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Bayesian Pricing of the Optimal-Replication Strategy for European Option in the JD(M)J Model[†]

A b s t r a c t. In incomplete markets replication strategies may not exist and pricing of derivatives is not an easy task. This paper presents an application of Bertsimas, Kogan and Lo's algorithm of determining an optimal-replication strategy. In the Merton model the likelihood function is a product of a mixture of infinite number of components. In the paper this number is assumed to be equal to a fixed value M+1. To determine the optimal strategy, we should estimate unknown parameters. To this end we resort to Bayesian estimation techniques. The presented methodology is exemplified by an empirical research.

K e y w o r d s: incomplete markets, Bayesian inference, jump-diffusion process, pricing of derivatives.

J E L Classification: C11, C15, C58, C61, C63, G17

Introduction

The Black-Scholes model assumes a continuous path of underlying instrument values. Pricing European options under the Black-Scholes model is an easy task (Black, Scholes, 1973). Unfortunately, if we additionally include a component responsible for jumps, we obtain a model of a risky instrument in an incomplete market. Then, the pricing of options is more of a challenge. In general, replication strategies do not exist (Lamberton, Lapeyre, 2000; Shreve, 2004). Bertsimas, Kogan and Lo (2001) propose an algorithm of determining an optimal-replication strategy. The optimality is understood in a mean-squared sense. Apart from the major drawbacks of the strategy – it is self-

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financing but it is not an admissible strategy, and its calculation may be time-consuming – it plays a significant role in hedging and pricing derivatives.

To determine the optimal-replication strategy, the unknown parameters of the model need to be estimated. In this paper we resort to the Bayesian estimation techniques. Accounting for the parameter uncertainty inherent to the estimated parameters, for which the Bayesian methodology is widely appreciated, is relevant not only to the estimation itself, but extends also to the pricing of optimal strategy, providing the researcher with a full (posterior) distribution of the strategy cost (instead of a single value).

1. The Jump-Diffusion Model with M-jumps

Let $(\Omega, \Im, P)$ denote a probability space. We consider a standard Wiener process $W = (W_t)_{t\geq 0}$, a Poisson process $N = (N_t)_{t\geq 0}$ with intensity $\lambda > 0$, and a family of independent random variables $Q = \{Q_j : j = 1,2,...\}$. The variables Q_j's have Gaussian distributions: $Q_j \sim N(\mu_Q, \sigma_Q^2)$. It is also assumed that σ–algebras generated by *W*, *N* and *Q* are independent.

In the Merton model (Merton, 1976) the price of a risky instrument is governed by a jump-diffusion process $P = (P_t)_{t\geq 0}$ which is the solution of the equation:

$$dP_t = \mu P_t dt + \sigma P_t dW_t + (e^Q - 1)P_t dN_t.$$

The first two elements on the right-hand side define a pure diffusion process. The last element corresponds to jumps. $(P_t)_{t\geq 0}$ is an adapted and right-continuous process. It can be shown that:

$$P_t = P_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t + \sum_{j=1}^{N_t} Q_j\right\}.$$

The price between consecutive jumps is governed by a geometric Brownian motion. The process P has a finite number of jumps on each interval $[0,t]$. The logarithm of the price is the solution of the equation:

$$d \ln P_t = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dW_t + QdN_t.$$

Hence, for a given time step $\Delta > 0$ we obtain:

$$\ln(P_{t+\Delta}) = \ln(P_t) + \left(\mu - \frac{1}{2}\sigma^2\right)\Delta + \sigma(W_{t+\Delta} - W_t) + \sum_{j=N_t+1}^{N_{t+\Delta}} Q_j.$$

It follows that the probability density function of logarithmic rates of return is given by (Hanson, Westman, 2002):

$$p_{\ln\left(\frac{P_{t+\Delta}}{P_t}\right)}(x) = \sum_{k=0}^{\infty} \exp(-\lambda\Delta)\tfrac{(\lambda\Delta)^k}{k!}\phi\left(x;\left(\mu-\tfrac{1}{2}\sigma^2\right)\Delta+\mu_Q k, \sigma^2\Delta+\sigma_Q^2 k\right), \quad (1)$$

where $\phi\left(\cdot;m,s^2\right)$ denotes the density of a normal distribution with mean m and variance s^2. Therefore, the likelihood function is given by the product of an infinite mixture of normal densities, which highly complicates the statistical inference for the model.

In order to define a jump-diffusion model with M jumps, let us consider a finite approximation of series (1):

$$\sum_{k=0}^{\infty} \exp(-\lambda\Delta)\frac{(\lambda\Delta)^k}{k!}\phi\left(x;\left(\mu-\frac{1}{2}\sigma^2\right)\Delta+\mu_Q k, \sigma^2\Delta+\sigma_Q^2 k\right),$$

$$\approx \sum_{k=0}^{M} \exp(-\lambda\Delta)\frac{(\lambda\Delta)^k}{k!}\phi\left(x;\left(\mu-\frac{1}{2}\sigma^2\right)\Delta+\mu_Q k, \sigma^2\Delta+\sigma_Q^2 k\right), \quad (2)$$

where $M \in \{0,1,2,...\}$ is a fixed constant. In the Black-Scholes framework, the process of logarithmic returns of risky instrument is governed by an arithmetic Brownian motion which is a pure diffusion process. Under $M = 0$ the above approximation collapses to the density of this arithmetic Brownian motion with time step Δ (Kloeden, Platen, 1992). In the general case the integral of sum (2) may not equal one. Therefore, to obtain a probability density function, let us normalize the approximation given by (2):

$$p(x_i|\theta) = \sum_{k=0}^{M} w_k \phi\left(x_i;\left(\mu-\frac{1}{2}\sigma^2\right)\Delta+\mu_Q k, \sigma^2\Delta+\sigma_Q^2 k\right), \quad (3)$$

with $w_k = \frac{(\lambda\Delta)^k}{k!}\left[\sum_{j=0}^{M}\frac{(\lambda\Delta)^j}{j!}\right]^{-1}$, $k = 0,...,M$, being the normalizing weights[1]. In the paper the logarithmic rates of return $x = (x_1, x_2, \ldots)$ are assumed to follow the distribution defined by (3), and the resulting model is termed the jump-diffusion model with *M* jumps, or JD(*M*)J, in short. The construction of the process restricts the number of jumps over any time interval Δ to *M*, with the magnitude of each jump model with normal distribution $N\left(\mu_Q, \sigma_Q^2\right)$. Finally, let us note that the jump-diffusion specification under study is some approximation to the Merton model. Therefore, and on a more statistical note, estimators of JD(M)J model parameters could be treated as approximations of the Merton model parameters.

[1] See Frühwirth-Schnatter (2006) for a thorough exposition on mixture modeling.

2. Bayesian Inference for the JD(M)J Model

In the JD(M)J model there are five unknown parameters $(\mu, \sigma^2, \lambda, \mu_Q, \sigma_Q{}^2) \in \Theta$, where $\Theta = \mathrm{R} \times (0,\infty) \times (0,\infty) \times \mathrm{R} \times (0,\infty) \subset \mathrm{R}^5$. The likelihood function is given by:

$$p(x|\theta) = \prod_{i=1}^{n} \sum_{k=0}^{M} w_k \phi\left(x_i; \left(\mu - \frac{1}{2}\sigma^2 \right)\Delta + \mu_Q k, \sigma^2 \Delta + \sigma_Q^2 k \right). \tag{4}$$

If we observe a path of some JD(M)J process we do not know whether the observations or which of them have resulted from jumps. Moreover, we are not able to (directly) identify an impact of the pure diffusion process and the jumps. In other words, we do not know which component of sum (3) is "responsible" for each observation. To solve this problem we introduce latent variables $Z = (Z_1, ..., Z_n)$ such that $Z_i \in \{0,1,...,M\}$ and $P(Z_i = j|\theta) = w_j$, where $i \in \{1,...,n\}$ and $j \in \{0,1,...,M\}$. By means of Z_i's, we can assess the contribution made by the jumps (as compared with the pure diffusion compound) to explain each of n observations. Increments of the Poisson process N are independent, variables $Z_1,...,Z_n$ are also independent. Consequently,

$$p(x_i|Z_i, \theta) = \phi\left(x_i; \left(\mu - \frac{1}{2}\sigma^2 \right)\Delta + \mu_Q Z_i, \sigma^2 \Delta + \sigma_Q^2 Z_i \right)$$

and

$$p(x|Z,\theta) = \prod_{i=1}^{n} \phi\left(x_i; \left(\mu - \frac{1}{2}\sigma^2 \right)\Delta + \mu_Q Z_i, \sigma^2 \Delta + \sigma_Q^2 Z_i \right).$$

It is advisable to consider the following reparametrization of the model parameters:

$$L = \lambda\Delta, \; h_\sigma = \tfrac{1}{\sigma^2}, \; h_Q = \tfrac{1}{\sigma_Q^2},$$

under which vector of all the n+15 unknown quantities is given by:

$$(Z,\theta) = (Z_1, ..., Z_n, \mu, h_\sigma, L, \mu_Q, h_Q).$$

The Bayesian model is defined by the joint distribution of the observables x, the hidden variables Z and the parameters θ:

$$p(x, Z, \theta) = p(x|Z,\theta)\, p(Z,\theta).$$

Under a common assumption of the mutual prior independence among the covariates of θ, the joint prior density is formed:

$$p(Z,\theta) = p(Z|\theta)\, p(\theta) = p(\mu)\, p(h_\sigma)\, p(L)\, p(\mu_Q)\, p(h_Q) \prod_{i=1}^{n} p(Z_i|\theta),$$

where:

$$P(Z_i = j|\theta) = w_j,\ w_k = \frac{(\lambda\Delta)^k}{k!}\left[\sum_{j=0}^{M}\frac{(\lambda\Delta)^j}{j!}\right]^{-1} \text{ for } k = 0,\ldots,M,$$

$$p(\mu) = \phi(\mu; m_\mu, s_\mu^2),$$

$$p(h_\sigma) \propto h_\sigma^{(\nu_\sigma-2)/2}\exp(-Ah_\sigma/2),$$

$$p(L) \propto L^{(\nu_L-2)/2}\exp(-L/2),$$

$$p(\mu_Q) = \phi(\mu_Q; m_Q, s_Q^2),$$

$$p(h_Q) \propto h_Q^{(\nu_Q-2)/2}\exp(-Bh_Q/2).$$

Such a choice of the prior structure (normal distributions for μ and μ_Q, the gamma distributions $Gamma\left(\frac{\nu_\sigma}{2},\frac{A}{2}\right)$ and $Gamma\left(\frac{\nu_Q}{2},\frac{B}{2}\right)$ for h_σ and h_Q, respectively, and the $\chi^2_{\nu_L}$ distribution for L) densities guarantees that the posterior density is a bounded function even though the likelihood function is unbounded (Lin, Huang, 2002). The prior structure is determined under $A = B = 1$, $\nu_L = 10\Delta$, $m_\mu = 0.01$, $s_\mu^2 = 1$, $\nu_\sigma = 5$, $m_Q = 0.01$, $s_Q^2 = 1$, $\nu_Q = 5$.

Posterior characteristics of the unknown quantities are calculated via the Markov Chain Monte Carlo (MCMC) methods (Gamerman, Lopes, 2006), combining the Gibbs sampler, the independence and the sequential Metropolis-Hastings algorithms, as well as the acceptance-rejection sampling. For more details on the technicalities we refer to Kostrzewski (2011).

3. The Optimal-Replication Strategy

Pricing and hedging derivatives are among investors' fundamental problems. Investors employ replication strategies to hedge derivatives. Unfortunately, in the case of incomplete markets such a strategy may not exist. Some idea is to create a self-financing strategy, the value of which is "close" (at maturity) to the one of the derivative's payoff function. We apply the results of Bertsimas, Kogan and Lo (2001) to define and calculate the optimal-replication strategy for portfolios comprising a risky asset and riskless bonds. The approach involves buying, selling, borrowing and lending the portfolio constituents. Let P_t and B_t denote price of the risky instrument and value of the riskless investment at $0 \le t \le T$ respectively. The payoff of an European option at maturity *T* is denoted by $F(P_T)$. Finally, let θ_t be the amount of stocks in the portfolio at time *t*. Then $V_t = \theta_t P_t + B_t$ is the value of the portfolio at *t*. Bertsimas, Kogan and Lo (2001) consider a mean-squared-error criterion to define the optimal-replication strategy θ_t^*, under which V_t^* is the value of the optimal portfolio. It follows that V_t^* and θ_t^* minimize:

$$\sqrt{E\left([V_T - F(P_T)]^2 \middle| V_0\right)}$$

over $\{V_0, \theta_t\}$. Moreover,

$$\varepsilon^* = \sqrt{\min_{\{V_0,\ \theta_t\}} E\left([V_T - F(P_T)]^2 \middle| V_0\right)}$$

constitutes the minimum replication error, that is an error of fitting the strategy into the payoff F at T. If the replication strategy exists, then $\varepsilon^* = 0$ and $V_T^* = F(P_T)$. The error ε^* is construed as a relative measure of the market incompleteness, with its relativity justified by ε^* corresponding only to a given derivative and a given model. To evaluate the optimal-replication strategy Bertsimas, Kogan and Lo (2001) make some additional assumptions:

1. There are no taxes and transaction costs.
2. Purchasing, selling, borrowing and shortsale are possible without any restrictions.
3. The borrowing and lending interest rate r is constant and equal zero.
4. P is a Markov process.
5. Trading takes place at known and fixed times $t_i \in \{t_0, \ldots, t_N\}$, where $0 = t_0 < t_N = T$.

To simplify the notation let $t_i \equiv i$. The aim is to calculate strategy $\theta^*(i, V_i, P_i)$, the initial value V_0^* of the optimal portfolio and the error ε^*. Let x_i be a logarithmic rate of return such that $P_{i+1} = P_i \exp\left(\ln\left(\frac{P_{i+1}}{P_i}\right)\right) = P_i \exp(x_i)$. Bellman's principle of optimality (Bertsekas, 1995) yields the following theorem:

Theorem 1 (Bertsimas, Kogan, Lo, 2001)

If $J_i(V_i, P_i) = \min_{\substack{\theta(k, V_k, P_k) \\ i \le k \le N-1}} E\left((V_N - F(P_N))^2 \middle| V_i, P_i\right)$, then:

$$J_N(V_N, P_N) = (V_N - F(P_N))^2,$$
$$J_i(V_i, P_i) = \min_{\theta(i,\ V_i,\ P_i)} E\left(J_{i+1}(V_{i+1}, P_{i+1}) \middle| V_i, P_i\right),$$

for $i = 0, \ldots, N-1$.

Theorem 1 suggests that the strategy is set recursively. The problem of optimal replication is solved via stochastic dynamic programming. The main results are formulated in the theorem below.

Theorem 2 (Bertsimas, Kogan, Lo, 2001)

Under the above assumptions:

1. There are functions: $a_i(P_i)$, $b_i(P_i)$ and $c(P_i)$, such that:

   $$J_i(V_i,P_i)-a_i(P_i)[V_i-b_i(P_i)]^2+c_i(P_i),\ i=0,...,N.$$

2. $\theta^*(i,V_i,P_i)=p_i(P_i)-V_i q_i(P_i)$, where functions a_i, b_i, c_i, p_i and q_i are evaluated recursively. Starting with $a_N(P_N)=1$, $b_N(P_N)=F(P_N)$, $c_N(P_N)=0$, the calculations for $i=N-1,...,0$ proceed as follows:

   $$p_i(P_i)=\frac{E\left[a_{i+1}(P_{i+1})\cdot b_{i+1}(P_{i+1})\cdot(P_{i+1}-P_i)|P_i\right]}{E\left[a_{i+1}(P_{i+1})\cdot(P_{i+1}-P_i)^2|P_i\right]},$$

   $$q_i(P_i)=\frac{E\left[a_{i+1}(P_{i+1})\cdot(P_{i+1}-P_i)|P_i\right]}{E\left[a_{i+1}(P_{i+1})\cdot(P_{i+1}-P_i)^2|P_i\right]},$$

   $$a_i(P_i)=E[a_{i+1}(P_{i+1})\cdot(1-q_i(P_i)\cdot(P_{i+1}-P_i))^2|P_i],$$

   $$b_i(P_i)=\frac{1}{a_i(P_i)}\cdot E[a_{i+1}(P_{i+1})\cdot(b_{i+1}(P_{i+1})-p_i(P_i)\cdot(P_{i+1}-P_i))\cdot$$
   $$\cdot(1-q_i(P_i)\cdot(P_{i+1}-P_i))|P_i],$$

   $$c_i(P_i)=E[a_{i+1}(P_{i+1})\cdot(b_{i+1}(P_{i+1})-p_i(P_i)\cdot(P_{i+1}-P_i))^2|P_i]+$$
   $$+E[c_{i+1}(P_{i+1})|P_i]-a_i(P_i)\cdot[b_i(P_i)]^2.$$

3. $a_i(P_i)>0$, $c_i(P_i)\geq 0$ for $i=N-1,...,0$.

4. Under the optimal-replication strategy $\theta^*(i,V_i,P_i)$ we obtain: $J_0(V_0,P_0)=a_0(P_0)[V_0-b_0(P_0)]^2+c_0(P_0)$, $V_0^*=b_0(P_0)$ and $\varepsilon^*=\sqrt{c_0(P_0)}$.

Properties (Bertsimas, Kogan, Lo, 2001)

a) The error of replication $\varepsilon^*=\sqrt{c_0(P_0)}$ is the same for put and call options.

b) If prices P_t follow a geometric Brownian motion and $N\to\infty$, then the cost V_0^* of the optimal-replication strategy converges to the Black-Scholes price.

c) $b_0(P_0)$ meets the put-call parity.

d) $\theta^*(i,V_i,P_i)$ is the self-financing strategy which does not guarantee $V_i^*\geq 0$.

e) The value of the optimal-replication strategy could be lower or higher at maturity *T* than the value of the payoff function.

The optimal-replication strategy could be less or more attractive than other strategies, e.g. the delta-hedging strategy. It is because the optimal-replication strategy is optimal only in the mean-squared-error sense. In general, calculation of expectations defined in Theorem 2 may not be straightforward. In the case of the JD(M)J models numerical techniques should be employed to approximate their values.

To calculate the cost of the optimal-replication strategy V_0^* and the relative measure of market incompleteness ε^* the conditional expectations defined in Theorem 2 need to be evaluated. Obviously, these are given by relevant integrals, as for instance:

$$E\left[a_{i+1}(P_{i+1})\cdot(P_{i+1}-P_i)\middle|P_i\right]=$$
$$=\int_{-\infty}^{\infty} a_{i+1}\left(P_i\exp(x)\right)\cdot\left(P_i\exp(x)-P_i\right)\cdot p\left(x\middle|P_i\right)dx=$$
$$=\sum_{k=0}^{M}\frac{w_k}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{a_{i+1}\left(P_i\exp\left(m_k+\sqrt{2}\sigma_k y\right)\right)\left(P_i\exp\left(m_k+\sqrt{2}\sigma_k y\right)-P_i\right)}{\exp\left(y^2\right)}dy,$$

where $w_k=\frac{(\lambda\Delta)^k}{k!}\left[\sum_{j=0}^{M}\frac{(\lambda\Delta)^j}{j!}\right]^{-1}$, $m_k=\left(\mu-\frac{1}{2}\sigma^2\right)\Delta+\mu_Q k$ and $\sigma_k^2=\sigma^2\Delta+\sigma_Q^2 k$. Analytical calculations of such formulae are difficult or positively impossible, which is why numerical approximations, such as the piecewise cubic Hermite interpolation and Gauss-Hermit quadrature, are utilized. All numerical calculations are carried out in R using the pracma and glmmML packages.

4. Empirical Studies

In this section we present the results of Bayesian estimation, model comparison and pricing of the optimal-replication strategy. The calculations are performed for two stock market indices WIG20 and S&P100.

The WIG20 is a stock market index comprising 20 biggest and most liquid companies on the Warsaw Stock Exchange (WSE)[2]. The considered time series x consists of 946 daily logarithmic rates of return on the WIG20 index closing quotations from June 5, 2007 to March 11, 2011[3].

The S&P100 index includes 100 leading US stocks recorded by Standard & Poor's. The considered data x contains 1,077 daily log-returns on the index over a period from April 2, 2007 to July 8, 2011[4].

Daily closing quotations of the WIG20 and S&P100 indices are presented in Figure 1, whereas Figure 2 plots the logarithmic rates of return.

[2] www.gpw.pl.

[3] The data were downloaded from www.gpwinfostrefa.pl.

[4] The data were downloaded from http://finance.yahoo.com/.

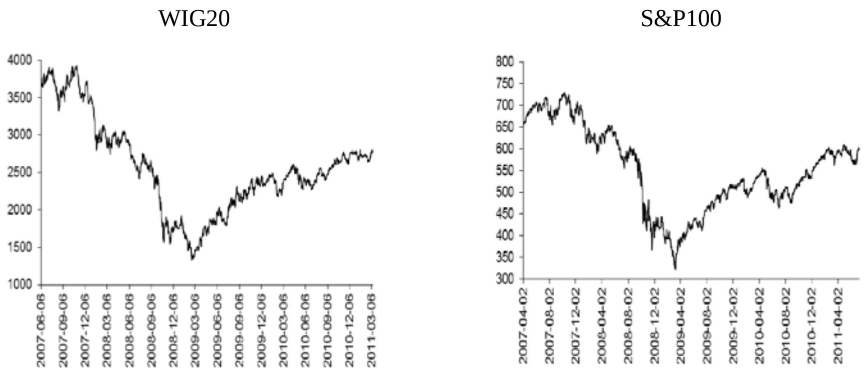


Figure 1. Daily closing quotations of the WIG20 and S&P100 indices

The horizontal lines present bands of plus/minus two or three standard deviations (dashed and dotted lines, respectively).

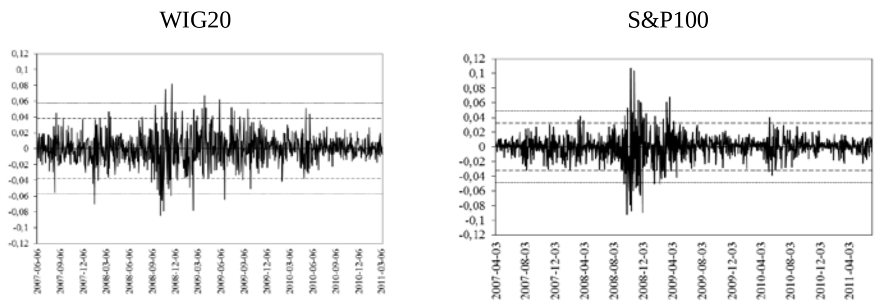


Figure 2. Daily logarithmic rates of return on the WIG20 and S&P100 closing quotations

As evidenced in Figure 2 the outlying log-returns on the S&P100 index are more prominent than the ones featured by the WIG20 series, which may hint at the jump component playing a more crucial role in modeling the former.

4.1. WIG20

It is assumed that time interval between consecutive observations equals $\Delta = 1/252$. For the WIG20 series we restrict the analysis to two model specifications: JD(0)J (i.e. a pure diffusion process) and JD(1)J (i.e. the one allowing for a single jump over a given time interval Δ).

4.1.1. General Results

Table 1 presents posterior means and standard deviations (in parentheses) of the parameters. The results are based on 600,000 and 1,000,000 draws of posterior distributions, preceded by 10,000 and 300,000 burn-in passes for M=0 and M=1, respectively. The results of the MCMC sampler are robust to the choice of the starting points. Convergence of the chains is confirmed by the CUMSUM statistics (Yu, Mykland, 1998), as well as the ergodic means and standard deviations plots.

Noteworthy, posterior characteristics of the pure diffusion parameters, i.e. μ and σ^2, are close to their JD(1)J counterparts, which may hint at there being no need for jumps to be accounted for. The conclusion is also supported by the close to zero posterior mean of the jump intensity λ, accompanied with relatively large posterior dispersion of the parameters

Table 1. Posterior means and standard deviations (in parentheses) of the parameters for the WIG20 index

Parameters	JD(0)J		JD(1)J	
λ	–		0.0557	(0.3053)
μ	-0.0364	(0.1556)	-0.0346	(0.1545)
μ_Q	–		0.0085	(0.9770)
σ^2	0.0917	(0.0042)	0.0919	(0.0044)
σ^2_Q	–		0.3520	(1.3100)

We now focus on the choice of the appropriate value of M. The model with the highest posterior probability is referred to as the best one. The best model points the value of M. We have to compare:

$$P(JD(0)J|x) = \frac{P(JD(0)J)P(x|JD(0)J)}{P(JD(0)J)P(x|JD(0)J) + P(JD(1)J)P(x|JD(1)J)}$$

and

$$P(JD(1)J|x) = 1 - P(JD(0)J|x).$$

The Newton-Raftery estimators (Newton, Raftery, 1994; Raftery, Newton, Satagopan, Krivitsky, 2007) are employed to assess the posterior probabilities $P(x|JD(0)J)$ and $P(x|JD(1)J)$. These estimators are consistent, but their asymptotic variances do not exist. In practice, the values of the estimator may not stable. A longer Monte Carlo chain (of 1,000,000 draws) was generated to increase the credibility of the estimator.

Under equal prior probabilities of each model, i.e. $P(JD(0)J) = P(JD(1)J)$, we obtain $P(JD(0)J \mid x) \approx P(JD(1)J \mid x)$. However, invoking Occam's razor that promotes parsimony (and thereby models with lower number of parameters) we set $P(JD(0)J) \propto 2^{-2}$ and $P(JD(1)J) \propto 2^{-5}$, which results in $P(JD(0)J \mid x) \approx 0.9$. The JD(0)J model is more likely a posteriori than the JD(1)J specification. In other words, jumps are non-essential in modeling dynamics of daily (closing) quotations of the WIG20 index[5].

4.1.2. Calculating the Optimal-Replication Strategy Cost

Under market completeness of the Black-Scholes model replication strategies do exist. Continuous trading opportunity is one of the model's underlying assumptions. However, in practice this assumption is quite unrealistic. If we limit trading opportunities to discrete times we get an incomplete model (Bertsimas, Kogan, Lo, 2001). In the JD(0)J framework and under the assumption of the fixed time Δ between consecutive trading times, the replication strategy may not exist. Further, we calculate the costs and errors of the optimal strategies for some European options.

Let us consider two European call options. The date of pricing the optimal strategy is March 14, 2011, and the maturity date *T* is March 18, 2011. Strike prices are equal $K = 2700$ and $K = 2800$. The closing quotation value of the WIG20 index on March 14, 2011 equals 2757.76. The first option is in the money and the second one is out of the money. The riskless interest rate is arbitrarily set at $r = 0.0362$ (*r* equals an arithmetic mean of WIBID ON and WIBOR ON on March 14, 2011). The theory of optimal-replication strategy was originally presented under the restriction of $r = 0$, but, fortunately, it could be generalized so as to incorporate any constant riskless rate $r > 0$. The pillar of the extension is normalization of all prices with the price of a zero-coupon bond (Bertsimas, Kogan, Lo, 2001).

Figures 3 and 4 display posterior distributions of the optimal-replication strategy cost V_0^* and the relative measure of the market incompleteness ε^* for each strike price. The histograms are calculated on the basis of 1,000 states of Markov chains.

The maturity *T* is specified as $4/252$, which may appear a short period of time, but is long enough to judge the convergence of the optimal-replication strategy to the replication strategy (the strategy exists in Black-Scholes framework). For the time being let us assume that the unknown parameters equal the assessed posterior means, i.e. $\mu = -0.03644597$ and $\sigma^2 = 0.09171522$. Let *l* denote the number of times the portfolio changes over the duration of the

[5] The Barndorff-Nielsen and Shephard's nonparametric test also rejects jumps in the considered time series (Barndorff-Nielsen, Shephard, 2006).

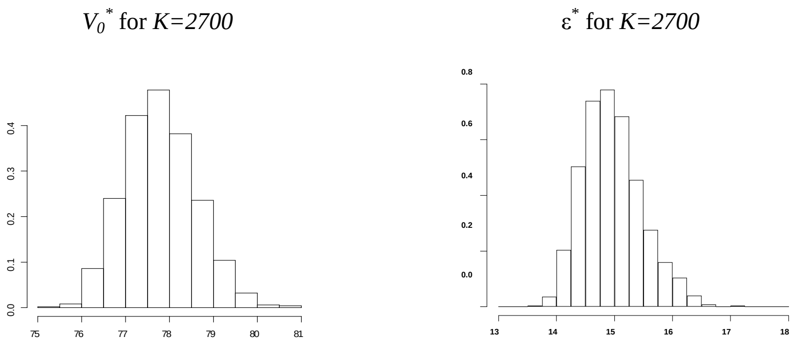


Figure 3. Histograms of the posterior distributions of V_0^* and ε^* for $K = 2700$

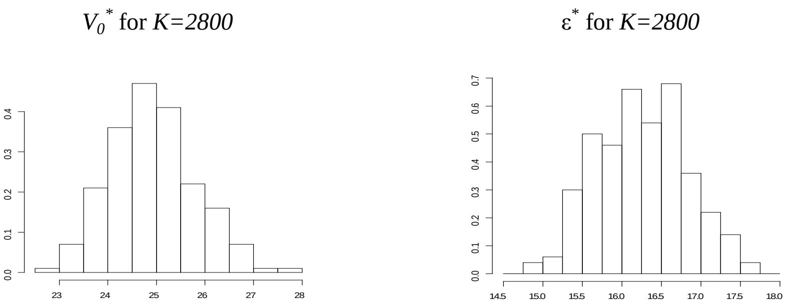


Figure 4. Histograms of the posterior distributions of V_0^* and ε^* for $K = 2800$

options (a strategy is a sequence of portfolios). Tables 2 and 3 present the cost of the optimal-replication strategy V_0^* and the relative measure of the market incompleteness ε^* for each strike price K. The prices of options calculated under the Black-Scholes assumptions are presented in the last row of Table 2. Recall that the market completeness of the Black-Scholes model warrants a zero replication error, i.e. $\varepsilon^* = 0$.

We note that as the number *l* of times the portfolio changes over the option duration increases the optimal-replication strategy cost converges to the Black-Scholes price. The relation is accompanied by a systematic decrease in the replication error (see Table 3), indicating that the market is “nearing” completeness.

The prices of the options on March 14, 2011 equaled 52 and 5, for strike prices *K*=2700 and *K*=2800, respectively. Posterior histograms and expected values of V_0^* suggest that hedging of the options by the optimal-replication is

Table 2. Values of V_0^* calculated under $\mu = -0.03644597$ and $\sigma^2 = 0.09171522$ for increasing values of l, along with the Black-Scholes (BS) prices

l	V_0^*	
	K=2700	K=2800
1	77.7705	26.1447
4	77.7768	24.7248
10	77.7477	25.0945
30	77.7488	25.0621
BS	77.7427	25.0372

Table 3. Values of ε^* calculated under $\mu = -0.03644597$ and $\sigma^2 = 0.09171522$ for increasing values of l

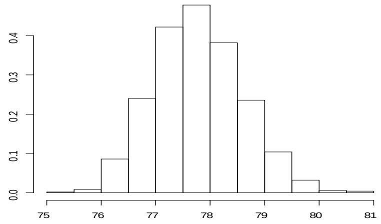
l	ε^*	
	K=2700	K=2800
1	27.8907	28.6173
4	15.1117	16.6562
10	9.8949	10.5713
30	5.8611	6.2656
100	3.2616	3.4959
200	2.3230	2.4952
500	1.4778	1.5844
BS	0	0

strategy expensive in comparison with the prices of the options. Note that the above results depend on estimation of the model's parameters and the choice of the observation set. If the estimation is based on a shorter series, avoiding the period of time with more volatile changes of the index, the estimation and pricing results are affected. We additionally consider a dataset from May 5, 2010 to March 11, 2011. Then the values of the relative measure of market incompleteness ε^* are smaller than in the case of the full sample, and so is the posterior mean of the volatility parameter σ, with its value declining from 0.03 in the case of the full sample model to 0.02 for the trimmed series. Figure 5 presents the costs of the optimal-replication strategy V_0^* for a strike price K=2700 and two sets of observations. However, the cost of the new strategy is still high (or the price of the option is low).

4.2. S&P100

Let us consider the S&P100 index and three model specifications: JD(0)J, JD(1)J and JD(10)J. It is assumed that the time interval between consecutive observations equals $\Delta = 1/252$.

V_0^* (05-Jun-2007 to 11-Mar-2011) V_0^* (05-May-2010 to 11-Mar-2011)



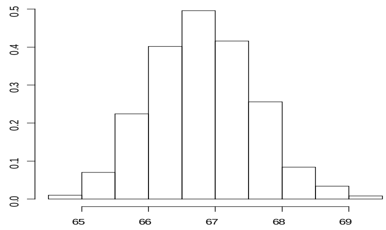


Figure 5. Costs of the optimal-replication strategy V_0^* for the strike price K=2700 and two sets of observations: 05-Jun-2007 to 11-Mar-2011 and 05-May-2010 to 11-Mar-2011

4.2.1. General Results

Table 4 contains results of Bayesian estimation - posterior means and standard deviations (in parentheses). The outcomes are based on 1,000,000 draws of the Markov chain and 25,000 burn-in passes. The results of the MCMC sampler are robust to the choice of the starting points. Convergence of the chains is confirmed by the CUMSUM statistics (Yu, Mykland, 1998), as well as the ergodic means and standard deviations plots.

Posterior means of the pure diffusion parameters μ and σ^2 calculated in the JD(1)J and JD(10)J models – though almost identical across the two specifications – differ quite substantially from their counterparts in the JD(0)J model (i.e. the one that precludes any jumps). Particularly, note that $E(\sigma^2 | y, \text{JD(0)J}) = 0.0677$ as opposed to $E(\sigma^2 | y, \text{JD}(M)\text{J}) = 0.047$ for M=1 and M=10. The difference is justified by the jump component "absorbing" some of the log-returns' volatility, whereas exclusion of jumps in the JD(0)J specification is compensated with a higher value of the volatility parameter's posterior mean.

Noteworthy, the posterior results for the JD(M)J specifications featuring M>0 are very close, which may be indicative of there being no empirical need for allowing for more than a single jump per Δ. Particularly, posterior means of the jump intensity parameter λ in both the JD(1)J and the JD(10)J model consistently imply that on average there are 4 jumps per year.

Table 4. Posterior means and standard deviations (in parentheses) of the JD(M)J models' parameters for the S&P100 index

Parameters	JD(0)J		JD(1)J		JD(10)J	
λ	–		4.4234	(1.3363)	4.3507	(1.3030)
μ	0.0145	(0.1256)	0.0426	(0.1089)	0.0430	(0.1085)
μ_Q	–		-0.0090	(0.1850)	-0.0087	(0.1845)
σ^2	0.0677	(0.0029)	0.0470	(0.0028)	0.0470	(0.0028)
σ^2_Q	–		0.5440	(1.1183)	0.5451	(1.3625)

Turning to the formal pair-wise model comparison, we calculate decimal logarithms of Bayes factors (Bernardo, Smith, 2002):

$$\log_{10}(B_{1,0}) = \log_{10}\left(\frac{P(\mathrm{JD}(1)\,\mathrm{J} \mid x)}{P(\mathrm{JD}(0)\,\mathrm{J} \mid x)}\right) \approx 17,$$

$$\log_{10}(B_{1,10}) = \log_{10}\left(\frac{P(\mathrm{JD}(1)\,\mathrm{J} \mid x)}{P(\mathrm{JD}(10)\,\mathrm{J} \mid x)}\right) \approx 1.7.$$

It appears that the JD(1)J specification beats the competition, being as much as ca. 17 orders of magnitude more likely a posteriori than the simplest model structure (JD(0)J)[6]. Although only marginally, the former is also favored against the other jump-diffusion specification, i.e. JD(10)J, which seems to be penalized for its excessively large number M=10 of jumps allowed per Δ. Admittedly, the result fits in well with the overall pursuit for parsimony.

4.2.2. Calculating the Optimal-replication Strategy Cost

We confine our further considerations to the JD(1)J model. It is known that markets are incomplete when sources of randomness outnumber the underlying traded risky instruments (Björk, 2004). In the JD(1)J specification there are three sources of randomness – the Wiener process W, the Poisson process N, and random variables Q. In our setting we consider a market with only one risky underlying instrument (a stock market index) accompanied by as much as three sources of randomness, so the market is incomplete. Therefore, we resort to the optimal-replication strategies for selected options.

Let us consider two European call options. A date of pricing of the optimal strategy is July 11, 2011, and the maturity date T of the options is July 15, 2011. Strike prices equal $K = 590$ and $K = 610$. The closing quotation of the

[6] The Barndorff-Nielsen and Shephard's nonparametric test reject the pure diffusion at significance level 0.05 (p-value equals 0.011).

S&P100 index on July 11, 2011 equals 588.15. Both of the options are out of the money. The riskless interest rate is set at $r = 0.0075$ and it equals the Fed Funds Discount Rate at the considered option duration.

Table 5 presents posterior means and standard deviations (in parentheses) of the optimal-replication strategy (initial) cost V_0^* and the relative measure of the market incompleteness ε^* for each strike price. On the day of the pricing, according to our knowledge, there were no transactions of selling the considered options.

Table 5. Posterior means (and standard deviations) of V_0^* and ε^* calculated for the S&P100 index as an underlying instrument

Quantity	K=590	K=610
V_0^*	12.12 (2.8927)	6.619 (3.0739)
ε^*	35.68 (11.6726)	35.47 (12.8357)

Table 6 contains quantiles of the posterior distributions of V_0^* and ε^*. In general, the call option with a lower strike price is more expensive than the option with a higher strike price. Note that the cost of the optimal-replication strategy V_0^* is higher for the more attractive option.

Table 6. Quantiles of the posterior distributions of V_0^* and ε^* calculated for the S&P100 index as an underlying instrument

Orders of the quantiles	K=590		K=610	
	V_0^*	ε^*	V_0^*	ε^*
5%	8.4071	20.2538	2.2598	16.2821
25%	9.9695	27.0364	4.6371	26.7550
50%	11.4782	32.7958	6.2447	33.4317
75%	13.5252	42.2325	8.1295	43.2222
95%	17.8427	57.2116	12.6826	58.7495

Figures 6 and 7 display histograms of the posterior distributions of the optimal-replication strategy cost V_0^* and the relative measure of the market incompleteness ε^*. These histograms are based on (only) 150 (randomly chosen) states of the Markov chains. The reason behind such a small sample is time-consuming calculations of the optimal-replication strategy for each parameter vector. These calculations took about twenty hours on a standard PC. The application of parallel calculations reduced that time to seven hours.

A fairly large dispersion of the posterior distributions of V_0^* and ε^* may stem from a relatively large parameter uncertainty (as evidenced by the posterior standard deviations).

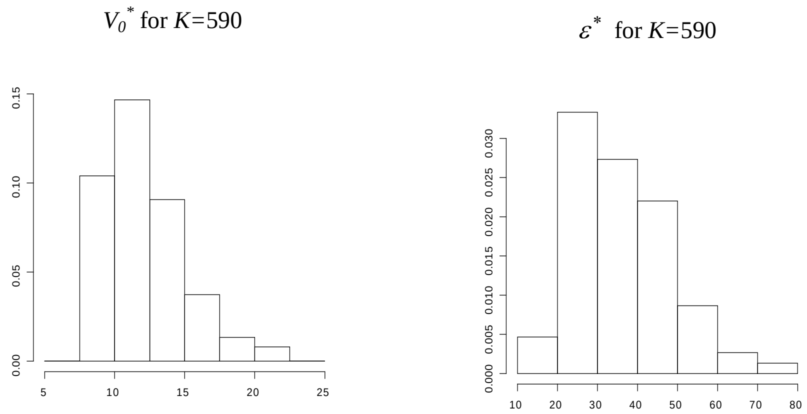


Figure 6. Histograms of the posterior distributions of V_0^* and ε^* for $K = 590$

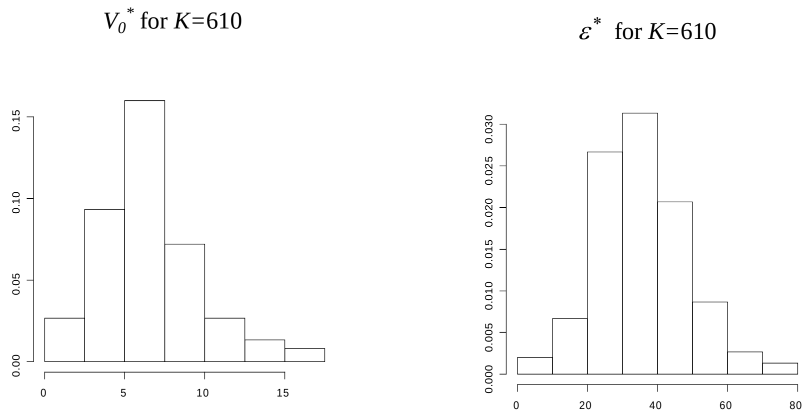


Figure 7. Histograms of the posterior distributions of V_0^* and ε^* for $K = 610$

5. Conclusions

This paper concerns the issue of option hedging in incomplete market models using stochastic dynamic programming and Bayesian statistics.

Familiar models of option pricing are complete. Unfortunately, the assumptions these structures usually rest upon are quite unrealistic. For instance, the Black-Scholes model is hinged upon continuous trading and continuous paths of a risky underlying instrument. Relaxing these assumptions leads to incomplete market models, such as the JD(*M*)J structures considered in the present study.

It is shown that incorporation of jumps in modeling financial time series may improve the model fit (as compared with a pure diffusion process). Unfortunately, the market incompleteness in the models featuring jumps (JD(M)J with M>0) renders the task of pricing and hedging derivatives more demanding. In general, as the replication strategy does not exist, the investor needs to resort to some optimal strategy. In the study we succeeded in employing the optimal-replication strategy algorithm, derived by Bertsimas, Kogan and Lo (2001), in the JD(M)J framework.

Contrary to what seems a common practice in the financial mathematics works, where the model's parameters are set arbitrarily, we estimate the parameters using Bayesian methodology, taking advantage of its accounting for the parameters uncertainty. Moreover, the results are further employed to infer upon the degree of market incompleteness as well as to price the optimal-replication strategy. Specifically, posterior densities (rather than point values solely) of both the optimal strategy costs along and its relative error are calculated (using the MCMC techniques), providing us with some insight into their uncertainty.

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Bayesowska wycena kosztu optymalnej strategii replikującej europejską opcję w modelu JD(M)J

Z a r y s t r e ś c i. Wycena opcji w modelu niezupełnym jest nietrywialnym zagadnieniem. Przykładem modelu niezupełnego jest wprowadzony przez Mertona model dyfuzji ze skokami. Gęstość logarytmu procesu dyfuzji ze skokami jest nieskończoną mieszanką rozkładów normalnych. W badaniu przyjęto, że liczba mieszanek jest skończona. Otrzymany w ten sposób model nazwano modelem JD(M)J. W praktyce parametry modelu są nieznane i wymagają estymacji. W badaniu zastosowano wnioskowanie bayesowskie. JD(M)J jest modelem niezupełnym dla którego, w ogólnym przypadku, nie można wskazać strategii replikujących instrumenty pochodne. W badaniu zaprezentowano algorytm wyznaczający optymalną w sensie średniokwadratowym strategię replikującą europejski instrument pochodny. Do zilustrowania omówionej teorii wykorzystano indeksy WIG20 i S&P100. Przedstawiona metodologia jest użyteczna dla inwestorów, którzy chcą uwzględnić w wycenie instrumentów pochodnych oraz analizach szacowania ryzyka „niepewność" estymacji parametrów modelu.

S ł o w a k l u c z o w e: rynki niezupełne, wnioskowanie bayesowskie, procesy dyfuzji ze skokami, wycena instrumentów pochodnych.