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On the minimum ergodic average and minimal systems

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ABSTRACT

We prove some equivalences associated with the case when the average lower time is minimal. In addition, we characterize the minimal systems by means of the positivity of invariant measures on open sets and also the minimum ergodic averages. Finally, we show that a minimal system admits an open set whose measure is minimal with respect to a set of ergodic measures and its value can be chosen in $[0,1]$.

RESUMEN

Demostramos algunas equivalencias asociadas con el caso cuando el tiempo inferior promedio es mínimo. Además, caracterizamos los sistemas minimales a través de la positividad de medidas invariantes en conjuntos abiertos y también los promedios ergódicos mínimos. Finalmente, mostramos que un sistema minimal admite un conjunto abierto cuya medida es mínima con respecto a un conjunto de medidas ergódicas y su valor puede ser elegido en $[0,1]$.

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1 Introduction

The main motivation of this paper is the result obtained by Jenkinson in [3] which states that given an invariant measure there exists a continuous function that achieves the maximum ergodic average using such measure. This result has been used in several recent works [1, 6, 7]. A minimizing version of this result is possible to obtain in a straightforward way. In this sense, given the behavior of uniquely ergodic systems, it is natural to ask whether this version admits any relation to minimal systems and time averages. The present paper addresses both problems in the following way. Firstly, we prove some equivalences associated with the case when the average lower time is minimal. We also characterize the minimal systems by means of the positivity of invariant measures on open sets and also the minimum ergodic averages (this result was inspired by Theorem 6.17 in [9]). Finally, we show that given a finite set of ergodic measures in a minimal system it is possible to find an open set whose measure is minimal and its value can be chosen in $[0,1]$. Let us state our results in a precise way.

Throughout this paper, the pair (X,d) denotes a compact metric space and $C(X)$ denotes the space of all continuous real-valued functions on X. We denote by $\mathcal{M}(X)$ the set of all Borel probability measures of X, provided with the weak* topology. Let $T: X \to X$ be a continuous transformation. Given μ an element of $\mathcal{M}(X)$, we say that μ is T-invariant if $\mu(T^{-1}(A)) = \mu(A)$ for every Borel subset A of X. We denote by $M_T(X)$ the set of T-invariant probability measures. A probability measure μ is called ergodic if $\mu(A) \in \{0,1\}$ for each T-invariant set A. Denote by $\mathcal{E}_T(X)$ the set of ergodic measures. For $x \in X$, let δ_x be denote the Dirac point measure of x defined by $\delta_x(A) = 1$ when $x \in A$ and $\delta_x(A) = 0$ otherwise.

Let $f : X \to \mathbb{R}$ be a continuous function, we say that an invariant measure μ is f-minimizing if the minimum ergodic average [4] defined by

$$\alpha(f) = \min\left\{\int_X f dm : m \in M_T(X)\right\},$$

satisfies $\alpha(f) = \displaystyle\int_X f d\mu$. Given $x \in X$, recall that the lower time average is

$$\underline{\tau}(x,f) = \liminf_{N\to\infty} \frac{1}{N}\sum_{i=0}^{N-1} f \circ T^i(x).$$

We consider the number $E(x,f) = \underline{\tau}(x,f) - \alpha(f)$. This number quantifies the non-minimal time average. Note that $E(x,f) \geq 0$.

Next we state our first result that characterizes the cases where the non-minimal time average is equal to zero totally and partially uniform. For this purpose, we must recall that (X,T) is said to be uniquely ergodic if there is a unique invariant probability measure on X.

Theorem 1.1. *Let $T : X \to X$ be a continuous transformation of a compact metric space. For every $x \in X$ and $f \in C(X)$, we have the following equivalences*

(1) $E \equiv 0$ *if and only if* (X, T) *is uniquely ergodic.*

(2) $E(\cdot, f) = 0$ *if and only if* $\alpha(f) = \int_X f d\mu$, *for all* $\mu \in \mathcal{M}_T(X)$.

(3) $E(x, \cdot) = 0$ *if and only if every ergodic measure is a limit point of the sequence* $\left\{ \frac{1}{n} \sum_{i=0}^{n-1} \delta_{T^i(x)} \right\}$.

Our next result shows a characterization of the minimal systems through the open sets and minimum ergodic averages. Recall that a dynamical system (X, T) is called minimal if X does not contain any non-empty, proper, closed T-invariant subset.

Theorem 1.2. *Let $T : X \to X$ be a continuous transformation of a compact metric space. The following statements are equivalents:*

(1) (X, T) *is a minimal system.*

(2) *For each non empty open set* $A \subset X$ *and each* $\mu \in \mathcal{M}_T(X)$, *we have* $\mu(A) > 0$.

(3) *Every non-zero* $f \in C(X)$ *with* $f \geq 0$ *satisfies* $\alpha(f) > 0$.

Finally, in the case of non-discrete minimal systems, it is satisfied that the minimum ergodic average reaches all values of $[0, 1]$ for continuous functions of norm one. (see Lemma 2.8). Motivated by this, we found a condition on the ergodic measures [2] to obtain a version of this result through open sets.

Theorem 1.3. *Let $T : X \to X$ be a continuous transformation of a non-discrete compact metric space. If (X, T) is minimal and $\mathcal{F}$ is a finite subset of $\mathcal{E}_T(X)$, then for every $r \in [0, 1]$ there is an open set A such that r is the minimun value of $\mu(A)$ whenever $\mu \in \mathcal{F}$.*

The paper is organized as follows. In Section 2, we will prove several results necessary for the proof of the main theorems. Finally, in Section 3, we will prove Theorems 1.1, 1.2 and 1.3.

2 Preliminary lemmas

Let X be a compact metric space and $T : X \to X$ be a continuous transformation. We denote the applications

$$\begin{aligned} \alpha : C(X) &\longrightarrow \mathbb{R} \\ f &\longmapsto \min_{\mu \in \mathcal{M}_T(X)} \int_X f d\mu, \end{aligned}$$

and

$$\begin{aligned} E : X \times C(X) &\longrightarrow [0, +\infty) \\ (x, f) &\longmapsto \underline{\tau}(x, f) - \alpha(f). \end{aligned}$$

Below are some properties of these applications that are straightforward from the definition. Let Z be a convex set of a vector space V. A subset F of Z is called face of Z if whenever $x, y \in Z$ and $\lambda x + (1-\lambda)y \in F$ with $0 < \lambda < 1$, then $\{x, y\} \subset F$.

Proposition 2.1. *We have the following properties*

(1) *α is continuous and T-invariant.*

(2) $\alpha(1) = 1$.

(3) $E(x, f) = E(x, f \circ T)$.

(4) *$\alpha(f) \leq \alpha(g)$ whenever $f \leq g$.*

(5) *The set $\left\{\mu \in M_T(X) : \alpha(f) = \int_X f d\mu\right\}$ is a non-empty closed face of $M_T(X)$.*

We will prove some additional properties of E

Lemma 2.2. *Let $T : X \to X$ be a continuous transformation of a compact metric space. It holds that $E(x, f) = 0$ for every $x \in X$ and $f \in C(X)$ if and only if the system (X, T) is uniquely ergodic.*

Proof. It is sufficient to prove that if $E \equiv 0$ then the system is uniquely ergodic. By Theorem 1 in [3] for every ergodic measure ν there exists an $f \in C(X)$ such that ν is the unique f-minimizing measure, that is, ν is the unique satisfying

$$\int_X f d\nu = \alpha(f).$$

Since $E(x, f) = 0$ for each $x \in X$, we obtain

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f \circ T^i(x) = \int_X f d\nu. \tag{2.1}$$

If the system is not uniquely ergodic, there exists $\omega \in \mathcal{E}_T(X)$ such that $\omega \neq \nu$. Let p be a generic point for ω. Using (2.1), we have

$$\int_X f d\omega = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f \circ T^i(p) = \int_X f d\nu < \int_X f d\omega.$$

It is a contradiction. So (X, T) is uniquely ergodic. □

Lemma 2.3. *Let $T : X \to X$ be a continuous transformation of a compact metric space. Given $f \in C(X)$. Then, $E(x, f) = 0$ for every $x \in X$ if and only if $\alpha(f) = \int_X f d\mu$, for all $\mu \in \mathcal{M}_T(X)$.*

Proof. By Proposition 2.1, we know that the set

$$H = \left\{ \nu \in \mathcal{M}_T(X) : \alpha(f) = \int_X f d\nu \right\},$$

is a non-empty closed face of $M_T(X)$. If $H \neq \mathcal{M}_T(X)$, then there is $\mu \in \mathcal{E}_T(X) \setminus H$. Let p be a generic point for μ, so

$$\int_X f d\mu = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f \circ T^i(p) = \alpha(f), \tag{2.2}$$

the last equality in (2.2) is a consequence of the hypothesis $E(p, f) = 0$. Thus $\mu \in H$, which is absurd.

Conversely, given $x \in X$ we can find a sequence $\{N_k\}$ such that the inferior mean sojourn time is written as

$$\underline{\tau}(x, f) = \lim_{k\to\infty} \frac{1}{N_k} \sum_{i=0}^{N_k-1} f \circ T^i(x), \tag{2.3}$$

and also $\left\{ N_k^{-1} \sum_{i=0}^{N_k-1} \delta_{T^i(x)} \right\}$ is convergent to $\nu \in \mathcal{M}_T(X)$. Therefore $E(x, f) = 0$ since

$$\underline{\tau}(x, f) = \int_X f d\nu = \alpha(f). \qquad \square$$

A consequence of the above result is the following

Corollary 2.4. *The set $\{f \in C(X) : E(x, f) = 0$ for every $x \in X\}$ is a closed linear subspace of $C(X)$.*

Lemma 2.5. *Let $T : X \to X$ be a continuous transformation of a compact metric space. Given $x \in X$. Then, $E(x, f) = 0$ for each $f \in C(X)$ if and only if every ergodic measure is a limit point of the sequence $\left\{ \frac{1}{n} \sum_{i=0}^{n-1} \delta_{T^i(x)} \right\}$.*

Proof. Denote by Λ the set of the limit points of the sequence $\left\{ \frac{1}{n} \sum_{i=0}^{n-1} \delta_{T^i(x)} \right\}$. Suppose there is $\mu \in \mathcal{E}_T(X) \setminus \Lambda$. By Theorem 1 in [3], for every ergodic measure μ there exists an $f \in C(X)$ such that μ is the unique with the property

$$\int_X f d\mu = \alpha(f).$$

Since $E(x, f) = 0$, we have

$$\int_X f d\mu = \alpha(f) = \underline{\tau}(x, f).$$

Moreover, using (2.3), there is a sequence $\{m_k\}$ in $\mathcal{M}(X)$ such that

$$\underline{\tau}(x, f) = \lim_{k\to\infty} \int_X f dm_k.$$

We can assume that $\{m_k\}$ converges to $\nu \in \mathcal{M}_T(X)$. Then $\int_X f d\mu = \int_X f d\nu$ with $\nu \neq \mu$. It is a contradiction.

Conversely, given $f \in C(X)$ there exists an ergodic measure μ such that $\alpha(f) = \int_X f d\mu$. On the other hand, there is a sequence $\{N_k\}$ satisfying

$$\mu = \lim_{k\to\infty} N_k^{-1} \sum_{i=0}^{N_k-1} \delta_{T^i(x)}.$$

Therefore

$$\alpha(f) \leq \underline{\tau}(x, f) \leq \lim_{k\to\infty} \frac{1}{N_k} \sum_{i=0}^{N_k-1} f \circ T^i(x) = \int_X f d\mu = \alpha(f),$$

so $E(x, f) = 0$. □

Now, we introduce the following auxiliary application

$$\begin{aligned} \varrho : \mathcal{T} \times \mathcal{C} &\longrightarrow [0, 1] \\ (A, \mathcal{F}) &\longmapsto \varrho(A, \mathcal{F}) = \min_{\mu\in\mathcal{F}} \mu(A), \end{aligned}$$

where $\mathcal{T}$ denotes the topology associated with X and $\mathcal{C}$ denotes the space of all closed subsets of $\mathcal{M}_T(X)$. We write $\varrho(A) = \varrho(A, \mathcal{M}_T(X))$. Note that $\varrho(A)$ can be interpreted as the capacity of an open set (see Lemma 4.1 in [5]).

Lemma 2.6. *Let $T : X \to X$ be a continuous transformation of a compact metric space. It holds that $\varrho(A) > 0$ for every non-empty open set A if and only if $\alpha(f) > 0$ for each non-zero $f \in C(X)$ with $f \geq 0$.*

Proof. Given a non-zero $f \in C(X)$ with $f \geq 0$. We can find a non-empty open A and a constant $c > 0$ that verify $f(x) \geq c$ for all $x \in A$. It follows that

$$\alpha(f) \geq \min_{\mu\in\mathcal{M}_T(X)} \int_A f d\mu \geq c\varrho(A) > 0.$$

Hence $\alpha(f) > 0$.

Conversely, let A be a non-empty open set in X. By Urysohn's Lemma choose $f \in C(X)$ with $0 \leq f \leq 1$, $f(p) = 1$ and $f = 0$ on A^c for some $p \in A$. If $\varrho(A) = 0$, there exists some $\nu \in \mathcal{M}_T(X)$ such that $\nu(A) = 0$, therefore

$$0 < \alpha(f) \leq \int_A f d\nu = 0.$$

It is a contradiction. □

Lemma 2.7. *Let $T : X \to X$ be a continuous transformation of a compact metric space. If (X, T) is a minimal system, then $\varrho(A) > 0$ for every non-empty open set A.*

Proof. Since (X, T) is minimal, for every non-empty open set A we have

$$X = \bigcup_{i=0}^{n} T^{-i}(A),$$

for some $n \in \mathbb{N}$, therefore $\varrho(A) \geq \frac{1}{n+1}$, that is, $\varrho(A) > 0$. □

Lemma 2.8. *Let $T : X \to X$ be a continuous transformation of a non-discrete compact metric space. If (X, T) is minimal, then given $r \in [0, 1]$, there is an $f \in C(X)$ with $\|f\|_\infty = 1$ such that $\alpha(f) = r$.*

Proof. Note that the set $B = \{f \in C(X) : \|f\|_\infty = 1\}$ is connected in $(C(X), \|.\|_\infty)$. Given $r \in (0, 1)$, by Lemma 2.7 and since (X, T) is non-discrete, we obtain a non-empty open set A with the following property $0 < \varrho(A) < r/2$. By Urysohn's Lemma choose $g \in C(X)$ with $0 \leq g \leq 1$, $g(p) = 1$ and $g = 0$ on A^c for some $p \in A$. Therefore

$$\alpha(g) = \min_{\mu \in \mathcal{M}_T(X)} \int_A g d\mu \leq \varrho(A) < r/2.$$

By Proposition 2.1, α is continuous on B and $\alpha(1) = 1$. So, there exists $f \in C(X)$ with $\|f\|_\infty = 1$ such that $\alpha(f) = r$. Now for the remaining cases it is sufficient to consider the constant functions $f \equiv 1$ and $g \equiv -1$. Then $\alpha(f) = 1$ and $\alpha(g) = -1$, it follows that there is $h \in B$ such that $\alpha(h) = 0$. □

If we denote

$$\widetilde{E}(x, A) = \underline{\tau}(x, \chi_A) - \varrho_A,$$

this value represents the non-minimal mean sojourn time on A. Also we can obtain that $\widetilde{E}(x, A) \in [0, 1]$.

Recall that a point $x \in X$ is periodic for $T : X \to X$ if $T^n(x) = x$ for some $n \in \mathbb{N}$ and the minimal such n is called the period of T. A point $x \in X$ is called pre-periodic if some iterate of x is periodic. We denote by $\mathcal{O}(x)$ the orbit of x.

Lemma 2.9. *Let $T : X \to X$ be a continuous transformation of a compact metric space. It holds that $\widetilde{E}(x, A) \equiv 0$ for every $x \in X$ and $A \in \mathcal{T}$ if and only if every point in X is pre-periodic and there is only one periodic orbit.*

Proof. It is enough to prove the sufficiency. First, we claim that $\widetilde{E} \equiv 0$ implies that each measure in $M_T(X)$ is atomic. Suppose there is a non-atomic μ invariant measure. Given $z \in X$, we can find open sets $\{V_n^z\}_{n\in\mathbb{N}}$ such that $T^n(z) \in V_n^z$ and $\mu(V_n^z) < 1/2^{n+1}$. Therefore, the open set $A_z = \bigcup_n V_n^z$ contains the orbit of z, so $\underline{\tau}(z, A_z) = 1$. Thus $\widetilde{E}(z, A_z) > 1/2$ since $\varrho(A_z) \leq \mu(A_z) < 1/2$. This proves our claim. Let ν be an ergodic measure. There is $p \in X$ with $\nu(p) > 0$. By the Poincaré's Recurrence (Theorem 1.2.4 in [8]), the point p is periodic. Given $x \in X$, if $X = \mathcal{O}(p)$, then there is nothing to prove. Otherwise, the open set $B = X \setminus \mathcal{O}(p)$ satisfies $\varrho_B = 0$, so $\underline{\tau}(x, B) = 0$. This implies that $\mathcal{O}(x) \not\subset B$. Hence, there exists a periodic point p such that for each $x \in X$ there exists $k \in \mathbb{N}$ satisfying $T^k(x) \in \mathcal{O}(p)$. □

3 Proof of the theorems

Proof of Theorem 1.1. The proof of this result is actually contained in the Lemmas 2.2, 2.3 and 2.5. □

Proof of Theorem 1.2. To prove that Item (1) implies Item (2), we use Lemma 2.7. To prove that Item (2) implies Item (1), assume that (X, T) is not a minimal system. There exists some point $x \in X$ whose orbit is not dense in X. We consider the non-empty open set $A = X \setminus \overline{\mathcal{O}(x)}$, so $\varrho(A) > 0$. On the other hand, there are a sequence $\{N_k\}$ and a measure $\mu \in \mathcal{M}_T(X)$ satisfying

$$\mu = \lim_{k \to \infty} N_k^{-1} \sum_{i=0}^{N_k - 1} \delta_{T^i(x)},$$

therefore

$$\varrho(A) \leq \mu(A) \leq \liminf_{k \to \infty} \frac{1}{n_k} |\{0 \leq i \leq N_k - 1 : T^i(x) \in A\}| = 0.$$

It is a contradiction. Finally, the Lemma 2.6 proves the equivalence between Item (2) and Item (3). □

Proof of Theorem 1.3. Suppose that there exists $r \in (0, 1)$ such that $\varrho(A, \mathcal{F}) \neq r$ for every open set A . We consider the set

$$\mathcal{Z} = \{B : B \text{ is open in } X \text{ and } 0 < \varrho(B, \mathcal{F}) < r\}.$$

By Lemma 2.7, we obtain that $\mathcal{Z}$ is non-empty since (X, T) is non-discrete. We partially order $\mathcal{Z}$ by inclusion. Assume $\{B_i\}_{i \in I} \subset \mathcal{Z}$ is a totally ordered subset of $\mathcal{Z}$ where I is infinite. An upper bound for the B_i's in $\mathcal{Z}$ is the open set $\mathfrak{B} = \bigcup_{i \in I} B_i$. Since I is infinite, we can suppose that $\mathbb{N} \subset I$. We choose an increasing sequence $\{A_j\}$ such that $\mathfrak{B} = \bigcup_{j \in \mathbb{N}} A_j$. If $\mathcal{F} = \{\mu_\ell\}_{\ell=1}^N$, then there exists ℓ such that the set $K_\ell = \{j \in \mathbb{N} : \varrho(A_j, \mathcal{F}) = \mu_\ell(A_j)\}$ is infinite. Thus, given $\nu \in \mathcal{F}$ we have

$$\nu(\mathfrak{B}) = \lim_{j \in K_\ell} \nu(A_j) \geq \lim_{j \in K_\ell} \mu_\ell(A_j) = \mu_\ell(\mathfrak{B}),$$

then $\varrho(\mathfrak{B},\mathcal{F}) = \mu_\ell(\mathfrak{B})$. On the other hand, since $\mathfrak{B} = \bigcup_{j\in\mathbb{N}} A_j$ using the regularity of the measure we have that there are $j \in \mathbb{N}$ and a compact K such that $K \subset A_j$ and $\mu_\ell(K) \leq \mu_\ell(A_j) < r$. So, $\mu_\ell(\mathcal{B}) \leq r$ but for the hypothesis $\varrho(A,\mathcal{F}) \neq r$. Hence $\varrho(\mathfrak{B},\mathcal{F}) = \mu_\ell(\mathfrak{B}) < r$, therefore $\mathfrak{B} \in \mathcal{Z}$. Zorn's lemma now tells us that $\mathcal{Z}$ contains a maximal element $\mathfrak{A}$. Let $\mu \in \mathcal{F}$ such that $\mu(\mathfrak{A}) = \varrho(\mathfrak{A},\mathcal{F}) < r$. Given $x \in X$, there is an open set U_x with $\mu(U_x) < r - \mu(\mathfrak{A})$. Hence

$$\varrho(\mathfrak{A}\cup U_x,\mathcal{F}) \leq \mu(\mathfrak{A}\cup U_x) \leq \mu(\mathfrak{A}) + \mu(U_x) < r.$$

Using the maximality of $\mathfrak{A}$ it is concluded that $U_x \subset \mathfrak{A}$ for every $x \in X$, so $\mathfrak{A} = X$. It implies $\varrho(\mathfrak{A},\mathcal{F}) = 1 > r$, which is absurd. □

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