a consequence, we have the inequality (4.6).

We estimate that

$$\begin{aligned} I &:= \int_M e^{-pu}((\chi_u^n - \chi^n) - \psi(\chi_u^{n-\alpha}\wedge\omega^\alpha - \chi^{n-\alpha}\wedge\omega^\alpha)) \\ &= \int_M e^{-pu}\Big(\frac{\chi_u^n}{\chi_u^{n-\alpha}\wedge\omega^\alpha} - \frac{\chi^n}{\chi^{n-\alpha}\wedge\omega^\alpha}\Big)\chi^{n-\alpha}\wedge\omega^\alpha \\ &\leq C\int_M e^{-pu}\omega^n. \end{aligned} \tag{4.8}$$

On the other hand, we have that by Stokes' theorem,

$$\begin{aligned} I &= \int_0^1\int_M e^{-pu}\frac{d}{dt}(\chi_{tu}^n - \psi\chi_{tu}^{n-\alpha}\wedge\omega^\alpha)dt \\ &= \int_0^1\int_M e^{-pu}\sqrt{-1}\partial\bar\partial u\wedge(n\chi_{tu}^{n-1} - (n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha)dt \\ &= \int_0^1\int_M d(e^{-pu}\sqrt{-1}\bar\partial u\wedge(n\chi_{tu}^{n-1} - (n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha))dt \\ &\quad -\int_0^1\int_M\sqrt{-1}\partial e^{-pu}\wedge\bar\partial u\wedge(n\chi_{tu}^{n-1} - (n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha)dt \\ &\quad +\int_0^1\int_M\sqrt{-1}e^{-pu}\bar\partial u\wedge\partial(n\chi_{tu}^{n-1} - (n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha)dt \\ &= p\int_0^1\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge(n\chi_{tu}^{n-1} - (n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha)dt \\ &\quad -\frac{1}{p}\int_0^1\int_M\sqrt{-1}\bar\partial e^{-pu}\wedge\partial(n\chi_{tu}^{n-1} - (n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha)dt \\ &= p\int_0^1\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge(n\chi_{tu}^{n-1} - (n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha)dt \\ &\quad -\frac{1}{p}\int_0^1\int_M d(\sqrt{-1}e^{-pu}\partial(n\chi_{tu}^{n-1} - (n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha))dt \\ &\quad +\frac{1}{p}\int_0^1\int_M e^{-pu}\sqrt{-1}\bar\partial\partial(n\chi_{tu}^{n-1} - (n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha))dt \\ &= p\int_0^1\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge(n\chi_{tu}^{n-1} - (n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha)dt \\ &\quad -\frac{1}{p}\int_0^1\int_M e^{-pu}\sqrt{-1}\partial\bar\partial(n\chi_{tu}^{n-1} - (n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha)dt, \end{aligned} \tag{4.9}$$

where we have used that $d = A + \partial + \bar\partial + \bar A$,

$$\bar\partial(e^{-pu}\sqrt{-1}\bar\partial u\wedge(n\chi_{tu}^{n-1} - (n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha)) = 0,$$

$$A(e^{-pu}\sqrt{-1}\bar\partial u\wedge(n\chi_{tu}^{n-1} - (n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha)) = 0,$$

$$\bar A(e^{-pu}\sqrt{-1}\bar\partial u\wedge(n\chi_{tu}^{n-1} - (n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha)) = 0,$$

$$\partial(\sqrt{-1}e^{-pu}\partial(n\chi_{tu}^{n-1} - (n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha)) = 0,$$

$$A(\sqrt{-1}e^{-pu}\partial(n\chi_{tu}^{n-1} - (n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha)) = 0,$$
$$\bar{A}(\sqrt{-1}e^{-pu}\partial(n\chi_{tu}^{n-1} - (n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha)) = 0$$

and from (2.9),

$$\begin{aligned}\bar{\partial}\partial(n\chi_{tu}^{n-1} - (n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha) &= -(\partial\bar{\partial} + A\bar{A} + \bar{A}A)(n\chi_{tu}^{n-1} - (n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha)\\ &= -\partial\bar{\partial}(n\chi_{tu}^{n-1} - (n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha)\end{aligned}$$

since we have

$$A(n\chi_{tu}^{n-1} - (n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha) = \bar{A}(n\chi_{tu}^{n-1} - (n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha) = 0.$$

We compute that for $0 \le t \le 1$,

$$\begin{aligned}
&-\frac{1}{p}\int_M e^{-pu}\sqrt{-1}\partial\bar{\partial}(n\chi_{tu}^{n-1} - (n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha)\\
=\ &-\frac{1}{p}\int_M e^{-pu}\sqrt{-1}\partial\left(n(n-1)\chi_{tu}^{n-2}\wedge(\bar{\partial}\chi + \sqrt{-1}t\bar{\partial}\partial\bar{\partial}u) - (n-\alpha)\bar{\partial}\psi\wedge\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha\right.\\
&\left.-(n-\alpha)(n-\alpha-1)\psi\chi_{tu}^{n-\alpha-2}\wedge(\bar{\partial}\chi + \sqrt{-1}t\bar{\partial}\partial\bar{\partial}u)\wedge\omega^\alpha - \alpha(n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^{\alpha-1}\wedge\bar{\partial}\omega\right)\\
=\ &-\frac{1}{p}\int_M \sqrt{-1}e^{-pu}\Big\{n(n-1)(n-2)\chi_{tu}^{n-3}\wedge(\partial\chi + t\sqrt{-1}\partial\partial\bar{\partial}u)\wedge(\bar{\partial}\chi + t\sqrt{-1}\bar{\partial}\partial\bar{\partial}u)\\
&+n(n-1)\chi_{tu}^{n-2}\wedge(\partial\bar{\partial}\chi + t\sqrt{-1}\partial\bar{\partial}\partial\bar{\partial}u) - (n-\alpha)\partial\bar{\partial}\psi\wedge\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha\\
&+(n-\alpha)(n-\alpha-1)\bar{\partial}\psi\wedge\chi_{tu}^{n-\alpha-2}\wedge(\partial\chi + t\sqrt{-1}\partial\partial\bar{\partial}u)\wedge\omega^\alpha\\
&+\alpha(n-\alpha)\bar{\partial}\psi\wedge\chi_{tu}^{n-\alpha-1}\wedge\omega^{\alpha-1}\wedge\partial\omega\\
&-(n-\alpha)(n-\alpha-1)\partial\psi\wedge\chi_{tu}^{n-\alpha-2}\wedge(\bar{\partial}\chi + t\sqrt{-1}\bar{\partial}\partial\bar{\partial}u)\wedge\omega^\alpha\\
&-(n-\alpha)(n-\alpha-1)(n-\alpha-2)\psi\chi_{tu}^{n-\alpha-3}\wedge(\partial\chi + t\sqrt{-1}\partial\partial\bar{\partial}u)\wedge(\bar{\partial}\chi + t\sqrt{-1}\bar{\partial}\partial\bar{\partial}u)\wedge\omega^\alpha\\
&-(n-\alpha)(n-\alpha-1)\psi\chi_{tu}^{n-\alpha-2}\wedge(\partial\bar{\partial}\chi + t\sqrt{-1}\partial\bar{\partial}\partial\bar{\partial}u)\wedge\omega^\alpha\\
&-\alpha(n-\alpha)(n-\alpha-1)\psi\chi_{tu}^{n-\alpha-2}\wedge(\bar{\partial}\chi + t\sqrt{-1}\bar{\partial}\partial\bar{\partial}u)\wedge\omega^{\alpha-1}\wedge\partial\omega\\
&-\alpha(n-\alpha)\partial\psi\wedge\chi_{tu}^{n-\alpha-1}\wedge\omega^{\alpha-1}\wedge\bar{\partial}\omega\\
&-\alpha(n-\alpha)(n-\alpha-1)\psi\chi_{tu}^{n-\alpha-2}\wedge(\partial\chi + t\sqrt{-1}\partial\partial\bar{\partial}u)\wedge\omega^{\alpha-1}\wedge\bar{\partial}\omega\\
&-\alpha(n-\alpha)(\alpha-1)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^{\alpha-2}\wedge\partial\omega\wedge\bar{\partial}\omega\\
&-\alpha(n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^{\alpha-1}\wedge\partial\bar{\partial}\omega\Big\}\\
\ge\ &-\frac{C}{p}\int_M e^{-pu}\chi_{tu}^{n-3}\wedge\omega^3 - \frac{C}{p}\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar{\partial}u\wedge\chi_{tu}^{n-3}\wedge\omega^2 - C\int_M e^{-pu}\chi_{tu}^{n-2}\wedge\omega^2\\
&-\frac{C}{p}\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar{\partial}u\wedge\chi_{tu}^{n-2}\wedge\omega\\
&-\frac{C}{p}\int_M e^{-pu}\chi_{tu}^{n-\alpha-1}\wedge\omega^{\alpha+1} - \frac{C}{p}\int_M e^{-pu}\chi_{tu}^{n-\alpha-2}\wedge\omega^{\alpha+2}\\
&-\frac{C}{p}\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar{\partial}u\wedge\chi_{tu}^{n-\alpha-2}\wedge\omega^{\alpha+1} - \frac{C}{p}\int_M e^{-pu}\chi_{tu}^{n-\alpha-3}\wedge\omega^{\alpha+3}\\
&-\frac{C}{p}\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar{\partial}u\wedge\chi_{tu}^{n-\alpha-3}\wedge\omega^{\alpha+2},
\end{aligned}\tag{4.10}$$

where we have used that for instance, by applying (3.12),

$$\begin{aligned}
&\int_M \sqrt{-1}e^{-pu}\chi_{tu}^{n-2}\wedge t\sqrt{-1}\partial\bar\partial\partial\bar\partial u\\
=\;&\int_M \sqrt{-1}e^{-pu}\chi_{tu}^{n-2}\wedge t\sqrt{-1}(\mathcal{T}_1 * \partial u + \mathcal{T}_2 * \bar\partial u)\\
\leq\;& C\int_M e^{-pu}\chi_{tu}^{n-2}\wedge\sqrt{-1}\partial u\wedge\bar\partial u\wedge\omega + C\int_M e^{-pu}\chi_{tu}^{n-2}\wedge\omega^2,
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
&\int_M \bar\partial\psi\wedge\chi_{tu}^{n-\alpha-2}\wedge t\sqrt{-1}\partial\partial\bar\partial u\wedge\omega^\alpha\\
=\;&\int_M \bar\partial\psi\wedge\chi_{tu}^{n-\alpha-2}\wedge t\sqrt{-1}\mathcal{T}_3 * \bar\partial u\wedge\omega^\alpha\\
\leq\;& C\int_M \chi_{tu}^{n-\alpha-2}\wedge\sqrt{-1}\partial u\wedge\bar\partial u\wedge\omega^{\alpha+1} + C\int_M \chi_{tu}^{n-\alpha-2}\wedge\omega^{\alpha+2},
\end{aligned} \tag{4.12}$$

$$\begin{aligned}
&\int_M \sqrt{-1}e^{-pu}\chi_{tu}^{n-3}\wedge\partial\chi\wedge t\sqrt{-1}\bar\partial\partial\bar\partial u\wedge\omega\\
=\;&\int_M \sqrt{-1}e^{-pu}\chi_{tu}^{n-3}\wedge\partial\chi\wedge t\sqrt{-1}\mathcal{T}_4 * \partial u\wedge\omega\\
\leq\;& C\int_M e^{-pu}\chi_{tu}^{n-3}\wedge\sqrt{-1}\partial u\wedge\bar\partial u\wedge\omega + C\int_M e^{-pu}\chi_{tu}^{n-3}\wedge\omega^3.
\end{aligned} \tag{4.13}$$

Since we have assumed that $\chi, \chi_u > 0$, then we have that $\chi_{tu} > 0$ for any $0 \leq t \leq 1$. Now we introduce the following crucial inequalities (cf. [6]):

Lemma 4.3. *For any $0 < t \leq 1$, $1 < l \leq n$, one has that*

$$\frac{l}{l-1}\int_0^t\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\chi_{su}^{l-1}\wedge\omega^{n-l}ds \geq \lambda\int_0^t\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\chi_{su}^{l-2}\wedge\omega^{n-l+1}ds, \tag{4.14}$$

and for any $0 < t \leq 1$, $1 \leq k \leq n$, one has that

$$\frac{k+1}{k}\int_0^t \chi_{su}^k\wedge\omega^{n-k}ds \geq \lambda\int_0^t \chi_{su}^{k-1}\wedge\omega^{n-k+1}ds, \tag{4.15}$$

where $\lambda > 0$ is the uniform constant in (4.5).

Proof. By using integration by parts and Gårding's inequality as in [6, (2.22)], we have that by using $\chi \geq \lambda\omega$,

$$\begin{aligned}
&\int_0^t\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\chi_{su}^{l-1}\wedge\omega^{n-l}ds\\
=\;&\int_0^t\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\chi_{su}^{l-2}\wedge(\chi + s\sqrt{-1}\partial\bar\partial u)\wedge\omega^{n-l}ds\\
\geq\;&\lambda\int_0^t\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\chi_{su}^{l-2}\wedge\omega^{n-l+1}ds\\
&+\frac{1}{l-1}\int_0^t\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge s\frac{d}{ds}\chi_{su}^{l-1}\wedge\omega^{n-l}ds\\
\geq\;&\lambda\int_0^t\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\chi_{su}^{l-2}\wedge\omega^{n-l+1}ds\\
&-\frac{1}{l-1}\int_0^t\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\chi_{su}^{l-1}\wedge\omega^{n-l}ds,
\end{aligned} \tag{4.16}$$

where we used that

$$\begin{aligned}
&\int_0^t \int_M e^{-pu}\sqrt{-1}\partial u \wedge \bar{\partial} u \wedge s\frac{d}{ds}\chi_{su}^{l-1} \wedge \omega^{n-l} ds \\
= \;& t\int_M e^{-pu}\sqrt{-1}\partial u \wedge \bar{\partial} u \wedge \chi_{tu}^{l-1} \wedge \omega^{n-l} - \int_0^t \int_M e^{-pu}\sqrt{-1}\partial u \wedge \bar{\partial} u \wedge \chi_{su}^{l-1} \wedge \omega^{n-l} ds \\
\geq \;& -\int_0^t \int_M e^{-pu}\sqrt{-1}\partial u \wedge \bar{\partial} u \wedge \chi_{su}^{l-1} \wedge \omega^{n-l} ds.
\end{aligned}$$

The inequality (4.16) gives the desired one (4.14). Next we compute that by using integration by parts and Gårding's inequality as in [6, (3.7)], for $1 \leq k \leq n$, using $\chi \geq \lambda\omega$,

$$\begin{aligned}
\int_0^t \chi_{su}^k \wedge \omega^{n-k} ds &= \int_0^t \chi_{su}^{k-1} \wedge (\chi + s\sqrt{-1}\partial\bar{\partial} u) \wedge \omega^{n-k} ds \\
&\geq \lambda \int_0^t \chi_{su}^{k-1} \wedge \omega^{n-k+1} ds + \frac{1}{k}\int_0^t s\frac{d}{ds}(\chi_{su}^k \wedge \omega^{n-k}) ds \\
&= \lambda \int_0^t \chi_{su}^{k-1} \wedge \omega^{n-k+1} ds + \frac{t}{k}\chi_{tu}^k \wedge \omega^{n-k} - \frac{1}{k}\int_0^t \chi_{su}^k \wedge \omega^{n-k} ds \\
&\geq \lambda \int_0^t \chi_{su}^{k-1} \wedge \omega^{n-k+1} ds - \frac{1}{k}\int_0^t \chi_{su}^k \wedge \omega^{n-k} ds,
\end{aligned}$$

which implies the inequality (4.15). □

By applying these inequalities (4.14) and (4.15) for $t = 1$ to the estimate (4.10), we obtain that

$$\begin{aligned}
&-\frac{1}{p}\int_0^1 \int_M e^{-pu}\sqrt{-1}\partial\bar{\partial}(n\chi_{tu}^{n-1} - (n-\alpha)\psi\chi_{tu}^{n-\alpha-1} \wedge \omega^\alpha) dt \\
\geq \;& -\frac{C}{p}\int_0^1 \int_M e^{-pu}\chi_{tu}^{n-1} \wedge \omega dt - \frac{C}{p}\int_0^1 \int_M e^{-pu}\sqrt{-1}\partial u \wedge \bar{\partial} u \wedge \chi_{tu}^{n-1} dt. \qquad (4.17)
\end{aligned}$$

Combining (4.17) with (4.9), we have that

$$\begin{aligned}
I \;\geq\; & p\int_0^1 \int_M e^{-pu}\sqrt{-1}\partial u \wedge \bar{\partial} u \wedge \Big\{\Big(n - \frac{C}{p^2}\Big)\chi_{tu}^{n-1} - (n-\alpha)\psi\chi_{tu}^{n-\alpha-1} \wedge \omega^\alpha\Big\} dt \\
& -\frac{C}{p}\int_0^1 \int_M e^{-pu}\chi_{tu}^{n-1} \wedge \omega dt. \qquad (4.18)
\end{aligned}$$

By the concavity of hyperbolic polynomials, for $0 < \tau < 1$, $1 \leq k \leq n$, we have (cf. [6, (2.13)])

$$\frac{1}{\tau}S_k^{\frac{1}{k}}(\chi_{\tau tu}) + \Big(1 - \frac{1}{\tau}\Big)S_k^{\frac{1}{k}}(\chi) \geq S_k^{\frac{1}{k}}(\chi_{tu}),$$

which gives

$$S_k(\chi_{\tau tu}) \geq \tau^k S_k(\chi_{tu}).$$

For $\tau = \frac{1}{2}$, $k = n-1$, we obtain that

$$\begin{aligned}
\int_0^1 \int_M e^{-pu}\chi_{tu}^{n-1} \wedge \omega dt &\leq 2^{n-1}\int_0^1 \int_M e^{-pu}\chi_{\frac{tu}{2}}^{n-1} \wedge \omega dt \\
&= 2^n \int_0^{\frac{1}{2}} \int_M e^{-pu}\chi_{tu}^{n-1} \wedge \omega dt. \qquad (4.19)
\end{aligned}$$

By combining (4.8), (4.18) and (4.19), we have that

$$\begin{aligned} & p\int_0^1\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar{\partial}u\wedge\Big\{\Big(n-\frac{C}{p^2}\Big)\chi_{tu}^{n-1}-(n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha\Big\}dt \\ \leq\ & C\int_M e^{-pu}\omega^n+\frac{C}{p}\int_0^{\frac{1}{2}}\int_M e^{-pu}\chi_{tu}^{n-1}\wedge\omega dt. \end{aligned} \tag{4.20}$$

Since we have $\chi_{tu}>0$ and

$$n\chi_{tu}^{n-1}-(n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha>0$$

for any $0\leq t\leq 1$, we can choose a sufficiently large p so that

$$n\chi_{tu}^{n-1}-(n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha-\frac{C}{p^2}\chi_{tu}^{n-1}>0.$$

Then we have that by the concavity of the quotient equation, for some $0<\delta<1$, we have (cf. [6, (3.10)])

$$n\chi_{tu}^{n-1}-(n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha>n\Big\{1-\frac{1}{(1+\delta-t\delta)^\alpha}\Big\}\chi_{tu}^{n-1},$$

hence for sufficiently large p,

$$\begin{aligned} & \int_0^1\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar{\partial}u\wedge\Big\{\Big(n-\frac{C}{p^2}\Big)\chi_{tu}^{n-1}-(n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha\Big\}dt \\ \geq\ & \int_0^{\frac{1}{2}}\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar{\partial}u\wedge\Big\{\Big(n-\frac{C}{p^2}\Big)\chi_{tu}^{n-1}-(n-\alpha)\psi\chi_{tu}^{n-\alpha-1}\wedge\omega^\alpha\Big\}dt \\ \geq\ & \int_0^{\frac{1}{2}}n\Big\{1-\frac{C}{np^2}-\frac{1}{(1+\delta-t\delta)^\alpha}\Big\}\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar{\partial}u\wedge\chi_{tu}^{n-1}dt \\ \geq\ & n\Big\{1-\frac{C}{np^2}-\frac{1}{(1+\frac{\delta}{2})^\alpha}\Big\}\int_0^{\frac{1}{2}}\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar{\partial}u\wedge\chi_{tu}^{n-1}dt. \end{aligned} \tag{4.21}$$

On the other hand, we compute by Stokes' theorem,

$$\begin{aligned} & \frac{1}{p}\int_0^{\frac{1}{2}}\int_M e^{-pu}\chi_{tu}^{n-1}\wedge\omega dt \\ =\ & \frac{1}{p}\int_0^{\frac{1}{2}}\int_0^t\frac{d}{ds}\Big(\int_M e^{-pu}\chi_{su}^{n-1}\wedge\omega\Big)dsdt+\frac{1}{2p}\int_M e^{-pu}\chi^{n-1}\wedge\omega \\ =\ & \frac{n-1}{p}\int_0^{\frac{1}{2}}\int_0^t\int_M e^{-pu}\sqrt{-1}\partial\bar{\partial}u\wedge\chi_{su}^{n-2}\wedge\omega dsdt+\frac{1}{2p}\int_M e^{-pu}\chi^{n-1}\wedge\omega \\ =\ & \frac{n-1}{p}\int_0^{\frac{1}{2}}\int_0^t\int_M d(e^{-pu}\sqrt{-1}\bar{\partial}u\wedge\chi_{su}^{n-2}\wedge\omega)dsdt \\ & -\frac{n-1}{p}\int_0^{\frac{1}{2}}\int_0^t\int_M\sqrt{-1}\partial e^{-pu}\wedge\bar{\partial}u\wedge\chi_{su}^{n-2}\wedge\omega dsdt \\ & +\frac{n-1}{p}\int_0^{\frac{1}{2}}\int_0^t\int_M e^{-pu}\sqrt{-1}\bar{\partial}u\wedge\partial(\chi_{su}^{n-2}\wedge\omega)dsdt+\frac{1}{2p}\int_M e^{-pu}\chi^{n-1}\wedge\omega \end{aligned}$$

$$\begin{aligned}
&= (n-1)\int_0^{\frac{1}{2}}\int_0^t\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar{\partial}u\wedge\chi_{su}^{n-2}\wedge\omega dsdt \\
&\quad -\frac{n-1}{p^2}\int_0^{\frac{1}{2}}\int_0^t\int_M \sqrt{-1}\bar{\partial}e^{-pu}\wedge\partial(\chi_{su}^{n-2}\wedge\omega)dsdt+\frac{1}{2p}\int_M e^{-pu}\chi^{n-1}\wedge\omega \\
&= (n-1)\int_0^{\frac{1}{2}}\int_0^t\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar{\partial}u\wedge\chi_{su}^{n-2}\wedge\omega dsdt \\
&\quad -\frac{n-1}{p^2}\int_0^{\frac{1}{2}}\int_0^t\int_M d(\sqrt{-1}e^{-pu}\partial(\chi_{su}^{n-2}\wedge\omega))dsdt \\
&\quad +\frac{n-1}{p^2}\int_0^{\frac{1}{2}}\int_0^t\int_M e^{-pu}\sqrt{-1}\bar{\partial}\partial(\chi_{su}^{n-2}\wedge\omega)dsdt+\frac{1}{2p}\int_M e^{-pu}\chi^{n-1}\wedge\omega \\
&= (n-1)\int_0^{\frac{1}{2}}\int_0^t\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar{\partial}u\wedge\chi_{su}^{n-2}\wedge\omega dsdt \\
&\quad -\frac{n-1}{p^2}\int_0^{\frac{1}{2}}\int_0^t\int_M e^{-pu}\sqrt{-1}\partial\bar{\partial}(\chi_{su}^{n-2}\wedge\omega)dsdt+\frac{1}{2p}\int_M e^{-pu}\chi^{n-1}\wedge\omega, \qquad (4.22)
\end{aligned}$$

where we used that as in the computation in (4.9),

$$d(e^{-pu}\sqrt{-1}\bar{\partial}u\wedge\chi_{su}^{n-2}\wedge\omega)=(\partial+\bar{\partial}+A+\bar{A})(e^{-pu}\sqrt{-1}\bar{\partial}u\wedge\chi_{su}^{n-2}\wedge\omega)=\partial(e^{-pu}\sqrt{-1}\bar{\partial}u\wedge\chi_{su}^{n-2}\wedge\omega),$$

$$d(\sqrt{-1}e^{-pu}\wedge\partial(\chi_{su}^{n-2}\wedge\omega))=(\partial+\bar{\partial}+A+\bar{A})(\sqrt{-1}e^{-pu}\wedge\partial(\chi_{su}^{n-2}\wedge\omega))=\bar{\partial}(\sqrt{-1}e^{-pu}\wedge\partial(\chi_{su}^{n-2}\wedge\omega)),$$

and

$$\bar{\partial}\partial(\chi_{su}^{n-2}\wedge\omega)=-(\partial\bar{\partial}+A\bar{A}+\bar{A}A)(\chi_{su}^{n-2}\wedge\omega)=-\partial\bar{\partial}(\chi_{su}^{n-2}\wedge\omega).$$

Applying (3.12), we estimate that as in (4.11)-(4.13) such as

$$\begin{aligned}
&\int_M \sqrt{-1}e^{-pu}\chi_{su}^{n-3}\wedge s\sqrt{-1}\partial\bar{\partial}\partial\bar{\partial}u\wedge\omega \\
&\leq C\int_M e^{-pu}\chi_{su}^{n-3}\wedge\sqrt{-1}\partial u\wedge\bar{\partial}u\wedge\omega^2+C\int_M e^{-pu}\chi_{su}^{n-3}\wedge\omega^3, \qquad (4.23)
\end{aligned}$$

$$\begin{aligned}
&\int_M e^{-pu}\chi_{su}^{n-4}\wedge s\sqrt{-1}\partial\partial\bar{\partial}u\wedge\bar{\partial}\chi\wedge\omega \\
&\leq C\int_M e^{-pu}\chi_{su}^{n-4}\wedge\sqrt{-1}\partial u\wedge\bar{\partial}u\wedge\omega^3+C\int_M e^{-pu}\chi_{su}^{n-4}\wedge\omega^4, \qquad (4.24)
\end{aligned}$$

$$\begin{aligned}
&\int_M e^{-pu}\chi_{su}^{n-4}\wedge\partial\chi\wedge s\sqrt{-1}\bar{\partial}\partial\bar{\partial}u\wedge\omega^2 \\
&\leq C\int_M e^{-pu}\chi_{su}^{n-4}\wedge\sqrt{-1}\partial u\wedge\bar{\partial}u\wedge\omega^3+C\int_M e^{-pu}\chi_{su}^{n-4}\wedge\omega^4. \qquad (4.25)
\end{aligned}$$

Then we estimate that by applying these estimates (4.23)-(4.25) and the inequalities (4.14)-(4.15),

$$\begin{aligned}
&\frac{n-1}{p^2}\int_0^t\int_M e^{-pu}\sqrt{-1}\partial\bar{\partial}(\chi_{su}^{n-2}\wedge\omega)ds \\
&= \frac{n-1}{p^2}\int_0^t\int_M e^{-pu}\sqrt{-1}\partial((n-2)\chi_{su}^{n-3}\wedge(\bar{\partial}\chi+s\sqrt{-1}\bar{\partial}\partial\bar{\partial}u)\wedge\omega)ds \\
&= \frac{n-1}{p^2}\int_0^t\int_M e^{-pu}\sqrt{-1}\Big\{(n-2)(n-3)\chi_{su}^{n-4}\wedge(\partial\chi+s\sqrt{-1}\partial\partial\bar{\partial}u)\wedge(\bar{\partial}\chi+s\sqrt{-1}\bar{\partial}\partial\bar{\partial}u)\wedge\omega \\
&\quad +(n-2)\chi_{su}^{n-3}\wedge(\partial\bar{\partial}\chi+s\sqrt{-1}\partial\bar{\partial}\partial\bar{\partial}u)\wedge\omega+(n-2)\chi_{su}^{n-3}\wedge(\bar{\partial}\chi+s\sqrt{-1}\bar{\partial}\partial\bar{\partial}u)\wedge\partial\omega
\end{aligned}$$

$$
\begin{aligned}
&+(n-2)\chi_{su}^{n-3}\wedge(\partial\chi+s\sqrt{-1}\partial\partial\bar\partial u)\wedge\bar\partial\omega+\chi_{su}^{n-2}\wedge\partial\bar\partial\omega\Big\}ds\\
\leq\ & \frac{C}{p^2}\int_0^t\int_M e^{-pu}\chi_{su}^{n-4}\wedge\omega^4ds+\frac{C}{p^2}\int_0^t\int_M e^{-pu}\chi_{su}^{n-4}\wedge\sqrt{-1}\partial u\wedge\bar\partial u\wedge\omega^3ds\\
&+\frac{C}{p^2}\int_0^t\int_M e^{-pu}\chi_{su}^{n-3}\wedge\omega^3ds+\frac{C}{p^2}\int_0^t\int_M e^{-pu}\chi_{su}^{n-3}\wedge\sqrt{-1}\partial u\wedge\bar\partial u\wedge\omega^2ds\\
&+\frac{C}{p^2}\int_0^t\int_M e^{-pu}\chi_{su}^{n-2}\wedge\omega^2ds\\
\leq\ & \frac{C_1}{p^2}\int_0^t\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\chi_{su}^{n-2}\wedge\omega ds+\frac{C_2}{p^2}\int_0^t\int_M e^{-pu}\chi_{su}^{n-2}\wedge\omega^2ds.
\end{aligned}\tag{4.26}
$$

By choosing p sufficiently large such that $\frac{C_1}{p^2}<n-1$, $\frac{C_2}{p}<\lambda\cdot\frac{n-1}{n}$, by combining (4.22) with (4.26), and applying (4.15) for $t=\frac{1}{2}$, $k=n-1$ such that

$$
\int_0^{\frac{1}{2}}\int_M e^{-pu}\chi_{tu}^{n-2}\wedge\omega^2dt\leq\frac{1}{\lambda}\cdot\frac{n}{n-1}\int_0^{\frac{1}{2}}\int_M e^{-pu}\chi_{tu}^{n-1}\wedge\omega dt,
$$

we obtain that for $0\leq t\leq\frac{1}{2}$,

$$
\begin{aligned}
&\frac{1}{p}\int_0^{\frac{1}{2}}\int_M e^{-pu}\chi_{tu}^{n-1}\wedge\omega dt\\
\leq\ & (n-1)\int_0^{\frac{1}{2}}\int_0^t\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\chi_{su}^{n-2}\wedge\omega dsdt+\frac{1}{2p}\int_M e^{-pu}\chi^{n-1}\wedge\omega\\
&+\frac{C_1}{p^2}\int_0^{\frac{1}{2}}\int_0^t\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\chi_{su}^{n-2}\wedge\omega dsdt+\frac{C_2}{p^2}\int_0^{\frac{1}{2}}\int_0^t\int_M e^{-pu}\chi_{su}^{n-2}\wedge\omega^2dsdt\\
\leq\ & (n-1)\int_0^{\frac{1}{2}}\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\chi_{tu}^{n-2}\wedge\omega dt\\
&+\frac{1}{2p}\cdot\frac{\lambda(n-1)}{n}\int_0^{\frac{1}{2}}\int_M e^{-pu}\chi_{tu}^{n-2}\wedge\omega^2dt+\frac{1}{2p}\int_M e^{-pu}\chi^{n-1}\wedge\omega\\
\leq\ & (n-1)\int_0^{\frac{1}{2}}\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\chi_{tu}^{n-2}\wedge\omega dt\\
&+\frac{1}{2p}\int_0^{\frac{1}{2}}\int_M e^{-pu}\chi_{tu}^{n-1}\wedge\omega dt+\frac{1}{2p}\int_M e^{-pu}\chi^{n-1}\wedge\omega
\end{aligned}\tag{4.27}
$$

which implies that we have by applying (4.14) for $t=\frac{1}{2}$, $l=n$,

$$
\begin{aligned}
&\frac{1}{2p}\int_0^{\frac{1}{2}}\int_M e^{-pu}\chi_{tu}^{n-1}\wedge\omega dt\\
\leq\ & (n-1)\int_0^{\frac{1}{2}}\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\chi_{tu}^{n-2}\wedge\omega dt+\frac{1}{2p}\int_M e^{-pu}\chi^{n-1}\wedge\omega\\
\leq\ & \frac{n}{\lambda}\int_0^{\frac{1}{2}}\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\chi_{tu}^{n-1}dt+\frac{1}{2p}\int_M e^{-pu}\chi^{n-1}\wedge\omega.
\end{aligned}\tag{4.28}
$$

Therefore, by combining (4.28) with (4.20), (4.21), we obtain that

$$
\begin{aligned}
&\Big[np\Big\{1-\frac{C}{np^2}-\frac{1}{(1+\frac{\delta}{2})^\alpha}\Big\}-C\frac{2n}{\lambda}\Big]\int_0^{\frac{1}{2}}\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar\partial u\wedge\chi_{tu}^{n-1}dt\\
\leq\ & C\int_M e^{-pu}\omega^n+\frac{C}{p}\int_M e^{-pu}\chi^{n-1}\wedge\omega\leq C\int_M e^{-pu}\omega^n.
\end{aligned}\tag{4.29}
$$

We choose p sufficiently large such that

$$\Big[n\Big\{1-\frac{C}{np^2}-\frac{1}{(1+\frac{\delta}{2})^{\alpha}}\Big\}-C\frac{2n}{\lambda p}\Big]>0.$$

By applying (4.14) for $t=\frac{1}{2}$ repeatedly, we obtain

$$\begin{aligned}\int_0^{\frac{1}{2}}\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar{\partial}u\wedge\chi_{tu}^{n-1}dt &\geq \lambda\frac{n-1}{n}\int_0^{\frac{1}{2}}\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar{\partial}u\wedge\chi_{tu}^{n-2}\wedge\omega dt\\ &\geq \lambda^2\frac{n-1}{n}\frac{n-2}{n-1}\int_0^{\frac{1}{2}}\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar{\partial}u\wedge\chi_{tu}^{n-3}\wedge\omega^2 dt\\ &\cdots\\ &\geq \frac{\lambda^{n-1}}{n}\frac{1}{2}\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar{\partial}u\wedge\omega^{n-1}. \end{aligned}\tag{4.30}$$

By combining (4.29) with (4.30), we finally obtain that for sufficiently large p,

$$p\int_M e^{-pu}\sqrt{-1}\partial u\wedge\bar{\partial}u\wedge\omega^{n-1}\leq C\int_M e^{-pu}\omega^n,$$

which tells us that there exists a sufficiently large p_0 such that for all $p\geq p_0$, the desired inequality (4.3) holds. □

The rest of the proof is similar to the ones in [7, 8]. In the following, we give a brief proof for reader's convenience. We introduce the definition of Gauduchon metrics on almost complex manifolds.

Definition 4.4. *Let (M^{2n},J) be an almost complex manifold. A metric g is called a Gauduchon metric on M if g is an almost Hermitian metric whose associated real $(1,1)$-form $\omega=\sqrt{-1}g_{i\bar{j}}\zeta^i\wedge\zeta^{\bar{j}}$ satisfies $d^*(Jd^*\omega)=0$, where d^* is the adjoint of d with respect to g, which is equivalent to $d(Jd(\omega^{n-1}))=0$, or $\partial\bar{\partial}(\omega^{n-1})=0$.*

One has the following well-known result.

Proposition 4.5 (cf. [2, Theorem 2.1], [3])**.** *Let (M^{2n},J,ω) be a compact almost Hermitian manifold with $n\geq 2$. Then there exists a smooth function σ, unique up to addition of a constant, such that the conformal almost Hermitian metric $e^{\sigma}\omega$ is Gauduchon.*

Thanks to Proposition 4.5, there exists a smooth function $\sigma: M\to\mathbb{R}$ with $\sup_M\sigma=0$ such that $\omega_G:=e^{\sigma}\omega$ is Gauduchon on M.

Lemma 4.6 (cf. [8, Lemma 2.3])**.** *Let M be a compact almost complex manifold of real dimension $2n$ $(n\geq 2)$ with a Gauduchon metric ω_G. If ϕ is a smooth nonnegative function on M with $\Delta_G\phi\geq -C_0$, where Δ_G is the Laplacian operator with respect to ω_G, then there exists a positive constant C_1, C_2 depending only on (M,ω_G) and C_0 such that*

$$\int_M|\partial\phi^{\frac{p+1}{2}}|^2_{\omega_G}\omega_G^n\leq C_1p\int_M\phi^p\omega_G^n\tag{4.31}$$

for any $p\geq 1$, and

$$\sup_M\phi\leq C_2\max\Big\{\int_M\phi\omega_G^n,1\Big\}.\tag{4.32}$$

Proof. We compute for $p \geq 1$, by Stokes' theorem,

$$\begin{aligned}
\int_M |\partial \phi^{\frac{p+1}{2}}|^2_{\omega_G}\omega_G^n &= n\int_M \sqrt{-1}\partial\phi^{\frac{p+1}{2}} \wedge \bar\partial\phi^{\frac{p+1}{2}} \wedge \omega_G^{n-1} \\
&= \frac{n(p+1)^2}{4}\int_M \sqrt{-1}\phi^{p-1}\partial\phi \wedge \bar\partial\phi \wedge \omega_G^{n-1} \\
&= \frac{n(p+1)^2}{4p}\int_M \sqrt{-1}\partial(\phi^p) \wedge \bar\partial\phi \wedge \omega_G^{n-1} \\
&= \frac{n(p+1)^2}{4p}\int_M \sqrt{-1}(\partial+\bar\partial+A+\bar A)(\phi^p\bar\partial\phi\wedge\omega_G^{n-1}) \\
&\quad -\frac{n(p+1)^2}{4p}\int_M \phi^p\sqrt{-1}\partial\bar\partial\phi\wedge\omega_G^{n-1} + \frac{n(p+1)}{4p}\int_M \sqrt{-1}\bar\partial(\phi^{p+1})\wedge\partial\omega_G^{n-1} \\
&= -\frac{(p+1)^2}{4p}\int_M \phi^p n\frac{\sqrt{-1}\partial\bar\partial\phi\wedge\omega_G^{n-1}}{\omega_G^n}\omega_G^n \\
&\quad +\frac{n(p+1)}{4p}\int_M \sqrt{-1}(\partial+\bar\partial+A+\bar A)(\phi^{p+1}\partial\omega_G^{n-1}) \\
&\quad -\frac{n(p+1)}{4p}\int_M \phi^{p+1}\sqrt{-1}\bar\partial\partial\omega_G^{n-1} \\
&= \frac{(p+1)^2}{4p}\int_M \phi^p(-\Delta_G\phi)\omega_G^n \\
&\leq C_1 p\int_M \phi^p\omega_G^n, \qquad (4.33)
\end{aligned}$$

where we used that $(\bar\partial + A + \bar A)(\phi^p\bar\partial\phi\wedge\omega_G^{n-1}) = 0$, $(\partial + A + \bar A)(\phi^{p+1}\partial\omega_G^{n-1}) = 0$, and that

$$\bar\partial\partial\omega_G^{n-1} = -(\partial\bar\partial + A\bar A + \bar A A)\omega_G^{n-1} = -\partial\bar\partial\omega_G^{n-1} = 0$$

since we have $A\omega_G^{n-1} = \bar A\omega_G^{n-1} = 0$.

We apply the Sobolev inequality: for $\beta := \frac{n}{n-1} > 1$, and for any smooth function f,

$$\Big(\int_M f^{2\beta}\omega^n\Big)^{\frac{1}{\beta}} \leq C\Big(\int_M |\partial f|^2_g\omega^n + \int_M f^2\omega^n\Big). \qquad (4.34)$$

Taking $\omega = \omega_G$ and $f = \phi^{\frac{q}{2}}$, where we put $q := p+1$, then for $q \geq 2$, we have that

$$\Big(\int_M \phi^{q\beta}\omega_G^n\Big)^{\frac{1}{\beta}} \leq Cq\max\Big\{\int_M \phi^q\omega_G^n, 1\Big\}.$$

By repeatedly replacing q by $q\beta$ and iterating, after setting $q = 2$, then we obtain that

$$\sup_M \phi \leq C\max\Big\{\Big(\int_M \phi^2\omega_G^n\Big)^{\frac{1}{2}}, 1\Big\} \leq C\max\Big\{\Big(\sup_M \phi\Big)^{\frac{1}{2}}\Big(\int_M \phi\omega_G^n\Big)^{\frac{1}{2}}, 1\Big\},$$

which gives us the desired estimate (4.32). ☐

By applying the inequality (4.3) and the Sobolev inequality (4.34), for any $p \geq p_0$, we obtain that

$$\|e^{-u}\|_{L^{p\beta}} \leq C^{\frac{1}{p}}p^{\frac{1}{p}}\|e^{-u}\|_{L^p},$$

and by the standard iteration, we have that

$$e^{-p_0 \inf_M u} \leq C \int_M e^{-p_0 u} \omega^n. \tag{4.35}$$

We need the following lemma, whose proof goes in the same way as in the Hermitian case.

Lemma 4.7 (cf. [7, Lemma 3.2], [8, Lemma 2.2])**.** *Let f be a smooth function on a compact almost Hermitian manifold (M, J, ω). Write $d\mu := \frac{\omega^n}{\int_M \omega^n}$. If there exists a constant C_1 such that*

$$e^{-\inf_M f} \leq e^{C_1} \int_M e^{-f} d\mu, \tag{4.36}$$

then

$$|\{f \leq \inf_M f + C_1 + 1\}| \geq \frac{e^{-C_1}}{4}, \tag{4.37}$$

where $|\cdot|$ denotes the volume of the set with respect to $d\mu$.

We apply Lemma 4.6 to $f = p_0 u$, and then since we have the inequality (4.35), there exist uniform constants C, $\delta > 0$ such that

$$|\{u \leq \inf_M u + C\}| \geq \delta. \tag{4.38}$$

Now, we define $\phi := u - \inf_M u$. Since it satisfies that $\Delta_G \phi = e^{-\sigma} \Delta \phi > -C$, where Δ is the Laplacian operator with respect to ω, we may apply Lemma 4.3 to the function ϕ. From the Poincaré inequality and the estimate (4.31) with $p = 1$, we obtain that

$$\|\phi - \underline{\phi}\|_{L^2} \leq C\Big(\int_M |\partial\phi|^2_{\omega_G} \omega_G^n\Big)^{\frac{1}{2}} \leq C\|\phi\|^{\frac{1}{2}}_{L^1}, \tag{4.39}$$

where we put $\underline{\phi} := \frac{1}{\int_M \omega_G^n} \int_M \phi \omega_G^n$.

By making use of (4.38), the set $S := \{\phi \leq C\}$ satisfies that $|S|_G \geq \delta$, where $|\cdot|_G$ denotes the volume of a set with respect to ω_G^n. Therefore, we obtain that

$$\delta \underline{\phi} \leq \int_S \underline{\phi} \omega_G^n \leq \int_S (|\phi - \underline{\phi}| + C) \omega_G^n \leq \int_M |\phi - \underline{\phi}| \omega_G^n + C,$$

which gives that by applying (4.39),

$$\|\phi\|_{L^1} \leq C(\|\phi - \underline{\phi}\|_{L^1} + 1) \leq C(\|\phi - \underline{\phi}\|_{L^2} + 1) \leq C(\|\phi\|^{\frac{1}{2}}_{L^1} + 1).$$

Hence, ϕ is uniformly bounded in L^1, and from (4.32) and (1.2), we obtain a uniform bound of u in the L^∞ norm. □

Acknowledgments

This work was supported by JSPS KAKENHI Grant Number JP21K13798.

References

[1] L. Chen, “Hessian equations of Krylov type on Kähler manifolds”, preprint, arXiv:2107.12035v3, 2021.

[2] J. Chu, V. Tosatti and B. Weinkove, “The Monge-Ampère equation for non-integrable almost complex structures”, J. Eur. Math. Soc., vol. 21, no. 7, pp. 1949–1984, 2019.

[3] P. Gauduchon, “Le théorème de l’excentricité nulle”, C. R. Acad. Sci. Paris Sér. A-B, vol. 285, no. 5, pp. A387–A390, 1977.

[4] A. Newlander and L. Nirenberg, “Complex analytic coordinates in almost complex manifolds”, Ann. of Math. (2), vol. 65, pp. 391–404, 1957.

[5] W. Sun, “On a class of fully nonlinear elliptic equations on closed Hermitian manifolds”, J. Geom. Anal., vol. 26, no. 3, pp. 2459–2473, 2016.

[6] W. Sun, “On a class of fully nonlinear elliptic equations on closed Hermitian manifolds II: L^∞ estimate”, Comm. Pure Appl. Math., vol. 70, no. 1, pp. 172–199, 2017.

[7] V. Tosatti and B. Weinkove, “Estimates for the complex Monge-Ampère‘ equation on Hermitian and balanced manifolds”, Asian J. Math., vol. 14, pp. 19–40, 2010.

[8] V. Tosatti and B. Weinkove, “The complex Monge-Ampère equation on compact Hermitian manifolds”, J. Amer. Math. Soc., vol. 23, pp. 1187–1195, 2010.

[9] Q. Tu and N. Xiang, “The Dirichlet problem for mixed Hessian equations on Hermitian manifolds”, preprint, arXiv:2201.05030v1, 2022.

[10] L. Vezzoni, “On Hermitian curvature flow on almost complex manifolds”, Differential Geom. Appl., vol. 29, no. 5, pp. 709–722, 2011.

[11] C.-J. Yu, “Nonpositively curved almost Hermitian metrics on products of compact almost complex manifolds”, Acta Math. Sin., vol. 31, no. 1, pp. 61–70, 2015.

[12] J. Zhang, “Monge-Ampère type equations on almost Hermitian manifolds”, preprint, arXiv:2101.00380, 2022.

[13] T. Zheng, “An almost complex Chern-Ricci flow”, J. Geom. Anal., vol. 28, no. 3, pp. 2129–2165, 2018.