ECONOMIC ANALYSIS AND WORKERS' MANAGEMENT, 1, XX (1986), 65—75

SOLUTION OF SOME PROGRAMMING PROBLEMS BY THE USE OF PIECEWISE LINEAR APPROXIMATION

*Ivan MEŠKO**

ABSTRACT

In special cases the function of more variables can be expressed by separable functions. Some nonlinear programming problems can be therefore solved using piecewise linear approximation. In this paper a linear approximation function of continuous of two and of three variables using 0—1 variables subject to additional linear constraints is given.

1. INTRODUCTION

Piecewise linear function can be explicitly expressed using 0—1 variables subject to some linear constraints [3], [5]. Supposing that the objective function and constraints of the nonlinear programming problem can be expressed by separable functions, the problem can then be approximated by the mixed integer programming problem.

Since [4]

$$4L_1(x)\,L_2(x) = 4\,(a_o + a_1 x_1 + \ldots + a_n x_n)\,(b_o + b_1 x_1 + \ldots + b_n x_n) = (L_1(x) + L_2(x))^2 - (L_1(x) - L_2(x))^2 = y_1^2 - y_2^2 \quad (1.1)$$

subject to

$$y_1 = L_1(x) + L_2(x), \qquad y_2 = L_1(x) - L_2(x),$$

any quadratic function can be expressed by separable functions subject to some linear constraints. Any programming problem with quadratic objective function and quadratic constraints can therefore be approximated in such a way. Since

$$4L_1(x)\,L_2(x)\,L_3(x) = ((L_1(x) + L_2(x))^2 - (L_1(x) - L_2(x))^2)\,L_3(x) \quad (1.2)$$

the product of three linear functions can be expressed by separable functions using the substitution

* University of Maribor.

$$u = t^2 = (L_1(x) + L_2(x))^2, \qquad v = w^2 = (L_1(x) - L_2(x))^2.$$

The product of more variables can be expressed similarly. Some multilinear programming problems [6] can be approximated and solved using this result.

Similarly the function

$$f(x) = \frac{L_1(x)}{L_2(x)} + g(x),$$

where g (x) is separable, can be expressed by separable functions. If we consider (1.1) and $tL_2(x) = 1$ [8], we get

$$f(x) = tL_1(x) + g(x) = \frac{1}{4}((t + L_1(x))^2 - (t - L_1(x))^2) + g(x)$$

subject to

$$(t + L_2(x))^2 - (t - L_2(x))^2 = 4.$$

This result can be useful in multicriteria programming [1], [9] when one or more of the objectives are expressed by linear fractional functions, the other objective functions and the constraints being separable. If the weighted sum of the objective functions is taken as the objective function, the programming problem can be approximated by the mixed integer programming problem. If there are three linear objective functions and linear constraints, then, using the expression given in the next part, for any function of three objective functions the programming problem can be approximated by the mixed integer programming problem.

2. EXTENDED PIECEWISE LINEAR FUNCTION

The notion of the piecewise linear function can be extended to functions of two variables. Assume that the values

$$f_{ij} = f(x_i, y_j), \qquad i = 0, 1, \ldots, m, \qquad j = 0, 1, \ldots, n$$

are given, where

$$x_i = a_o + a_1 + \ldots + a_i, \qquad i = 0, 1, \ldots, m,$$
$$y_j = b_o + b_1 + \ldots + b_j, \qquad j = 0, 1, \ldots, n,$$

$$a_i > 0, \qquad i = 1, \ldots, m,$$
$$b_j > 0, \qquad j = 1, \ldots, n.$$

It can be shown [11] that the function

$$g(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{n} (f_{i-1j-1} (v_{i-1j-1} - v_{i-1j}) + (f_{i-1j} - f_{i-1j-1}) y_{ij}$$

$$+ (f_{ij-1} - f_{i-1j-1}) x_{ij} + (f_{ij} - f_{i-1j}) x'_{ij}) \tag{2.1}$$

subject to

$$x = \sum_{i=0}^{m-1} a_i u_i + \sum_{i=1}^{m} \sum_{j=1}^{n} (x_{ij} + x'_{ij}) a_i$$

$$y = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} b_j v_{ij} + \sum_{i=1}^{m} \sum_{j=1}^{n} y_{ij} b_j$$

$$v_{ij} - v_{ij+1} \geqq 0, \qquad v_{mj} = v_{in} = 0,$$
$$i = 0, 1, \ldots, m, \qquad j = 0, 1, \ldots, n-1$$

$$u_i - u_{i+1} = v_{io}, \qquad i = 0, 1, \ldots, m-1, \qquad u_o = 1, \qquad u_m = 0$$

$$x_{ij} + x'_{ij} + y_{ij} \leqq 2 (v_{i-1j-1} - v_{i-1j}),$$
$$i = 1, \ldots, m, \qquad j = 1, \ldots, n$$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x'_{ij} \leqq t \leqq \sum_{i=1}^{m} \sum_{j=1}^{n} (x_{ij} + y_{ij}) \leqq 1$$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} (x_{ij} + x'_{ij}) \leqq 1$$

on the triangles A (x_{i-1}, y_{j-1}), B (x_i, y_{j-1}), D (x_{i-1}, y_j) and on the triangles B (x_i, y_{j-1}), C (x_i, y_j), D (x_{i-1}, y_j) is linear and it satisfies the condition

$$g(x_i, y_j) = f_{ij}, \qquad i = 0, 1, \ldots, m, \qquad j = 0, 1, \ldots, n.$$

Here the variables x_{ij}, x'_{ij} and y_{ij} are non-negative and u_i, v_{ij} and t are 0—1 variables.

Similarly the notion of the piecewise linear function can be extended to functions of three variables. Assume that the values

$$f_{ijk} = f(x_i, y_j, z_k), \qquad i = 0, 1, \ldots, h, \qquad j = 0, 1, \ldots, m,$$
$$k = 0, \ldots, n$$

are given, where

$$x_i = a_o + a_1 + \ldots + a_i, \quad i = 0, 1, \ldots, h, \quad a_i > 0, \quad i = 1, \ldots, h,$$
$$y_j = b_o + b_1 + \ldots + b_j, \quad j = 0, 1, \ldots, m, \quad b_j > 0, \quad j = 1, \ldots, m,$$
$$z_k = c_o + c_1 + \ldots + c_k, \quad k = 0, 1, \ldots, n, \quad c_k > 0, \quad k = 1, \ldots, n.$$

Consider the substitution

$$x = \sum_{i=0}^{h-1} a_i u_i + \sum_{i=1}^{h} \sum_{j=1}^{m} \sum_{k=1}^{n} (x_{ijk} + x'_{ijk} + x''_{ijk}) a_i \tag{2.2}$$

$$y = \sum_{i=0}^{h-1} \sum_{j=0}^{m-1} b_j v_{ij} + \sum_{i=1}^{h} \sum_{j=1}^{m} \sum_{k=1}^{n} (y_{ijk} + y'_{ijk}) b_j \tag{2.3}$$

$$z = \sum_{i=0}^{h-1} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} c_k w_{ijk} + \sum_{i=1}^{h} \sum_{j=1}^{m} \sum_{k=1}^{n} (z_{ijk} + z'_{ijk}) c_k \tag{2.4}$$

subject to

$$u_o = 1, \quad u_h = 0, \quad v_{hj} = v_{im} = 0, \quad i = 0, 1, \ldots, h, \quad j = 0, 1, \ldots, m \tag{2.5}$$

$$w_{ijk-1} - w_{ijk} \geqq 0, \quad i = 0, 1, \ldots, h-1, \quad j = 0, 1, \ldots, m-1, \quad k = 1, \ldots, n \tag{2.6}$$

$$w_{ijn} = w_{imk} = w_{hjk} = 0, \quad i = 0, 1, \ldots, h, \quad j = 0, 1, \ldots, m, \quad k = 0, 1, \ldots, n \tag{2.7}$$

$$u_i - u_{i+1} = v_{io}, \quad i = 0, 1, \ldots, h-1 \tag{2.8}$$

$$v_{ij} - v_{ij+1} = w_{ijo}, \quad i = 0, 1, \ldots, h-1, \quad j = 0, 1, \ldots, m-1 \tag{2.9}$$

$$x_{ijk} + x'_{ijk} + x''_{ijk} + y_{ijk} + y'_{ijk} + z_{ijk} + z'_{ijk} \leqq 3\,(w_{i-1j-1k-1} - w_{i-1j-1k}),$$
$$i = 1, \ldots, h, \quad j = 1, \ldots, m, \quad k = 1, \ldots, n \tag{2.10}$$

$$\sum_{i=1}^{h} \sum_{j=1}^{m} \sum_{k=1}^{n} x'_{ijk} \leqq t_1 \leqq \sum_{i=1}^{h} \sum_{j=1}^{m} \sum_{k=1}^{n} (x_{ijk} + y_{ijk} + z_{ijk}) \leqq 1 \tag{2.11}$$

$$\sum_{i=1}^{h} \sum_{j=1}^{m} \sum_{k=1}^{n} x''_{ijk} \leqq t_2 \leqq \sum_{i=1}^{h} \sum_{j=1}^{m} \sum_{k=1}^{n} (x_{ijk} + x'_{ijk} + z_{ijk}) \leqq 1 \tag{2.12}$$

$$\sum_{i=1}^{h}\sum_{j=1}^{m}\sum_{k=1}^{n} y'_{ijk} \leqq t_3 \leqq \sum_{i=1}^{h}\sum_{j=1}^{m}\sum_{k=1}^{n} (x_{ijk} + y_{ijk}) \tag{2.13}$$

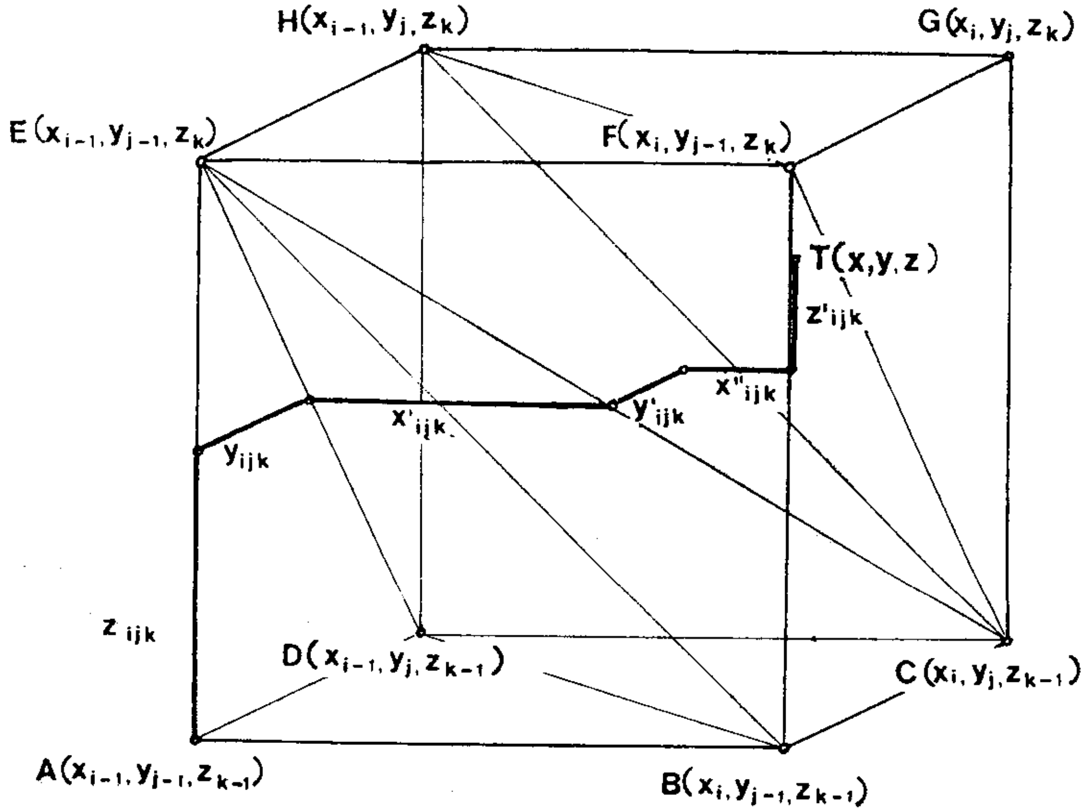
$$\sum_{i=1}^{h}\sum_{j=1}^{m}\sum_{k=1}^{n} z'_{ijk} \leqq t_4 \leqq \sum_{i=1}^{h}\sum_{j=1}^{m}\sum_{k=1}^{n} (x'_{ijk} + x''_{ijk} + y'_{ijk}) \leqq 1 \tag{2.14}$$

$$\sum_{i=1}^{h}\sum_{j=1}^{m}\sum_{k=1}^{n} (x_{ijk} + x'_{ijk} + x''_{ijk}) \leqq 1 \tag{2.15}$$

$$\sum_{i=1}^{h}\sum_{j=1}^{m}\sum_{k=1}^{n} (y_{ijk} + y'_{ijk}) \leqq 1 \tag{2.16}$$

$$\sum_{i=1}^{h}\sum_{j=1}^{m}\sum_{k=1}^{n} (z_{ijk} + z'_{ijk}) \leqq 1 \tag{2.17}$$

where u_i, v_{ij}, w_{ijk} and t_i are 0—1 variables and x_{ijk}, x'_{ijk}, x''_{ijk}, y_{ijk}, y'_{ijk}, z_{ijk}, and z'_{ijk} are non-negative.



Figure

Theorem. The function

$$g(x, y, z) = \sum_{i=1}^{h} \sum_{j=1}^{m} \sum_{k=1}^{n} (f_{i-1j-1k-1}\ (w_{i-1j-1k-1} - w_{i-1j-1k})$$
$$+ (f_{ij-1k-1} - f_{i-1j-1k-1})\, x_{ijk} + (f_{ijk-1} - f_{i-1jk-1})\, x'_{ijk}$$
$$+ (f_{ij-1k} - f_{i-1j-1k})\, x''_{ijk} + (f_{i-1jk-1} - f_{i-1j-1k-1})\, y_{ijk}$$
$$+ (f_{i-1jk} - f_{i-1j-1k})\, y'_{ijk} + (f_{i-1j-1k} - f_{i-1j-1k-1})\, z_{ijk}$$
$$+ (f_{ijk} - f_{ijk-1})\, z'_{ijk}) \tag{2.18}$$

subject to (2.2)—(2.17), on tetrahedrons S_1 (A, B, D, E), S_2 (B, C, D, E), S_3 (B, C, E, F), S_4 (C, D, E, H), S_5 (C, E, F, H) and S_6 (C, F, G, H) (see figure) is linear and satisfies the condition

$$g(x_i, y_j, z_k) = f_{ijk}, \qquad i = 0, 1, \ldots, h, \qquad j = 0, 1, \ldots, m,$$
$$k = 0, 1, \ldots, n.$$

Scheme of the proof. From (2.5)—(2.9) it follows

$$u_i = 1, \quad i < p, \qquad u_i = 0, \quad i \geqq p$$
$$v_{p-1j} = 1, \quad j < q, \qquad v_{ij} = 0 \quad \text{otherwise}$$
$$w_{p-1q-1k} = 1, \quad k < r, \qquad w_{ijk} = 0 \quad \text{otherwise.}$$

Therefore from (2.10) it follows

$$x_{ijk} = x'_{ijk} = x''_{ijk} = y_{ijk} = y'_{ijk} = z_{ijk} = z'_{ijk} = 0,$$
$$i \neq p \quad \text{or} \quad j \neq q \quad \text{or} \quad k \neq r.$$

If

$$x_{i-1} < x < x_i, \qquad y_{j-1} < y < y_j, \qquad z_{k-1} < z < z_k, \tag{2.19}$$

from (2.2)—(2.4) and (2.15)—(2.17) it follows

$$p = i, \qquad q = j, \qquad r = k.$$

For any choice of 0—1 variables t_1, t_2, t_3 and t_4 from (2.11)—(2.14) follows four independent equations. Adding (2.2)—(2.4) there are seven linear equations for x_{pqr}, x'_{pqr}, x''_{pqr}, y_{pqr}, y'_{pqr}, z_{pqr} and z'_{pqr}. For internal points of each of six tetrahedrons the variables $t_1, \ldots, t_4$ are by (2.11)—(2.14) and (2.2)—(2.4) unique determined and the received equations are not contradictive.

If $x = x_i$, $y = y_j$, $z = z_k$, from (2.2)—(2.4) and (2.15)—(2.17) it follows $p = i$ or $i + 1$, $q = j$ or $j + 1$, $r = k$ or $k + 1$ respectively but the value of (2.18) is independent of the choice of p, q and r. The linearity of (2.18) follows from the linearity of (2.2)—(2.4) and (2.18) if the 0—1 variables are not changed.

3. CHANCE-CONSTRAINED PROGRAMMING PROBLEMS

Consider the programming problem

$$max\,(c_1 x_1 + c_2 x_2 + \ldots + c_n x_n) \tag{3.1}$$

subject to non-negative variables and

$$a_{i1} x_1 + a_{i2} x_2 + \ldots + a_{in} x_n \leqq a_{io}, \qquad i = 1, \ldots, m. \tag{3.2}$$

Let some of elements c_j and a_{ij} be random variables. Therefore, some data are not known with certainty at the time when the decision variables x_j must be determined and the programming problem (3.1)—(3.2) must be transformed. We get a special and, for applications, important case of a chance-constrained programming problem [2] if we take

$$max\,E\;(c_1 x_1 + c_2 x_2 + \ldots + c_n x_n)$$

subject to non-negative variables and

$$a_{i1} x_1 + a_{i2} x_2 + \ldots + a_{in} x_n \leqq a_{io}, \qquad i \in D, \tag{3.3}$$

$$P\,(r_{i1} x_1 + r_{i2} x_2 + \ldots + r_{in} x_n \leqq r_{io}) \geqq p_i, \qquad i \in S. \tag{3.4}$$

Here a_{ij} $(j = 0, 1, \ldots, n)$ and p_i, $i \in S$, are known, c_j $(j = 1, \ldots, n)$ are random variables with the mean $\bar{c}_j$, r_{ij} $(j = 0, 1, \ldots, n)$ are independent normal random variables with the mean $\bar{r}_{ij}$ and variance s^2_{ij}. Constants p_i, $i \in S$, satisfy the condition $0 < p_i < 1$. If r_{ij} is a fixed constant, we take $r_{ij} = \bar{r}_{ij}$, $s_{ij} = 0$.

In the given case the variable

$$h_i = r_{io} - r_{i1} x_1 - r_{i2} x_2 - \ldots - r_{in} x_n, \qquad i \in S$$

is a normal random variable with the mean

$$\bar{h}_i = \bar{r}_{io} - \bar{r}_{i1} x_1 - \bar{r}_{i2} x_2 - \ldots - \bar{r}_{in} x_n, \tag{3.5}$$

and variance

$$s^2_i = s^2_{io} + s^2_{i1} x^2_1 + s^2_{i2} x^2_2 + \ldots + s^2_{in} x^2_n. \tag{3.6}$$

Therefore the constraint (3.4) can be written as

$$P\,(h_i \geqq 0) = \frac{1}{\sqrt{2\pi}} \int_{-\bar{h}_i/s_i}^{\infty} \exp\left(\frac{t^2}{2}\right) dt = F\left(\frac{\bar{h}_i}{s_i}\right) \tag{3.7}$$

where

$$F\left(\frac{\bar{h}_i}{s_i}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\bar{h}_i/s_i} exp\left(-\frac{t^2}{2}\right) dt. \qquad (3.8)$$

Because F is a strictly increasing function the inverse function F^{-1} can be used. From (3.7) it follows

$$F\left(\frac{\bar{h}_i}{s_i}\right) \geqq p_i$$

$$\frac{\bar{h}_i}{s_i} \geqq F^{-1}(p_i)$$

Considering $y_i = s^2_i$ and $s_i > 0$ we get

$$\bar{h}_i \geqq \sqrt{y_i}\, F^{-1}(p_i)$$

Considering (3.5) the constraint (3.4) can be written in the separable form

$$\bar{r}_{io} - \bar{r}_{i1} x_1 - \bar{r}_{i2} x_2 - \ldots - \bar{r}_{in} x_n \geqq \sqrt{y_i}\; F^{-1}(p_i) \qquad i \in S \qquad (3.9)$$

subject to

$$y_i = s^2_{io} + s^2_{i1} x^2_1 + s^2_{i2} x^2_2 + \ldots + s^2_{in} x^2_n, \qquad i \in S. \qquad (3.10)$$

The programming problem is received in the form

$$max\,(\bar{c}_1 x_1 + \bar{c}_2 x_2 + \ldots + \bar{c}_n x_n)$$

subject to non-negative variables and the constraints (3.3), (3.9) and (3.10). The objective function and (3.3) are linear, since (3.9) and (3.10) are separable, the programming problem can be approximated by the mixed integer programming problem [12].

If r_{ij} $(j = 0, 1, \ldots, n)$ are dependent random variables, they can be transformed into independent random variables using the principal components. Since such a transformation is linear, the mixed integer programming problem can also be used in this case.

Chance-constrained programming can often be used for optimizing the business process. Because of deficiency of computer programs for solving such programming problems, they are mostly approximated by a linear programming problem. If only c_j are random variables, approximation of this kind is available. The transformation of the model is simple if only r_{io} is a random variable. Then the constraint

$$P\,(r_{i1}x_1 + r_{i2}x_2 + \ldots + r_{in}x_n \leqq r_{io}) \geqq p_i$$

is equivalent to the constraint

$$\overline{r}_{io} - \overline{r}_{i1}x_1 - \overline{r}_{i2}x_2 - \ldots - r_{in}x_n \geqq s_{io}F^{-1}(p_i)$$

In general case the solution set of constraints (3.4) is not convex. If r_{ij} ($j = 0, 1, \ldots, n$) have a joint $n + 1$-dimensional normal distribution, then the solution set of the constraint

$$P\,(r_{i1}x_1 + r_{i2}x_2 + \ldots + r_{in}x_n \geqq r_{io}) \geqq p_i \geqq 0.5$$

is convex [7]. If $p_i < 0.5$ the convexity cannot be proved. It is therefore not easy to solve the general chance-constrained programming problem.

4. SIMPLE RECOURSE MODEL

Because of the nature of the stochastic restrictions, in some cases they cannot be expressed by (3.4) if the model corresponds with a business process. The deviation from the equation

$$r_{i1}x_1 + r_{i2}x_2 + \ldots + r_{in}x_n = r_{io}, \qquad i \in S$$

can often be compensated. If the cost of the compensation in such a stochastic constraint is proportional with the deviation from the equation, we get the simple recourse model [7], [13] in the form

$$min\,E\,(\sum_j c_j x_j + \sum_{i \in S} (p_i u_i + q_i|v_i|)) \tag{4.1}$$

subject to non-negative variables, (3.3) and

$$u_i - v_i = r_{io} - r_{i1}x_1 - r_{i2}x_2 - \ldots - r_{in}x_n = h_i, \qquad i \in S, \tag{4.2}$$

where

x_j is the quantity of the j-th production activity,
c_j is the unit cost of the j-th production activity,
p_i is the unit compensation cost if $h_i > 0$,
q_i is the unit compensation cost if $h_i < 0$,
$u_i = h_i$ if $h_i \geqq 0$,
$v_i = h_i$ if $h_i \leqq 0$,
r_{io} is the available stochastic capacity and
r_{ij} is the stochastic normative.

If r_{ij} ($j = 0, 1, \ldots, n$) are independent normal random variables, the expected value z in (4.1) can be achieved considering (3.5), (3.6) and (3.8) [10].

$$z = \sum_{j=1}^{n} \bar{c}_j x_j + \sum_{i \in S} (p_i \bar{h}_i + (p_i + q_i) \left(\frac{s_i}{\sqrt{2\pi}} exp\left(- \frac{\bar{h}_i^2}{2s_i^2} \right) \right.$$

$$\left. \left. - \bar{h}_i F \left(- \frac{\bar{h}_i}{s_i} \right) \right) \right). \quad (4.3)$$

Consider (3.10) in (4.3) we get the problem (3.1)—(3.2) in the form

$$max \left(\sum_{j=1}^{n} \bar{c}_j x_j + \sum_{i \in S} (p_i \bar{h}_i + (p_i + q_i) \right.$$

$$\left. \left(\sqrt{\frac{y_i}{2\pi}} exp \left(- \frac{\bar{h}_i^2}{2y_i} \right) - \bar{h}_i F \left(- \frac{\bar{h}_i}{\sqrt{y_i}} \right) \right) \right) \right) \quad (4.4)$$

subject to $x_j \geqq 0$ $(j = 1, \ldots, n)$, (3.3), (3.5) and (3.10). All constraints except (3.10) are linear and (3.10) is separable. Since (4.4) is presented as a sum of functions of two variables, (2.1) can be used. In general cases we thus get a large-scale mixed integer programming problem. The piecewise linear approxmation can be useful in special cases which are important for applications.

Received: 25. 10. 1985.
Revised: 6. 12. 1985.

REFERENCES

[1] Arih, L., Primjena metoda višekriterijalnog programiranja na višefazni proizvodni proces. Doktorska disertacija, Ekonomski fakultet Zagreb, Maribor, 1984.

[2] Charnes, A. and W. W. Cooper, Chance-Constrained Programming. Management Science 6, 1959, 73—79.

[3] Dück, W., Diskrete Optimierung. Vieweg, Braunschweig, 1977.

[4] Hadley, G., Nonlinear and Dynamic Programming. Addison-Wesley, Reading, Massachusetts, 1964.

[5] Healy, W. C., Multiple Choice Programming. Operations Research 12, 1964, 122—138.

[6] Indihar, S., Multilineare Programmierung. Proceedings in Operations Research 9, Physica Verlag, Würzburg—Wien, 1980, 490—496.

[7] Kall, P., Stochastic Linear Programming. Springer, Berlin—Heidelberg — New York, 1976.

[8] Martić, Lj., Nelinearno programiranje. Informator, Zagreb, 1973.

[9] Martić, Lj., Višekriterijalno programiranje. Informator, Zagreb, 1978.

[10] Meško, I., Computerprogramm für konvexe und stochastische lineare Optimierung. Operations Research Proceedings 1981, Springer, Berlin—Heidelberg—New York, 1982, 595—601.

[11] Meško, I., Mješovit cjelobrojni model za optimalizaciju poslovanja. SYM-OP-IS '85, Herceg Novi, 1985., 51—58.

[12] Mitra, G., Theorie and Application of Mathematical Programming. Academic Press, London—New York—San Francisco, 1976.

[13] Wets, R., Stochastic Programming: Solution Techniques and Approximation Schemes. Mathematical Programming, The State of the Art, Bonn 1982, Springer, Berlin—Heidelberg—New York — Tokyo, 1983, 566—603.

REŠEVANJE NEKATERNIH PROBLEMOV OPTIMIRANJA Z ODSEKOMA LINEARNIMI APROKSIMACIJAMI

Ivan MEŠKO

Rezime

Odsekoma linearno funkcijo več spremenljivk je mogoče podobno kot odsekoma linearno funkcijo ene spremenljivke eksplicitno izraziti s pomočjo 0—1 spremenljivk in dodatnih omejitev. Funkcija treh spremenljivk je izražena v obliki (2.18) pri pogojih (2.2) — (2.17).

V posebnih primerih je mogoče nelinearno funkcijo več spremenljivk izraziti s separabilnimi funkcijami. Taka izražava je prikazana za produkt več linearnih funkcij in za količnik dveh linearnih funkcij. Zato je mogoče nekatere modele za večkrterijsko in multilinearno optimiranje aproksimirati z mešanim celoštevilskim linearnim modelom. Z uporabo navedenih izražav je mogoče aproksimativno reševati v praksi najpogosteje nastopajoče probleme stohastičnega optimiranja.