Hom-Jordan and Hom-alternative bimodules

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Abstract: In this paper, Hom-Jordan and Hom-alternative bimodules are introduced. It is shown that Jordan and alternative bimodules are twisted via endomorphisms into Hom-Jordan and Hom-alternative bimodules respectively. Some relations between Hom-associative bimodules, Hom-Jordan and Hom-alternative bimodules are given.

Key words: Bimodules, alternative algebras, Jordan algebras, Hom-alternative algebras, Hom-Jordan algebras, Hom-associative algebras.

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1. Introduction

Algebras where the identities defining the structure are twisted by a homomorphism are called Hom-algebras. They have been intensively investigated in the literature recently. Hom-algebra started from Hom-Lie algebras introduced and discussed in [6, 10, 11, 12], motivated by quasi-deformations of Lie algebras of vector fields, in particular q-deformations of Witt and Virasoro algebras. Hom-associative algebras were introduced in [15] while Hom-alternative and Hom-Jordan algebras are introduced in [14], [23] as twisted generalizations of alternative and Jordan algebra respectively. The reader is referred to [20] for applications of alternative algebras to projective geometry, buildings, and algebraic groups and to [4, 9, 16, 19] for discussions about the important roles of Jordan algebras in physics, especially quantum mechanics.

The anti-commutator of a Hom-alternative algebra gives rise to a Hom-Jordan algebra [23]. Starting with a Hom-alternative algebra $(A, \cdot, \alpha)$, it is known that the Jordan product

$$x * y = \frac{1}{2}(x \cdot y + y \cdot x)$$

gives a Hom-Jordan algebra $A^+ = (A, *, \alpha)$. In other words, Hom-alternative algebras are Hom-Jordan-admissible [23].

The notion of bimodule for a class of algebras defined by multilinear identities has been introduced by Eilenberg [3]. If $\mathcal{H}$ is in the class of associative algebras or in the one of Lie algebras then this notion is the familiar one for which we are in possession of well-worked theories. The study of bimodule (or representation) of Jordan algebras was initiated by N. Jacobson [7]. Subsequently the alternative case was considered by Schafer [17].

Modules over an ordinary algebra has been extended to the ones of Hom-algebras in many works [2, 18, 21, 22].

The aim of this paper is to introduce Hom-alternative bimodules and Hom-Jordan bimodules and to discuss about some findings. The paper is organized as follows. In section two, we recall basic notions related to Hom-algebras and modules over Hom-associative algebras. Section three is devoted to the introduction of Hom-alternative bimodules . Proposition 3.7 shows that from a given Hom-alternative bimodule, a sequence of this kind of bimodules can be obtained. Theorem 3.8 establishes that, an alternative bimodule gives rise to a bimodule over the corresponding twisted algebra. It is also proved that a direct sum of a Hom-alternative algebra and a module over this Hom-algebra is again a Hom-alternative algebra (Theorem 3.11). In section four, we introduce Hom-Jordan modules and attest similar results as in the previous section. Furthermore, it is proved that a Hom-Jordan special left and right module, with an additional condition, has a bimodule structure over this Hom-algebra (Theorem 4.10). Finally, Proposition 4.12 shows that a bimodule over a Hom-associative algebra has a bimodule structure over its plus Hom-algebra. All vector spaces are assumed to be over a fixed ground field $\mathbb{K}$ of characteristic 0.

2. Preliminaries

We recall some basic notions introduced in [6, 15, 21] related to Hom-algebras and while dealing of any binary operation we will use juxtaposition in order to reduce the number of braces, i.e., e.g., for "$\cdot$", $xy \cdot \alpha(z)$ means $(x \cdot y) \cdot \alpha(z)$. Also, for the map $\mu : A^{\otimes 2} \to A$, we will write sometimes $\mu(a \otimes b)$ as $\mu(a, b)$ or ab for $a, b \in A$ and if V is another vector space, $\tau_1 : A \otimes V \to V \otimes A$ (resp. $\tau_2 : V \otimes A \to A \otimes V$) denote the twist isomorphism $\tau_1(a \otimes v) = v \otimes a$ (resp. $\tau_2(v \otimes a) = a \otimes v$).

DEFINITION 2.1. A Hom-module is a pair (M, α_M) consisting of a $\mathbb{K}$-module M and a linear self-map $\alpha_M : M \to M$. A morphism $f : (M, \alpha_M) \to (N, \alpha_N)$ of Hom-modules is a linear map $f : M \to N$ such that $f \circ \alpha_M = \alpha_N \circ f$.

DEFINITION 2.2. ([15, 21]) A Hom-algebra is a triple (A, μ_A, α_A) in which (A, α_A) is a Hom-module, $\mu : A^{\otimes 2} \to A$ is a linear map. The Hom-algebra (A, μ, α) is said to be multiplicative if $\alpha \circ \mu = \mu \circ \alpha^{\otimes 2}$ (multiplicativity). A morphism $f : (A, \mu_A, \alpha_A) \to (B, \mu_B, \alpha_B)$ of Hom-algebras is a morphism of the underlying Hom-modules such that $f \circ \mu_A = \mu_B \circ f^{\otimes 2}$.

An important class of Hom-algebras that is considered here is the one of Hom-alternative algebras. These algebras have been introduced in [14] and more studied in [23].

DEFINITION 2.3. Let (A, μ, α) be a Hom-algebra.

(i) The Hom-associator of A is the linear map $as_A : A^{\otimes 3} \to A$ defined as $as_A = \mu \circ (\mu \otimes \alpha - \alpha \otimes \mu)$. A multiplicative Hom-algebra (A, μ, α) is said to be Hom-associative algebra if $as_A = 0$.

(ii) A Hom-alternative algebra [14] is a multiplicative Hom-algebra (A, μ, α) that satisfies

$$as_A(x, x, y) = 0 \qquad \text{(left Hom-alternativity)}, \qquad (1)$$

$$as_A(x, y, y) = 0 \qquad \text{(right Hom-alternativity)} \qquad (2)$$

for all $x, y \in A$.

(iii) Let (A, μ, α) be a Hom-alternative algebra. A Hom-subalgebra of (A, μ, α) is a linear subspace H of A, which is closed for the multiplication μ and invariant by α, that is, $\mu(x, y) \in H$ and $\alpha(x) \in H$ for all $x, y \in H$. If furthermore $\mu(a, b) \in H$ and $\mu(b, a) \in H$ for all $(a, b) \in A \times H$, then H is called a two-sided Hom-ideal of A.

Now, we prove:

PROPOSITION 2.4. Let (A, μ, α) be a Hom-alternative algebra and I be a two-sided Hom-ideal of (A, μ, α). Then $(A/I, \bar{\mu}, \bar{\alpha})$ is a Hom-alternative algebra where $\bar{\mu}(\bar{x}, \bar{y}) = \overline{\mu(x, y)}$ and $\bar{\alpha}(\bar{x}) = \overline{\alpha(x)}$ for all $\bar{x}, \bar{y} \in A/I$.

Proof. First, note that the multiplicativity of $\bar{\mu}$ with respect to $\bar{\alpha}$ follows from the one of μ with respect to α. Next, pick $\bar{x}, \bar{y} \in A/I$. Then the left Hom-alternativity (1) in $(A/I, \bar{\mu}, \bar{\alpha})$ is proved as follows

$$as_{A/I}(\bar{x}, \bar{x}, \bar{y}) = \bar{\mu}(\bar{\mu}(\bar{x}, \bar{x}), \bar{\alpha}(\bar{y})) - \bar{\mu}(\bar{\alpha}(\bar{x}), \bar{\mu}(\bar{x}, \bar{y}))$$

$$= \overline{\mu(\mu(x, x)\alpha(y)) - \mu(\alpha(x), \mu(x, y))} = \overline{as_A(x, x, y))} = \bar{0}.$$

Similarly, we get (2) and therefore $(A/I, \bar{\mu}, \bar{\alpha})$ is a Hom-alternative algebra. ∎

As Hom-alternative algebras, Hom-Jordan algebras are fundamental objects of this paper. They appear as cousins of Hom-alternative algebras and these two Hom-algebras are related as Jordan and alternative algebras.

DEFINITION 2.5. ([23])   (i) A Hom-Jordan algebra is a multiplicative Hom-algebra (A, μ, α) such that $\mu \circ \tau = \mu$ (commutativity of μ) and the so-called Hom-Jordan identity holds

$$as_A(\mu(x, x,), \alpha(y), \alpha(x)) = 0, \forall\ (x, y) \in A^2 \qquad (3)$$

where, $\tau : A^{\otimes 2} \to A^{\otimes 2}$, $\tau(a \otimes b) = b \otimes a$, is the twist isomorphism.

(ii) Let (A, μ, α) be a Hom-Jordan algebra. A Hom-subalgebra of (A, μ, α) is a linear subspace H of A, which is closed for the multiplication μ and invariant by α, that is, $\mu(x, y) \in H$ and $\alpha(x) \in H$ for all $x, y \in H$. If furthermore $\mu(a, b) \in H$ for all $(a, b) \in A \times H$, then H is called a two-sided Hom-ideal (or simply Hom-ideal) of A [5].

Similarly as a Hom-alternative algebra case, if H is a Hom-ideal of a Hom-Jordan algebra (A, μ, α), then $(A/H, \bar{\mu}, \bar{\alpha})$ is a Hom-Jordan algebra where $\bar{\mu}(\bar{x}, \bar{y}) = \overline{\mu(x, y)}$ for all $\bar{x}, \bar{y} \in A/H$ and $\bar{\alpha} : A/H \to A/H$ is naturally induced by α, inherits a Hom-Jordan algebra structure, which is named quotient Hom-Jordan algebra.

Remark 2.6. In [14] Makhlouf defined a Hom-Jordan algebra as a commutative multiplicative Hom-algebra satisfying $as_A(x^2, y, \alpha(x)) = 0$, which becomes the identity (3) if y is replaced by $\alpha(y)$.

The proof of the following result can be found in [23] where the product $*$, differs from the one given here by a factor of $\frac{1}{2}$.

PROPOSITION 2.7. *Let (A, μ, α) be a Hom-alternative algebra. Then $A^+ = (A, *, \alpha)$ is a Hom-Jordan algebra where $x * y = xy + yx$ for all $x, y \in A$.*

EXAMPLE 2.8. From the eight-dimensional Hom-alternative algebra $O_\alpha = (O, \mu_\alpha, \alpha)$ with basis $\{e_0, e_{1,2}, e_3, e_4, e_5, e_6, e_7\}$ [23, Example 3.19], constructed from the octonion algebra which is an eight-dimensional alternative algebra, we obtain, the Hom-Jordan algebra $O_\alpha^+ = (O, * = \mu_\alpha + \mu_\alpha \circ \tau, \alpha)$ where the non zero products are: $e_0 * e_0 = 2e_0$, $e_0 * e_1 = e_1 * e_0 = 2e_5$, $e_0 * e_2 = e_2 * e_0 = 2e_6$, $e_0 * e_3 = e_3 * e_0 = 2e_7$, $e_0 * e_4 = e_4 * e_0 = 2e_1$, $e_0 * e_5 = e_5 * e_0 = 2e_2$, $e_0 * e_6 = e_6 * e_0 = 2e_3$, $e_0 * e_7 = e_7 * e_0 = 2e_4$, $e_1 * e_1 = e_2 * e_2 = e_3 * e_3 =$

$e_4 * e_4 = e_5 * e_5 = e_6 * e_6 = e_7 * e_7 = -2e_0$ and the twisting map α is given by $\alpha(e_0) = e_0$, $\alpha(e_1) = e_5$, $\alpha(e_2) = e_6$, $\alpha(e_3) = e_7$, $\alpha(e_4) = e_1$, $\alpha(e_5) = e_2$, $\alpha(e_6) = e_3$, $\alpha(e_7) = e_4$.

A. Makhlouf proved that the plus algebra of any Hom-associative algebra is a Hom-Jordan algebra as defined in [14]. Here, we prove the same result for the Hom-Jordan algebra as defined in [23] (see also Definition 2.5 above).

PROPOSITION 2.9. *Let $(A, \cdot, \alpha)$ be a Hom-associative algebra. Then $A^+ = (A, *, \alpha)$ is a Hom-Jordan algebra where $x * y = xy + yx$ for all $x, y \in A$.*

Proof. The commutativity of $*$ is obvious. We compute the Hom-Jordan identity as follows:

$$as_{A^+}\big(x^2, \alpha(x), \alpha(y)\big)$$
$$= (x^2 * \alpha(y)) * \alpha^2(x) - \alpha(x^2) * (\alpha(y) * \alpha(x))$$
$$= (x^2 \cdot \alpha(y)) \cdot \alpha^2(x) + (\alpha(y) \cdot x^2) \cdot \alpha^2(x) + \alpha^2(x) \cdot (x^2 \cdot \alpha(y))$$
$$\quad + \alpha^2(x) \cdot (\alpha(y) \cdot x^2) - \alpha(x^2) \cdot (\alpha(y) \cdot \alpha(x)) - \alpha(x^2) \cdot (\alpha(x) \cdot \alpha(y))$$
$$\quad - (\alpha(y) \cdot \alpha(x)) \cdot \alpha(x^2) - (\alpha(x) \cdot \alpha(y)) \cdot \alpha(x^2) \text{ (by a direct computation)}$$
$$= (\alpha(y) \cdot x^2) \cdot \alpha^2(x) + \alpha^2(x) \cdot (x^2 \cdot \alpha(y)) - \alpha(x^2) \cdot (\alpha(x) \cdot \alpha(y))$$
$$\quad - (\alpha(y) \cdot \alpha(x)) \cdot \alpha(x^2) \quad \text{(by the Hom-associativity)}$$
$$= (\alpha(y) \cdot x^2) \cdot \alpha^2(x) + \alpha^2(x) \cdot (x^2 \cdot \alpha(y)) - (\alpha(x) \cdot \alpha(x)) \cdot \alpha(x \cdot y)$$
$$\quad - \alpha(yx) \cdot (\alpha(x) \cdot \alpha(x)) \quad \text{(by the multiplicativity)}$$
$$= (\alpha(y) \cdot x^2) \cdot \alpha^2(x) + \alpha^2(x) \cdot (x^2 \cdot \alpha(y)) - \alpha^2(x) \cdot (\alpha(x) \cdot (x \cdot y))$$
$$\quad - ((yx) \cdot \alpha(x)) \cdot \alpha^2(x) \quad \text{(by the Hom-associativity)}$$
$$= 0 \quad \text{(by the Hom-associativity)} .$$

Then $A^+ = (A, *, \alpha)$ is a Hom-Jordan algebra. ∎

EXAMPLES 2.10. (i) Consider the three-dimensional Hom-associative algebra $\mathcal{A} = (A, \mu_A, \alpha_A)$ over $\mathbb{K}$ with basis (e_1, e_2, e_3) defined by $\mu_A(e_1, e_1) = e_1$, $\mu_A(e_2, e_2) = e_2$, $\mu_A(e_3, e_3) = e_1$, $\mu_A(e_1, e_3) = \mu_A(e_3, e_1) = -e_3$ and $\alpha_A(e_1) = e_1$, $\alpha_A(e_3) = -e_3$ (see [24, Theorem 3.12], Hom-algebra A'^3_3). Using the product $*$ in Proposition 2.9, the triple $\mathcal{A}^+ = (A, *, \alpha_A)$ is Hom-Jordan algebra where, $e_1 * e_1 = 2e_1$, $e_2 * e_2 = 2e_2$, $e_3 * e_3 = 2e_1$, $e_1 * e_3 = e_3 * e_1 = -2e_3$.

(ii) From the three-dimensional Hom-associative algebra $\mathcal{B} = (B, \mu_B, \alpha_B)$ over $\mathbb{K}$ with basis (e_1, e_2, e_3) defined by $\mu_B(e_1, e_1) = e_1$, $\mu_B(e_2, e_2) = e_1$, $\mu_B(e_3, e_3) = e_3$, $\mu_B(e_1, e_2) = \mu_B(e_2, e_1) = -e_2$ and $\alpha_B(e_1) = e_1$, $\alpha_B(e_2) = -e_2$ (see [24, Theorem 3.12], Hom-algebra A'^3_5). Then the triple $\mathcal{B}^+ = (B, *, \alpha_B)$ is a Hom-Jordan algebra, where "$*$" is the product in Proposition 2.9 and $e_1 * e_1 = 2e_1$, $e_2 * e_2 = 2e_1$, $e_3 * e_3 = 2e_3$, $e_1 * e_2 = e_2 * e_1 = -2e_2$.

Let us consider the following definitions which will be used in next sections.

DEFINITION 2.11. Let (A, μ, α_A) be any Hom-algebra.

(i) A Hom-module (V, α_V) is called an A-bimodule if it comes equipped with a left and a right structure maps on V that is morphisms $\rho_l : A \otimes V \to V$, $a \otimes v \mapsto a \cdot v$ and $\rho_r : V \otimes A \to V$, $v \otimes a \mapsto v \cdot a$ of Hom-modules respectively.

(ii) A morphism $f : (V, \alpha_V, \rho_l, \rho_r) \to (W, \alpha_W, \rho'_l, \rho'_r)$ of A-bimodules is a morphism of the underlying Hom-modules such that

$$f \circ \rho_l = \rho'_l \circ (Id_A \otimes f) \qquad \text{and} \qquad f \circ \rho_r = \rho'_r \circ (f \otimes Id_A).$$

(iii) Let (V, α_V) be an A-bimodule with structure maps ρ_l and ρ_r. Then the module Hom-associator of V is a trilinear map $as_{A,V}$ defined as:

$$as_{A,V} \circ Id_{V \otimes A \otimes A} = \rho_r \circ (\rho_r \otimes \alpha_A) - \rho_l \circ (\alpha_V \otimes \mu),$$
$$as_{A,V} \circ Id_{A \otimes V \otimes A} = \rho_r \circ (\rho_l \otimes \alpha_A) - \rho_l \circ (\alpha_A \otimes \rho_r),$$
$$as_{A,V} \circ Id_{A \otimes A \otimes V} = \rho_l \circ (\mu \otimes \alpha_V) - \rho_l \circ (\alpha_A \otimes \rho_l).$$

Remark 2.12. The module Hom-associator given above is a generalization of the one given in [2].

Now, let consider the following notion for Hom-associative algebras.

DEFINITION 2.13. Let (A, μ, α_A) be a Hom-associative algebra and (M, α_M) be a Hom-module.

(i) A left Hom-associative A-module structure on M consists of a morphism $\rho : A \otimes M \to M$ of Hom-modules, such that

$$\rho \circ (\alpha_A \otimes \rho) = \rho \circ (\mu \otimes \alpha_M) \tag{4}$$

(ii) A right Hom-associative A-module structure on M consists of a morphism $\rho : M \otimes A \to M$ of Hom-modules, such that

$$\rho \circ (\alpha_M \otimes \mu) = \rho \circ (\rho \otimes \alpha_A) \tag{5}$$

(iii) A Hom-associative A-bimodule structure on M consists of two structure maps $\rho_l : A \otimes M \to M$ and $\rho_r : M \otimes A \to M$ such that (M, α_M, ρ_l) is a left A-module, (M, α_M, ρ_r) is a right A-module and that the following Hom-associativity (or operator commutativity) condition holds:

$$\rho_l \circ (\alpha_A \otimes \rho_r) = \rho_r \circ (\rho_l \otimes \alpha_A) \tag{6}$$

Remark 2.14. Actually, left Hom-associative A-module, right Hom-associative A-module and Hom-associative A-bimodule have been already introduced in [21, 22] where they are called left A-module, right A-module and A-bimodule respectively. The expressions, used in Definition 2.13 for these notions, are motivated by the unification of our terminologies.

3. HOM-ALTERNATIVE BIMODULES

In this section, we give the definition of Hom-alternative (bi)modules. We prove that from a given Hom-alternative bimodule, a sequence of this kind of bimodules can be constructed. It is also proved that a direct sum of a Hom-alternative algebra and a bimodule over this Hom-algebra is a Hom-alternative algebra called a split null extension of the considered Hom-algebra.

First, we start by the following notion, due to [2], where it is called a module over a left (resp. right) Hom-alternative algebra. However, we call it a Hom-alternative left (resp. right) module in this paper.

DEFINITION 3.1. Let (A, μ, α_A) be a Hom-alternative algebra.

(i) A left Hom-alternative A-module is a Hom-module (V, α_V) with a left structure map $\rho_l : A \otimes V \to V$, $a \otimes v \mapsto a \cdot v$ such that

$$as_{A,V}(x, y, v) = -as_{A,V}(y, x, v) \qquad \text{for all } x, y \in A \text{ and } v \in V.$$

(ii) A right Hom-alternative A-module is a Hom-module (V, α_V) with a right structure map $\rho_r : V \otimes A \to V$, $v \otimes a \mapsto v \cdot a$ such that

$$as_{A,V}(v, x, y) = -as_{A,V}(v, y, x) \qquad \text{for all } x, y \in A \text{ and } v \in V.$$

Now, as a generalization of alternative bimodules [8, 17], one has:

DEFINITION 3.2. Let (A, μ, α_A) be a Hom-alternative algebra. A Hom-alternative A-bimodule is a Hom-module (V, α_V) with a (left) structure map $\rho_l : A \otimes V \to V$, $a \otimes v \mapsto a \cdot v$ and a (right) structure map $\rho_r : V \otimes A \to V$, $v \otimes a \mapsto v \cdot a$ such that the following equalities hold:

$$as_{A,V}(a, v, b) = -as_{A,V}(v, a, b) = as_{A,V}(b, a, v) = -as_{A,V}(a, b, v) \qquad (7)$$

for all $(a, b, v) \in A^{\times 2} \times V$.

Remarks 3.3.   (i) The relation (7) is equivalent to

$$as_{A,V}(a, v, b) = -as_{A,V}(v, a, b) = as_{A,V}(b, a, v) = -as_{A,V}(b, v, a)$$

or since the field's characteristic is 0 to

$$as_{A,V}(a, v, b) = -as_{A,V}(v, a, b) = as_V(b, a, v) \quad \text{and} \quad as_{A,V}(a, a, v) = 0 \,.$$

(ii) If $\alpha_A = Id_A$ and $\alpha_V = Id_V$ then V is the so-called alternative bimodule for the alternative algebra (A, μ) [8, 17].

EXAMPLES 3.4. Here are some examples of Hom-alternative A-bimodules.
(i) Let (A, μ, α_A) be a Hom-alternative algebra. Then (A, α_A) is a Hom-alternative A-bimodule where the structure maps are $\rho_l(a, b) = \mu(a, b)$ and $\rho_r(a, b) = \mu(b, a)$. More generally, if B is a two-sided Hom-ideal of (A, μ, α_A), then (B, α_A) is a Hom-alternative A-bimodule where the structure maps are $\rho_l(a, x) = \mu(a, x)$ and $\rho_r(x, b) = \mu(x, b)$ for all $x \in B$ and $(a, b) \in A^{\times 2}$.
(ii) If (A, μ) is an alternative algebra and M is an alternative A-bimodule [8] in the usual sense, then (M, Id_M) is a Hom-alternative $\mathbb{A}$-bimodule where $\mathbb{A} = (A, \mu, Id_A)$ is a Hom-alternative algebra.
(iii) If $f : (A, \mu_A, \alpha_A) \to (B, \mu_B, \alpha_B)$ is a surjective morphism of Hom-alternative algebras, then (B, α_B) becomes a Hom-alternative A-bimodule via f, i.e, the structure maps are defined as $\rho_l : (a, b) \mapsto \mu_B(f(a), b)$ and $\rho_r : (b, a) \mapsto \mu_B(b, f(a))$ for all $(a, b) \in A \times B$. Indeed one can remark that $as_{A,B} \circ (Id_A \otimes f \otimes Id_A) = f \circ as_A$.

In order to give another example of Hom-alternative bimodules , let us consider the following

DEFINITION 3.5. An abelian extension of Hom-alternative algebras is a short exact sequence of Hom-alternative algebras

$$0 \to (V, \alpha_V) \xrightarrow{i} (A, \mu_A, \alpha_A) \xrightarrow{\pi} (B, \mu_B, \alpha_B) \to 0$$

where (V, α_V) is a trivial Hom-alternative algebra, i and π are morphisms of Hom-algebras. Furthermore, if there exists a morphism $s : (B, \mu_B, \alpha_B) \to (A, \mu_A, \alpha_A)$ such that $\pi \circ s = id_B$ then the abelian extension is said to be split and s is called a section of π.

EXAMPLE 3.6. Given an abelian extension as in the previous definition, the Hom-module (V, α_V) inherits a structure of a Hom-alternative B-bimodule and the actions of the Hom-algebra (B, μ_B, α_B) on V are as follows. For any $x \in B$, there exist $\tilde{x} \in A$ such that $x = \pi(\tilde{x})$. Let x acts on $v \in V$ by $x \cdot v := \mu_A(\tilde{x}, i(v))$ and $v \cdot x := \mu_A(i(v), \tilde{x})$. These are well-defined, as another lift $\tilde{x}'$ of x is written $\tilde{x}' = \tilde{x} + v'$ for some $v' \in V$ and thus $x \cdot v = \mu_A(\tilde{x}, i(v)) = \mu_A(\tilde{x}', i(v))$ and $v \cdot x = \mu_A(i(v), \tilde{x}) = \mu_A(i(v), \tilde{x}')$ because V is trivial. The actions property follow from the Hom-alternativity identity. In case these actions of B on V are trivial, one speaks of a central extension.

The following result describes a sequence of Hom-alternative bimodules by twisting the structure maps of a given bimodule over this Hom-algebra.

PROPOSITION 3.7. Let (A, μ, α_A) be a Hom-alternative algebra and (V, α_V) be a Hom-alternative A-bimodule with the structure maps ρ_l and ρ_r. Then the maps

$$\rho_l^{(n)} = \rho_l \circ (\alpha_A^n \otimes Id_V)$$
$$\rho_r^{(n)} = \rho_r \circ (Id_V \otimes \alpha_A^n)$$

give the Hom-module (V, α_V) the structure of a Hom-alternative A-bimodule that we denote by $V^{(n)}$

Proof. It is clear that $\rho_l^{(n)}$ and $\rho_r^{(n)}$ are structure maps on $V^{(n)}$. Next, observe that for all $x, y \in A$ and $v \in V$,

$$\begin{aligned}
as_{A,V^{(n)}}(x, v, y) &= \rho_r^{(n)}(\rho_l^{(n)}(x, v), \alpha_A(y)) - \rho_l^{(n)}(\alpha_A(x), \rho_r^{(n)}(v, y)) \\
&= \rho_r(\rho_l(\alpha_A^n(x), v), \alpha_A^{n+1}(y)) - \rho_l(\alpha_A^{n+1}(x), \rho_r(v, \alpha_A^n(y))) \\
&= as_{A,V}(\alpha_A^n(x), v, \alpha_A^n(y))
\end{aligned}$$

and similarly

$$as_{A,V^{(n)}}(v,x,y) = as_{A,V}(v,\alpha_A^n(x),\alpha_A^n(y)),$$
$$as_{A,V^{(n)}}(y,x,v) = as_{A,V}(\alpha_A^n(y),\alpha_A^n(x),v),$$
$$as_{A,V^{(n)}}(x,y,v) = as_{A,V}(\alpha_A^n(x),\alpha_A^n(y),v).$$

Therefore, equalities of (7) in $V^{(n)}$ derive from the one in V. ∎

We know that alternative algebras can be deformed into Hom-alternative algebras via an endomorphism. The following result shows that alternative bimodules can be deformed into Hom-alternative bimodules via an endomorphism. This provides a large class of examples of Hom-alternative bimodules.

THEOREM 3.8. *Let (A,μ) be an alternative algebra, V be an alternative A-bimodule with the structure maps ρ_l and ρ_r, α_A be an endomorphism of the alternative algebra A and α_V be a linear self-map of V such that $\alpha_V \circ \rho_l = \rho_l \circ (\alpha_A \otimes \alpha_V)$ and $\alpha_V \circ \rho_r = \rho_r \circ (\alpha_V \otimes \alpha_A)$.*
Write A_{α_A} for the Hom-alternative algebra $(A,\mu_{\alpha_A},\alpha_A)$ and V_{α_V} for the Hom-module (V,α_V). Then the maps

$$\tilde{\rho}_l = \alpha_V \circ \rho_l \qquad and \qquad \tilde{\rho}_r = \alpha_V \circ \rho_r$$

give the Hom-module V_{α_V} the structure of a Hom-alternative A_{α_A}-bimodule.

Proof. Trivially, $\tilde{\rho}_l$ and $\tilde{\rho}_r$ are structure maps on V_{α_V}. The proof of (7) for V_{α_V} follows directly by the fact that $as_{A,V_{\alpha_V}} = \alpha_V^2 \circ as_{A,V}$ and the relation (7) in V. ∎

COROLLARY 3.9. *Let (A,μ) be an alternative algebra, V be an alternative A-bimodule with the structure maps ρ_l and ρ_r, α_A an endomorphism of the alternative algebra A and α_V be a linear self-map of V such that $\alpha_V \circ \rho_l = \rho_l \circ (\alpha_A \otimes \alpha_V)$ and $\alpha_V \circ \rho_r = \rho_r \circ (\alpha_V \otimes \alpha_A)$.*
Write A_{α_A} for the Hom-alternative algebra $(A,\mu_{\alpha_A},\alpha_A)$ and V_{α_V} for the Hom-module (V,α_V). Then the maps

$$\tilde{\rho}_l^{(n)} = \rho_l \circ \left(\alpha_A^{n+1} \otimes \alpha_V\right) \qquad and \qquad \tilde{\rho}_r^{(n)} = \rho_r \circ \left(\alpha_V \otimes \alpha_A^{n+1}\right)$$

give the Hom-module V_α the structure of a Hom-alternative A_{α_A}-bimodule for each $n \in \mathbb{N}$.

LEMMA 3.10. *Let (A, μ, α_A) be a Hom-alternative algebra and (V, α_V) be a Hom-alternative A-bimodule with the structure maps ρ_l and ρ_r. Then the following relation*

$$as_{A,V}(v, a, a) = 0 \qquad (8)$$

holds for all $a \in A$ and $v \in V$.

Proof. Using (7), for all $(a, b) \in A^{\times 2}$ and $v \in V$ we have $-as_{A,V}(v, a, b) = as_{A,V}(a, v, b)$ and $as_{A,V}(v, b, a) = -as_{A,V}(a, b, v)$. Moreover again from (7), we get $as_{A,V}(a, v, b) = -as_{A,V}(a, b, v)$ and then $-as_{A,V}(v, a, b) = as_{A,V}(v, b, a)$. It follows that $as_{A,V}(v, a, a) = 0$ since the field $\mathbb{K}$ is of characteristic 0. ∎

The following result shows that a direct sum of a Hom-alternative algebra and a bimodule over this Hom-algebra, is still a Hom-alternative, called the split null extension determined by the given bimodule.

THEOREM 3.11. *Let (A, μ, α_A) be a Hom-alternative algebra and (V, α_V) be a Hom-alternative A-bimodule with the structure maps ρ_l and ρ_r. Defining on $A \oplus V$ the bilinear map $\tilde{\mu} : (A \oplus V)^{\otimes 2} \to A \oplus V$, $\tilde{\mu}(a+m, b+n) := ab + a \cdot n + m \cdot b$ and the linear map $\tilde{\alpha} : A \oplus V \to A \oplus V$, $\tilde{\alpha}(a+m) := \alpha_A(a) + \alpha_V(m)$, then $E = (A \oplus V, \tilde{\mu}, \tilde{\alpha})$ is a Hom-alternative algebra.*

Proof. The multiplicativity of $\tilde{\alpha}$ with respect to $\tilde{\mu}$ follows from the one of α with respect to μ and the fact that ρ_l and ρ_r are morphisms of Hom-modules. Next

$$
\begin{aligned}
as_E(a + m, &a + m, b + n) \\
&= \tilde{\mu}(\tilde{\mu}(a+m, a+m), \tilde{\alpha}(b+n)) - \tilde{\mu}(\tilde{\alpha}(a+m), \tilde{\mu}(a+m, b+n)) \\
&= \tilde{\mu}(a^2 + a \cdot m + m \cdot a, \alpha_A(b) + \alpha_V(n)) \\
&\quad - \tilde{\mu}(\alpha_A(a) + \alpha_V(m), ab + a \cdot n + m \cdot b) \\
&= a^2 \alpha_A(b) + a^2 \cdot \alpha_V(n) + (a \cdot m) \cdot \alpha_A(b) + (m \cdot a) \cdot \alpha_A(b) \\
&\quad - \alpha_A(a)(ab) - \alpha_A(a) \cdot (a \cdot n) - \alpha_A(a) \cdot (m \cdot b) - \alpha_V(m) \cdot (ab) \\
&= \underbrace{as_A(a, a, b)}_{0} + \underbrace{as_V(a, a, n)}_{0} + \underbrace{as_{A,V}(a, m, b) + as_{A,V}(m, a, b)}_{0} \\
&\qquad \text{(by (1), Remarks 3.3 and (7))} \\
&= 0.
\end{aligned}
$$

Similarly, we compute

$$
\begin{aligned}
as_E(a + &m, b + n, b + n) \\
&= \tilde{\mu}(\tilde{\mu}(a + m, b + n), \tilde{\alpha}(b + n)) - \tilde{\mu}(\tilde{\alpha}(a + m), \tilde{\mu}(b + n, b + n)) \\
&= \tilde{\mu}(ab + a \cdot n + m \cdot b, \alpha_A(b) + \alpha_V(n)) \\
&\quad - \tilde{\mu}(\alpha_A(a) + \alpha_V(m), b^2 + b \cdot n + b \cdot b) \\
&= (ab)\alpha_A(b) + (ab) \cdot \alpha_V(m) + (a \cdot n) \cdot \alpha_A(b) + (m \cdot b) \cdot \alpha_A(b) \\
&\quad - \alpha_A(a)(b^2) - \alpha_A(a) \cdot (b \cdot n) - \alpha_A(a) \cdot (n \cdot b) - \alpha_V(m) \cdot b^2 \\
&= \underbrace{as_A(a, b, b)}_{0} + \underbrace{as_{A,V}(a, b, n) + as_{A,V}(a, n, b)}_{0} + \underbrace{as_{A,V}(m, b, b)}_{0} \\
&\qquad \text{(by (2), (7) and (8))} \\
&= 0 \,.
\end{aligned}
$$

We then conclude that $(A \oplus V, \tilde{\mu}, \tilde{\alpha})$ is a Hom-alternative algebra. ∎

Remark 3.12. Consider the split null extension $A \oplus V$ determined by the Hom-alternative bimodule (V, α_V) of the Hom-alternative algebra (A, μ, α_A) in the previous theorem. Write elements $a + v$ of $A \oplus V$ as (a, v). Then, there is an injective homomorphism of Hom-modules $i : V \to A \oplus V$ given by $i(v) = (0, v)$ and a surjective homomorphism of Hom-modules $\pi : A \oplus V \to A$ given by $\pi(a, v) = a$. Moreover $i(V)$ is a two-sided Hom-ideal of $A \oplus V$ such that $A \oplus V / i(V) \cong A$. On the other hand, there is a morphism of Hom-algebras $\sigma : A \to A \oplus V$ given by $\sigma(a) = (a, 0)$ which is clearly a section of π. Hence, we obtain the abelian split exact sequence of Hom-alternative algebras and (V, α_V) is a Hom-alternative A-bimodule via π.

4. HOM-JORDAN BIMODULES

In this section, we study Hom-Jordan bimodules. It is observed that similar results for Hom-alternative bimodules hold for Hom-Jordan bimodules. Some of them require an additional condition. Furthermore, relations between Hom-associative bimodules and Hom-Jordan bimodules are given on the one hand, and on the other hand, relations between left (resp. right) Hom-alternative modules and left(resp. right) special Hom-Jordan modules are proved. First, we have:

DEFINITION 4.1. Let (A, μ, α_A) be a Hom-Jordan algebra.

(i) A right Hom-Jordan A-module is a Hom-module (V, α_V) with a right structure map $\rho_r : V \otimes A \to V$, $v \otimes a \mapsto v \cdot a$ such that the following conditions hold:

$$\alpha_V(v \cdot a) \cdot \alpha_A(bc) + \alpha_V(v \cdot b) \cdot \alpha_A(ca) + \alpha_V(v \cdot c) \cdot \alpha_A(ab)$$
$$= (\alpha_V(v) \cdot bc) \cdot \alpha_A^2(a) + (\alpha_V(v) \cdot ca) \cdot \alpha_A^2(b) \qquad (9)$$
$$+ (\alpha_V(v) \cdot ab) \cdot \alpha_A^2(c) \, ,$$

$$\alpha_V(v \cdot a) \cdot \alpha_A(bc) + \alpha_V(v \cdot b) \cdot \alpha_A(ca) + \alpha_V(v \cdot c) \cdot \alpha_A(ab)$$
$$= ((v \cdot a) \cdot \alpha_A(b)) \cdot \alpha_A^2(c) + ((v \cdot c) \cdot \alpha_A(b)) \cdot \alpha_A^2(a) \qquad (10)$$
$$+ \alpha_V^2(v) \cdot ((ac)\alpha_A(b))$$

for all $a, b, c \in A$ and $v \in V$.

(ii) A left Hom-Jordan A-module is a Hom-module (V, α_V) with a left structure map $\rho_l : A \otimes V \to V$, $a \otimes v \mapsto a \cdot v$ such that the following conditions hold:

$$\alpha_A(bc) \cdot \alpha_V(a \cdot v) + \alpha_A(ca) \cdot \alpha_V(b \cdot v) + \alpha_A(ab) \cdot \alpha_V(c \cdot v)$$
$$= \alpha_A^2(a) \cdot (bc \cdot \alpha_V(v)) + \alpha_A^2(b) \cdot (ca \cdot \alpha_V(v)) \qquad (11)$$
$$+ \alpha_A^2(c) \cdot (ab \cdot \alpha_V(v)) \, ,$$

$$\alpha_A(bc) \cdot \alpha_V(a \cdot v) + \alpha_A(ca) \cdot \alpha_V(b \cdot v) + \alpha_A(ab) \cdot \alpha_V(c \cdot v)$$
$$= \alpha_A^2(c) \cdot (\alpha_A(b) \cdot (a \cdot v)) + \alpha_A^2(a) \cdot (\alpha_A(b) \cdot (c \cdot v)) \qquad (12)$$
$$+ ((ac)\alpha_A(b)) \cdot \alpha_V^2(v)$$

for all $a, b, c \in A$ and $v \in V$.

The following result allows to introduce the notion of right special Hom-Jordan modules.

THEOREM 4.2. *Let (A, μ, α_A) be a Hom-Jordan algebra, (V, α_V) be a Hom-module and $\rho_r : V \otimes A \to V$, $a \otimes v \mapsto v \cdot a$, be a bilinear map satisfying*

$$\alpha_V \circ \rho_r = \rho_r \circ (\alpha_V \otimes \alpha_A) \qquad (13)$$

and

$$\alpha_V(v) \cdot (ab) = (v \cdot a) \cdot \alpha_A(b) + (v \cdot b) \cdot \alpha_A(a) \qquad (14)$$

for all $(a, b) \in A^{\times 2}$ and $v \in V$. Then (V, α, ρ_r) is a right Hom-Jordan A-module called a right special Hom-Jordan A-module.

Proof. It suffices to prove (9) and (10). For all $(a, b) \in A^{\times 2}$ and $v \in V$, we have:

$$\alpha_V(v \cdot a) \cdot \alpha_A(bc) + \alpha_V(v \cdot b) \cdot \alpha_A(ca) + \alpha_V(v \cdot c) \cdot \alpha_A(ab)$$

$$= \alpha_V(v \cdot a) \cdot \alpha_A(b)\alpha_A(c) + \alpha_V(v \cdot b) \cdot \alpha_A(c)\alpha_A(a)$$

$$+ \alpha_V(v \cdot c) \cdot \alpha_A(a)\alpha_A(b) \quad \text{(multiplicativity)}$$

$$= ((v \cdot a) \cdot \alpha_A(b)) \cdot \alpha^2(c) + ((v \cdot a) \cdot \alpha_A(c)) \cdot \alpha^2(b)$$

$$+ ((v \cdot b) \cdot \alpha_A(c)) \cdot \alpha^2(a) + ((v \cdot b) \cdot \alpha_A(a)) \cdot \alpha^2(c)$$

$$+ ((v \cdot c) \cdot \alpha_A(a)) \cdot \alpha^2(b) + ((v \cdot c) \cdot \alpha_A(b)) \cdot \alpha^2(a) \quad \text{(by (14))}$$

$$= [\alpha_V(v) \cdot ab - (v \cdot b)\alpha_A(a)] \cdot \alpha^2(c) + ((v \cdot a) \cdot \alpha_A(c)) \cdot \alpha^2(b)$$

$$+ [\alpha_V(v) \cdot bc - (v \cdot c)\alpha_A(b)] \cdot \alpha^2(a) + ((v \cdot b) \cdot \alpha_A(a)) \cdot \alpha^2(c)$$

$$+ [\alpha_V(v) \cdot ca - (v \cdot a) \cdot \alpha_A(c)] \cdot \alpha^2(b)$$

$$+ ((v \cdot c) \cdot \alpha_A(b)) \cdot \alpha^2(a) \quad \text{(again by (14))}$$

$$= (\alpha_V(v) \cdot bc) \cdot \alpha_A^2(a) + (\alpha_V(v) \cdot ca) \cdot \alpha_A^2(b) + (\alpha_V(v) \cdot ab) \cdot \alpha_A^2(c)$$

and thus, we get (9). Finally, (10) is proved as follows:

$$\alpha_V(v \cdot a) \cdot \alpha_A(bc) + \alpha_V(v \cdot b) \cdot \alpha_A(ca) + \alpha_V(v \cdot c) \cdot \alpha_A(ab)$$

$$= \alpha_V(v \cdot a) \cdot \alpha_A(b)\alpha_A(c) + \alpha_V(v \cdot b) \cdot \alpha_A(c)\alpha_A(a)$$

$$+ \alpha_V(v \cdot c) \cdot \alpha_A(a)\alpha_A(b) \quad \text{(multiplicativity)}$$

$$= ((v \cdot a) \cdot \alpha_A(b)) \cdot \alpha_A^2(c) + ((v \cdot a) \cdot \alpha_A(c)) \cdot \alpha_A^2(b)$$

$$+ ((v \cdot b) \cdot \alpha_A(c)) \cdot \alpha_A^2(a) + ((v \cdot b) \cdot \alpha_A(a)) \cdot \alpha_A^2(c)$$

$$+ ((v \cdot c) \cdot \alpha_A(a)) \cdot \alpha_A^2(b) + ((v \cdot c) \cdot \alpha_A(b)) \cdot \alpha_A^2(a) \quad \text{(by ((14))}$$

$$= ((v \cdot a) \cdot \alpha_A(b)) \cdot \alpha_A^2(c) + [\alpha_V(v) \cdot ac - ((v \cdot c) \cdot \alpha_A(a)] \cdot \alpha_A^2(b)$$

$$+ ((v \cdot b) \cdot \alpha_A(c)) \cdot \alpha_A^2(a) + ((v \cdot b) \cdot \alpha_A(a)) \cdot \alpha_A^2(c)$$

$$+ ((v \cdot c) \cdot \alpha_A(a)) \cdot \alpha_A^2(b) + ((v \cdot c) \cdot \alpha_A(b)) \cdot \alpha_A^2(a) \quad \text{(again by (14))}$$

$$= ((v \cdot a) \cdot \alpha_A(b)) \cdot \alpha_A^2(c) + \alpha_V^2(v) \cdot ((ac)\alpha_A(b))$$

$$- (\alpha_V(v) \cdot \alpha_A(b)) \cdot \alpha_A(ac) + ((v \cdot b) \cdot \alpha_A(c)) \cdot \alpha_A^2(a)$$

$$+ ((v \cdot b) \cdot \alpha_A(a)) \cdot \alpha_A^2(c) + ((v \cdot c) \cdot \alpha_A(b)) \cdot \alpha_A^2(a) \quad \text{(again by (14))}$$

$$= ((v \cdot a) \cdot \alpha_A(b)) \cdot \alpha_A^2(c) + \alpha_V^2(v) \cdot ((ac)\alpha_A(b)) - (\alpha_V(v \cdot b) \cdot \alpha_A(ac)$$
$$+ ((v \cdot b) \cdot \alpha_A(c)) \cdot \alpha_A^2(a) + ((v \cdot b) \cdot \alpha_A(a)) \cdot \alpha_A^2(c)$$
$$+ ((v \cdot c) \cdot \alpha_A(b)) \cdot \alpha_A^2(a) \quad \text{(by (13))}$$
$$= ((v \cdot a) \cdot \alpha_A(b)) \cdot \alpha_A^2(c) + \alpha_V^2(v) \cdot ((ac)\alpha_A(b)) - ((v \cdot b) \cdot \alpha_A(a)) \cdot \alpha_A^2(c)$$
$$- ((v \cdot b) \cdot \alpha_A(c)) \cdot \alpha_A^2(a) + ((v \cdot b) \cdot \alpha_A(c)) \cdot \alpha_A^2(a)$$
$$+ ((v \cdot b) \cdot \alpha_A(a)) \cdot \alpha_A^2(c) + ((v \cdot c) \cdot \alpha_A(b)) \cdot \alpha_A^2(a) \quad \text{(by (14))}$$
$$= ((v \cdot a) \cdot \alpha_A(b)) \cdot \alpha_A^2(c) + ((v \cdot c) \cdot \alpha_A(b)) \cdot \alpha_A^2(a) + \alpha_V^2(v) \cdot ((ac)\alpha_A(b))$$

which is (10). ∎

Similarly, the following result can be proved.

THEOREM 4.3. *Let* (A, μ, α_A) *be a Hom-Jordan algebra,* (V, α_V) *be a Hom-module and* $\rho_l : A \otimes V \to V$, $v \otimes a \mapsto a \cdot v$, *be a bilinear map satisfying*

$$\alpha_V \circ \rho_l = \rho_l \circ (\alpha_A \otimes \alpha_V)$$

and

$$(ab) \cdot \alpha_V(v) = \alpha_A(a) \cdot (b \cdot v) + \alpha_A(b) \cdot (a \cdot v) \tag{15}$$

for all $(a, b) \in A^{\times 2}$ *and* $v \in V$. *Then* (V, α, ρ_l) *is a left Hom-Jordan A-module called a left special Hom-Jordan A-module.*

It is well known that the plus algebra of any Hom-alternative algebra is a Hom-Jordan algebra. The next result shows that any left (resp. right) Hom-alternative module a is also a left (resp. right) module over its plus Hom-algebra.

PROPOSITION 4.4. *Let* (A, μ, α_A) *be a Hom-alternative algebra and* (V, α_V) *be a Hom-module.*

(i) *If* (V, α_V) *is a right Hom-alternative A-module with the structure map* ρ_r *then* (V, α_V) *is a right special Hom-Jordan* A^+*-module with the same structure map* ρ_r.

(ii) *If* (V, α_V) *is a left Hom-alternative A-module with the structure map* ρ_l *then* (V, α_V) *is a left special Hom-Jordan* A^+*-module with the same structure map* ρ_l.

Proof. It suffices to prove (14) and (15).

(i) If (V, α_V) is a right Hom-alternative A-module with the structure map ρ_r, then for all $(x, y, v) \in A \times A \times V$, $as_{A,V}(v, x, y) = -as_{A,V}(v, y, x)$ by (8), i.e., $\alpha_V(v) \cdot (xy) + \alpha_V(v) \cdot (yx) = (v \cdot x) \cdot \alpha_A(y) + (v \cdot y) \cdot \alpha_A(x)$. Thus $\alpha_V(v) \cdot (x * y) = \alpha_V(v) \cdot (xy) + \alpha_V(v) \cdot (yx) = (v \cdot x) \cdot \alpha_A(y) + (v \cdot y) \cdot \alpha_A(x)$. Therefore (V, α_V) is a right special Hom-Jordan A^+-module by Theorem 4.2.

(ii) If (V, α_V) is a left Hom-alternative A-module with the structure map ρ_l, then for all $(x, y, v) \in A \times A \times V$, $as_{A,V}(x, y, v) = -as_{A,V}(y, x, v)$ by Remarks 3.3 and then $(xy) \cdot \alpha_V(v) + (yx) \cdot \alpha_V(v) = \alpha_A(x) \cdot (y \cdot v) + \alpha_A(y) \cdot (x \cdot v)$. Thus $(x * y) \cdot \alpha_V(v) = (xy) \cdot \alpha_V(v) + (yx) \cdot \alpha_V(v) = \alpha_A(x) \cdot (y \cdot v) + \alpha_A(y) \cdot (x \cdot v)$. Therefore (V, α_V) is a left special Hom-Jordan A^+-module by Theorem 4.3. ∎

Now, we give the definition of a Hom-Jordan bimodule.

DEFINITION 4.5. Let (A, μ, α_A) be a Hom-Jordan algebra. A Hom-Jordan A-bimodule is a Hom-module (V, α_V) with a left structure map $\rho_l : A \otimes V \to V$, $a \otimes v \mapsto a \cdot v$ and a right structure map $\rho_r : V \otimes A \to V$, $v \otimes a \mapsto v \cdot a$, such that the following conditions hold:

$$\rho_r \circ \tau_1 = \rho_l, \tag{16}$$

$$\alpha_V(v \cdot a) \cdot \alpha_A(bc) + \alpha_V(v \cdot b) \cdot \alpha_A(ca) + \alpha_V(v \cdot c) \cdot \alpha_A(ab)$$
$$= (\alpha_V(v) \cdot bc) \cdot \alpha_A^2(a) + (\alpha_V(v) \cdot ca) \cdot \alpha_A^2(b) \tag{17}$$
$$+ (\alpha_V(v) \cdot ab) \cdot \alpha_A^2(c),$$

$$\alpha_V(v \cdot a) \cdot \alpha_A(bc) + \alpha_V(v \cdot b) \cdot \alpha_A(ca) + \alpha_V(v \cdot c) \cdot \alpha_A(ab)$$
$$= ((v \cdot a) \cdot \alpha_A(b)) \cdot \alpha_A^2(c) + ((v \cdot c) \cdot \alpha_A(b)) \cdot \alpha_A^2(a) \tag{18}$$
$$+ ((ac)\alpha_A(b)) \cdot \alpha_V^2(v),$$

for all $a, b, c \in A$ and $v \in V$.

In term of the module Hom-associator, using the relation (16) and the fact that the structure maps are morphisms, the relations (17) and (18) are respectively

$$\circlearrowleft_{(a,b,c)} as_{A,V}(\alpha_A(a), \alpha_V(v), bc) = 0, \tag{19}$$

$$as_{A,V}(v \cdot a, \alpha_A(b), \alpha_A(c)) + as_{A,V}(v \cdot c, \alpha_A(b), \alpha_A(a))$$
$$+ as_{A,V}(ac, \alpha_A(b), \alpha_V(v)) = 0. \tag{20}$$

Remarks 4.6. (i) One can note that (17) and (18) are the same identities as (9) and (10) respectively.

(ii) Since $\rho_r \circ \tau_1 = \rho_l$, nothing is lost in dropping one of the compositions. Thus the term Hom-Jordan module can be used for Hom-Jordan bimodule.

(iii) Since the field is of characteristic 0, the identity (19) implies

$$as_{A,V}(\alpha_A(a), \alpha_V(v), a^2) = 0\,.$$

(iv) If $\alpha_A = Id_A$ and $\alpha_V = Id_V$ then V is reduced to the so-called Jordan module of the Jordan algebra (A, μ) [7, 8].

EXAMPLES 4.7. Here are some examples of Hom-Jordan bimodules.

(i) Let (A, μ, α_A) be a Hom-Jordan algebra. Then (A, α_A) is a Hom-Jordan A-bimodule where the structure maps are $\rho_l = \rho_r = \mu$. More generally, if B is a Hom-ideal of (A, μ, α_A), then (B, α_A) is a Hom-Jordan A-bimodule where the structure maps are $\rho_l(a, x) = \mu(a, x) = \mu(x, a) = \rho_r(x, a)$ for all $(a, x) \in A \times B$.

(ii) If (A, μ) is a Jordan algebra and M is a Jordan A-bimodule [8] in the usual sense then (M, Id_M) is a Hom-Jordan $\mathbb{A}$-bimodule where $\mathbb{A} = (A, \mu, Id_A)$ is a Hom-Jordan algebra.

(iii) If $f : (A, \mu_A, \alpha_A) \to (B, \mu_B, \alpha_B)$ is a surjective morphism of Hom-Jordan algebras, then (B, α_B) becomes a Hom-Jordan A-bimodule via f, i.e, the structure maps are defined by $\rho_l : (a, b) \mapsto \mu_B(b, f(a))$ and $\rho_r : (b, a) \mapsto \mu_B(f(a), b)$ for all $(a, b) \in A \times B$.

As in the case of Hom-alternative algebras, in order to give another example of Hom-Jordan bimodules, let us consider the following

DEFINITION 4.8. An abelian extension of Hom-Jordan algebras is a short exact sequence of Hom-Jordan algebras

$$0 \to (V, \alpha_V) \xrightarrow{i} (A, \mu_A, \alpha_A) \xrightarrow{\pi} (B, \mu_B, \alpha_B) \to 0$$

where (V, α_V) is a trivial Hom-Jordan algebra, i and π are morphisms of Hom-algebras. Furthermore, if there exists a morphism $s : (B, \mu_B, \alpha_B) \to (A, \mu_A, \alpha_A)$ such that $\pi \circ s = id_B$ then the abelian extension is said to be split and s is called a section of π.

EXAMPLE 4.9. Given an abelian extension as in the previous definition, the Hom-module (V, α_V) inherits a structure of a Hom-Jordan B-bimodule

and the actions of the Hom-algebra (B, μ_B, α_B) on V are as follows. For any $x \in B$, there exist $\tilde{x} \in A$ such that $x = \pi(\tilde{x})$. Let x acts on $v \in V$ by $x \cdot v := \mu_A(\tilde{x}, i(v))$ and $v \cdot x := \mu_A(i(v), \tilde{x})$. These are well-defined, as another lift $\tilde{x'}$ of x is written $\tilde{x'} = \tilde{x} + v'$ for some $v' \in V$ and thus $x \cdot v = \mu_A(\tilde{x}, i(v)) = \mu_A(\tilde{x'}, i(v))$ and $v \cdot x = \mu_A(i(v), \tilde{x}) = \mu_A(i(v), \tilde{x'})$ because V is trivial. The actions property follow from the Hom-Jordan identity. In case these actions of B on V are trivial, one speaks of a central extension.

The next result shows that a special left and right Hom-Jordan module has a Hom-Jordan bimodule structure under a specific condition.

THEOREM 4.10. *Let (A, μ, α_A) be a Hom-Jordan algebra and (V, α_V) be both a left and a right special Hom-Jordan A-module with the structure maps ρ_1 and ρ_2 respectively such that the Hom-associativity (or operator commutativity) condition holds*

$$\rho_2 \circ (\rho_1 \otimes \alpha_A) = \rho_1 \circ (\alpha_A \otimes \rho_2). \tag{21}$$

Define the bilinear maps $\rho_l : A \otimes V \to V$ and $\rho_r : V \otimes A \to V$ by

$$\rho_l = \rho_1 + \rho_2 \circ \tau_1 \quad \text{and} \quad \rho_r = \rho_1 \circ \tau_2 + \rho_2. \tag{22}$$

Then $(V, \alpha_V, \rho_l, \rho_r)$ is a Hom-Jordan A-bimodule.

Proof. It is clear that ρ_l and ρ_r are structure maps and (16) holds. To prove relations (17) and (18), let put $\rho_l(a \otimes v) := a \diamond v$, i.e., $a \diamond v = a \cdot v + v \cdot a$ for all $(a, v) \in A \times V$. We have then $\rho_r(v \otimes a) := v \diamond a = a \cdot v + v \cdot a$ for all $(a, v) \in A \times V$. Therefore for all $(a, b, v) \in A \times A \times V$, we have

$$\alpha_V(v \diamond a) \diamond \alpha_A(bc) + \alpha_V(v \diamond b) \diamond \alpha_A(ca) + \alpha_V(v \diamond c) \diamond \alpha_A(ab)$$

$$= \alpha_V(v \cdot a) \cdot \alpha_A(bc) + \alpha_V(a \cdot v) \cdot \alpha_A(bc) + \alpha_A(bc) \cdot \alpha_V(v \cdot a)$$

$$+ \alpha_A(bc) \cdot \alpha_V(a \cdot v) + \alpha_V(v \cdot b) \cdot \alpha_A(ca) + \alpha_V(b \cdot v) \cdot \alpha_A(ca)$$

$$+ \alpha_A(ca) \cdot \alpha_V(v \cdot b) + \alpha_A(ca) \cdot \alpha_V(b \cdot v) + \alpha_V(v \cdot c) \cdot \alpha_A(ab)$$

$$+ \alpha_V(c \cdot v) \cdot \alpha_A(ab) + \alpha_A(ab) \cdot \alpha_V(v \cdot c) + \alpha_A(ab) \cdot \alpha_V(c \cdot v)$$

$$\text{(by a straightforward computation)}$$

$$= \{\alpha_V(v \cdot a) \cdot \alpha_A(bc) + \alpha_V(v \cdot b) \cdot \alpha_A(ca) + \alpha_V(v \cdot c) \cdot \alpha_A(ab)\}$$

$$+ \{\alpha_A(bc) \cdot (\alpha_V(v) \cdot \alpha_A(a)) + \alpha_A(ca) \cdot (\alpha_V(v) \cdot \alpha_A(b))$$

$$+ \alpha_A(ab) \cdot (\alpha_V(v) \cdot \alpha_A(c))\} + \{\alpha_A(bc) \cdot \alpha_V(a \cdot v) + \alpha_A(ca) \cdot \alpha_V(b \cdot v)$$

$$+ \alpha_A(ab) \cdot \alpha_V(c \cdot v)\} + \{(\alpha_A(a) \cdot \alpha_V(v)) \cdot \alpha_A(bc)$$
$$+ (\alpha_A(b) \cdot \alpha_V(v)) \cdot \alpha_A(ca) + (\alpha_A(c) \cdot \alpha_V(v)) \cdot \alpha_A(ab)\}$$

(rearranging terms and noting that ρ_1 and ρ_2 are morphisms)

$$= \{(\alpha_V(v) \cdot bc) \cdot \alpha_A^2(a) + (\alpha_V(v) \cdot ca) \cdot \alpha_A^2(b) + (\alpha_V(v) \cdot ab) \cdot \alpha_A^2(c)\}$$
$$+ \{(bc \cdot \alpha_V(v)) \cdot \alpha_A^2(a) + (ca \cdot \alpha_V(v)) \cdot \alpha_A^2(b) + (ab \cdot \alpha_V(v)) \cdot \alpha_A^2(c)\}$$
$$+ \{\alpha_A^2(a) \cdot (bc \cdot \alpha_V(v)) + \alpha_A^2(b) \cdot (ca \cdot \alpha_V(v)) + \alpha_A^2(c) \cdot (ab \cdot \alpha_V(v))\}$$
$$+ \{\alpha_A^2(a) \cdot (\alpha_V(v) \cdot bc) + \alpha_A^2(b) \cdot (\alpha_V(v) \cdot ca) + \alpha_A^2(c) \cdot (\alpha_V(v) \cdot ab)\}$$

(by (9), (11) and (21))

$$= \{(\alpha_V(v) \diamond bc) \cdot \alpha_A^2(a) + (\alpha_V(v) \diamond ca) \cdot \alpha_A^2(b) + (\alpha_V(v) \diamond ab) \cdot \alpha_A^2(c)\}$$
$$+ \{\alpha_A^2(a) \cdot (\alpha_V(v) \diamond bc) + \alpha_A^2(b) \cdot (\alpha_V(v) \diamond ca) + \alpha_A^2(c) \cdot (\alpha_V(v) \diamond ab)\}$$

(by the definition of $\diamond$)

$$= (\alpha_V(v) \diamond bc) \diamond \alpha_A^2(a) + (\alpha_V(v) \diamond ca) \diamond \alpha_A^2(b) + (\alpha_V(v) \diamond ab) \diamond \alpha_A^2(c)$$

(again by the definition of $\diamond$).

Therefore, we get (17). Finally, we have:

$$\alpha_V(v \diamond a) \diamond \alpha_A(bc) + \alpha_V(v \diamond b) \diamond \alpha_A(ca) + \alpha_V(v \diamond c) \diamond \alpha_A(ab)$$
$$= \alpha_V(v \cdot a) \cdot \alpha_A(bc) + \alpha_V(a \cdot v) \cdot \alpha_A(bc) + \alpha_A(bc) \cdot \alpha_V(v \cdot a)$$
$$+ \alpha_A(bc) \cdot \alpha_V(a \cdot v) + \alpha_V(v \cdot b) \cdot \alpha_A(ca) + \alpha_V(b \cdot v) \cdot \alpha_A(ca)$$
$$+ \alpha_A(ca) \cdot \alpha_V(v \cdot b) + \alpha_A(ca) \cdot \alpha_V(b \cdot v) + \alpha_V(v \cdot c) \cdot \alpha_A(ab)$$
$$+ \alpha_V(c \cdot v) \cdot \alpha_A(ab) + \alpha_A(ab) \cdot \alpha_V(v \cdot c) + \alpha_A(ab) \cdot \alpha_V(c \cdot v)$$

(by a straightforward computation)

$$= \{\alpha_V(v \cdot a) \cdot \alpha_A(bc) + \alpha_V(v \cdot b) \cdot \alpha_A(ca) + \alpha_V(v \cdot c) \cdot \alpha_A(ab)\}$$
$$+ \{(\alpha_V(a \cdot v) \cdot \alpha_A(b)\alpha_A(c) + (\alpha_V(b \cdot v)) \cdot \alpha_A(c)\alpha_A(a)$$
$$+ (\alpha_V(c \cdot v) \cdot \alpha_A(a)\alpha_A(b)\} + \{\alpha_A(bc) \cdot \alpha_V(a \cdot v) + \alpha_A(ca) \cdot \alpha_V(b \cdot v)$$
$$+ \alpha_A(ab) \cdot \alpha_V(c \cdot v)\} + \{\alpha_A(b)\alpha_A(c) \cdot \alpha_V(v \cdot a)$$
$$+ \alpha_A(c)\alpha_A(a) \cdot \alpha_V(v \cdot b) + \alpha_A(a)\alpha_A(b) \cdot \alpha_V(v \cdot c)\}$$

(rearranging terms and using the multiplicativity of α_A)

$$= \{\underbrace{((v \cdot a) \cdot \alpha_A(b)) \cdot \alpha_A^2(c)}_{1} + \underbrace{((v \cdot c) \cdot \alpha_A(b)) \cdot \alpha_A^2(a)}_{2} + \underbrace{\alpha_V^2(v) \cdot ((ac)\alpha_A(b))}_{5}\}$$

$$+ \{\underbrace{((a \cdot v) \cdot \alpha_A(b)) \cdot \alpha_A^2(c)}_{1} + ((a \cdot v) \cdot \alpha_A(c)) \cdot \alpha_A^2(b)$$

$$+ ((b \cdot v) \cdot \alpha_A(c)) \cdot \alpha_A^2(a) + ((b \cdot v) \cdot \alpha_A(a)) \cdot \alpha_A^2(c)$$

$$+ ((c \cdot v) \cdot \alpha_A(a)) \cdot \alpha_A^2(b) + \underbrace{((c \cdot v) \cdot \alpha_A(b)) \cdot \alpha_A^2(a)}_{2}\}$$

$$+ \{\underbrace{\alpha_A^2(c) \cdot (\alpha_A(b) \cdot (a \cdot v))}_{3} + \underbrace{\alpha_A^2(a) \cdot (\alpha_A(b) \cdot (c \cdot v))}_{4}$$

$$+ \underbrace{((ac)\alpha_A(b)) \cdot \alpha_V^2(v)}_{5}\} + \{\alpha_A^2(b) \cdot (\alpha_A(c) \cdot (v \cdot a))$$

$$+ \underbrace{\alpha_A^2(c) \cdot (\alpha_A(b) \cdot (v \cdot a))}_{3} + \alpha_A^2(a) \cdot (\alpha_A(c) \cdot (v \cdot b))$$

$$+ \alpha_A^2(c) \cdot (\alpha_A(a) \cdot (v \cdot b)) + \underbrace{\alpha_A^2(a) \cdot (\alpha_A(b) \cdot (v \cdot c))}_{4}$$

$$+ \alpha_A^2(b) \cdot (\alpha_A(a) \cdot (v \cdot c))\} \quad \text{(by (10), (12), (14) and (15))}$$

$$= ((v \diamond a) \cdot \alpha_A(b)) \cdot \alpha_A^2(c) + ((v \diamond c) \cdot \alpha_A(b)) \cdot \alpha_A^2(a)$$

$$+ \alpha_A^2(c) \cdot (\alpha_A(b) \cdot (v \diamond a)) + \alpha_A^2(a) \cdot (\alpha_A(b) \cdot (v \diamond c))$$

$$+ \alpha_V^2(v) \diamond ((ac)\alpha_A(b)) + ((a \cdot v) \cdot \alpha_A(c)) \cdot \alpha_A^2(b)$$

$$+ ((b \cdot v) \cdot \alpha_A(c)) \cdot \alpha_A^2(a) + ((b \cdot v) \cdot \alpha_A(a)) \cdot \alpha_A^2(c)$$

$$+ ((c \cdot v) \cdot \alpha_A(a)) \cdot \alpha_A^2(b) + \alpha_A^2(b) \cdot (\alpha_A(c) \cdot (v \cdot a))$$

$$+ \alpha_A^2(a) \cdot (\alpha_A(c) \cdot (v \cdot b)) + \alpha_A^2(c) \cdot (\alpha_A(a) \cdot (v \cdot b))$$

$$+ \alpha_A^2(b) \cdot (\alpha_A(a) \cdot (v \cdot c))$$

$$= ((v \diamond a) \cdot \alpha_A(b)) \cdot \alpha_A^2(c) + ((v \diamond c) \cdot \alpha_A(b)) \cdot \alpha_A^2(a)$$

$$+ \alpha_A^2(c) \cdot (\alpha_A(b) \cdot (v \diamond a)) + \alpha_A^2(a) \cdot (\alpha_A(b) \cdot (v \diamond c))$$

$$+ \alpha_V^2(v) \diamond ((ac)\alpha_A(b)) + (\alpha_A(a) \cdot (v \cdot c)) \cdot \alpha_A^2(b)$$

$$+ (\alpha_A(b) \cdot (v \cdot c)) \cdot \alpha_A^2(a) + (\alpha_A(b) \cdot (v \cdot a)) \cdot \alpha_A^2(c)$$

$$+ (\alpha_A(c) \cdot (v \cdot a)) \cdot \alpha_A^2(b) + \alpha_A^2(b) \cdot ((c \cdot v) \cdot \alpha_A(a))$$

$$+ \alpha_A^2(a) \cdot ((c \cdot v) \cdot \alpha_A(b)) + \alpha_A^2(c) \cdot ((a \cdot v) \cdot \alpha_A(b))$$

$$+ \alpha_A^2(b) \cdot ((a \cdot v) \cdot \alpha_A(c)) \quad \text{(by (21))}$$

$$= ((v \diamond a) \cdot \alpha_A(b)) \cdot \alpha_A^2(c) + ((v \diamond c) \cdot \alpha_A(b)) \cdot \alpha_A^2(a)$$
$$+ \alpha_A^2(c) \cdot (\alpha_A(b) \cdot (v \diamond a)) + \alpha_A^2(a) \cdot (\alpha_A(b) \cdot (v \diamond c))$$
$$+ \alpha_V^2(v) \diamond ((ac)\alpha_A(b)) + \underbrace{\alpha_A^2(a) \cdot ((v \cdot c) \cdot \alpha_A(b))}_{6}$$
$$+ \underbrace{(\alpha_A(b) \cdot (v \cdot c)) \cdot \alpha_A^2(a)}_{7} + \underbrace{(\alpha_A(b) \cdot (v \cdot a)) \cdot \alpha_A^2(c)}_{8}$$
$$+ \underbrace{\alpha_A^2(c) \cdot ((v \cdot a) \cdot \alpha_A(b))}_{9} + \underbrace{(\alpha_A(b) \cdot (c \cdot v)) \cdot \alpha_A^2(a)}_{7}$$
$$+ \underbrace{\alpha_A^2(a) \cdot ((c \cdot v) \cdot \alpha_A(b))}_{6} + \underbrace{\alpha_A^2(c) \cdot ((a \cdot v) \cdot \alpha_A(b))}_{9}$$
$$+ \underbrace{(\alpha_A(b) \cdot (a \cdot v)) \cdot \alpha_A^2(c)}_{8} \quad \text{(again by (21))}$$
$$= \underbrace{((v \diamond a) \cdot \alpha_A(b)) \cdot \alpha_A^2(c)}_{10} + \underbrace{((v \diamond c) \cdot \alpha_A(b)) \cdot \alpha_A^2(a)}_{11}$$
$$+ \underbrace{\alpha_A^2(c) \cdot (\alpha_A(b) \cdot (v \diamond a))}_{13} + \underbrace{\alpha_A^2(a) \cdot (\alpha_A(b) \cdot (v \diamond c))}_{12}$$
$$+ \alpha_V^2(v) \diamond ((ac)\alpha_A(b)) + \underbrace{\alpha_A^2(a) \cdot ((v \diamond c) \cdot \alpha_A(b))}_{12}$$
$$+ \underbrace{(\alpha_A(b) \cdot (v \diamond c)) \cdot \alpha_A^2(a)}_{11} + \underbrace{(\alpha_A(b) \cdot (v \diamond a)) \cdot \alpha_A^2(c)}_{10}$$
$$+ \underbrace{\alpha_A^2(c) \cdot ((v \diamond a) \cdot \alpha_A(b))}_{13}$$
$$= ((v \diamond a) \diamond \alpha_A(b)) \cdot \alpha_A^2(c) + ((v \diamond c) \diamond \alpha_A(b)) \cdot \alpha_A^2(a)$$
$$+ \alpha_A^2(a) \cdot ((v \diamond c) \diamond \alpha_A(b)) + \alpha_A^2(c) \cdot ((v \diamond a) \diamond \alpha_A(b))$$
$$+ \alpha_V^2(v) \diamond ((ac)\alpha_A(b))$$
$$= ((v \diamond a) \diamond \alpha_A(b)) \diamond \alpha_A^2(c) + ((v \diamond c) \diamond \alpha_A(b)) \diamond \alpha_A^2(a)$$
$$+ \alpha_V^2(v) \diamond ((ac)\alpha_A(b))$$

which is (18). ∎

The following result will be used below. It gives a relation between Hom-associative modules and special Hom-Jordan modules.

LEMMA 4.11. *Let (A, μ, α_A) be a Hom-associative algebra and (V, α_V) be a Hom-module.*

(i) *If (V, α_V) is a right Hom-associative A-module with the structure maps ρ_r then (V, α_V) is a right special Hom-Jordan A^+-module with the same structure map ρ_r.*

(ii) *If (V, α_V) is a left Hom-associative A-module with the structure maps ρ_l then (V, α_V) is a left special Hom-Jordan A^+-module with the same structure map ρ_l.*

Proof. It also suffices to prove (14) and (15).

(i) If (V, α_V) is a right Hom-associative A-module with the structure map ρ_r then for all $(x, y, v) \in A \times A \times V$, $\alpha_V(v) \cdot (a*b) = \alpha_V(v) \cdot (ab) + \alpha_V(v) \cdot (ba) = (v \cdot a) \cdot \alpha_A(b) + (v \cdot b) \cdot \alpha_A(a)$ where the last equality holds by (5). Then (V, α_V) is a right special Hom-Jordan A^+-module.

(ii) If (V, α_V) is a left Hom-associative A-module with the structure map ρ_l then for all $(x, y, v) \in A \times A \times V$, $(a*b) \cdot \alpha_V(v) = (ab) \cdot \alpha_V(v) + (ba) \cdot \alpha_V(v) = \alpha_A(a) \cdot (b \cdot v) + \alpha_A(b) \cdot (a \cdot v)$ where the last equality holds by (4). Then (V, α_V) is a left special Hom-Jordan A^+-module. ∎

Now, we prove that a Hom-associative module gives rise to a Hom-Jordan module for its plus Hom-algebra.

PROPOSITION 4.12. *Let (A, μ, α_A) be a Hom-associative algebra and $(V, \rho_1, \rho_2, \alpha_V)$ be a Hom-associative A-bimodule. Then $(V, \rho_l, \rho_r, \alpha_V)$ is a Hom-Jordan A^+-bimodule where ρ_l and ρ_r are defined as in (22).*

Proof. The proof follows from Lemma 4.11 , the Hom-associativity condition (6) and Theorem 4.10. ∎

The following elementary result will be used below. It gives a property of a module Hom-associator.

LEMMA 4.13. *Let (A, μ, α_A) be a Hom-Jordan algebra and (V, α_V) be an Hom-Jordan A-bimodule with the structure maps ρ_l and ρ_r. Then*

$$\alpha_V^n \circ as_{A,V} \circ Id_{A \otimes V \otimes A} = as_{A,V} \circ (\alpha_A^{\otimes n} \otimes \alpha_V^{\otimes n} \otimes \alpha_A^{\otimes n}). \qquad (23)$$

Proof. Using twice the fact that ρ_l and ρ_r are morphisms of Hom-modules, we get

$$\alpha_V^n \circ as_{A,V} \circ Id_{A \otimes V \otimes A}$$
$$= \alpha_V^n \circ (\rho_r \circ (\rho_l \otimes \alpha_A) - \rho_l \circ (\alpha_A \otimes \rho_r))$$
$$= \alpha_V^n \circ \rho_r \circ (\rho_l \otimes \alpha_A) - \alpha_V^n \circ \rho_l \circ (\alpha_A \otimes \rho_r) \quad \text{(linearity of } \alpha_V^n)$$
$$= \rho_r \circ (\alpha_V^n \circ \rho_l \otimes \alpha_A^{n+1}) - \rho_l \circ (\alpha_A^{n+1} \otimes \alpha_V^n \circ \rho_r)$$
$$= \rho_r \circ (\rho_l \circ (\alpha_A^n \otimes \alpha_V^n) \otimes \alpha_A^{n+1}) - \rho_l \circ (\alpha_A^{n+1} \otimes \rho_r \circ (\alpha_V^n \otimes \alpha_A^n))$$
$$= (\rho_r \circ (\rho_l \otimes \alpha_A) - \rho_l \circ (\alpha_A \otimes \rho_r)) \circ (\alpha_A^{\otimes n} \otimes \alpha_V^{\otimes n} \otimes \alpha_A^{\otimes n})$$
$$= as_{A,V} \circ (\alpha_A^{\otimes n} \otimes \alpha_V^{\otimes n} \otimes \alpha_A^{\otimes n}).$$

That ends the proof. ∎

The next result is similar to the one of Proposition 3.7, but an additional condition is needed.

PROPOSITION 4.14. *Let (A, μ, α_A) be a Hom-Jordan algebra and (V, α_V) be a Hom-Jordan A-bimodule with the structure maps ρ_l and ρ_r. Suppose that there exists $n \in \mathbb{N}$ such that $\alpha_V^n = Id_V$. Then the maps*

$$\rho_l^{(n)} = \rho_l \circ (\alpha_A^n \otimes Id_V), \tag{24}$$
$$\rho_r^{(n)} = \rho_r \circ (Id_V \otimes \alpha_A^n) \tag{25}$$

give the Hom-module (V, α_V) the structure of a Hom-Jordan A-bimodule that we denote by $V^{(n)}$.

Proof. Since the structure map ρ_l is a morphism of Hom-modules, we get:

$$\alpha_V \circ \rho_l^{(n)} = \alpha_V \circ \rho_l \circ (\alpha_A^n \otimes Id_V) \quad \text{(by (24))}$$
$$= \rho_l \circ (\alpha_A^{n+1} \otimes \alpha_V)$$
$$= \rho_l \circ (\alpha_A^n \otimes Id_V) \circ (\alpha_A \otimes \alpha_V)$$
$$= \rho_l^{(n)} \circ (\alpha_A \otimes \alpha_V)$$

Then, $\rho_l^{(n)}$ is a morphism. Similarly, we get that $\rho_r^{(n)}$ is a morphism and that (16) holds for $V^{(n)}$. Next, we compute

$$\circlearrowleft_{(a,b,c)} as_{A,V^{(n)}}(\alpha_A(a), \alpha_V(v), ab)$$

$$=\circlearrowleft_{(a,b,c)}\{\rho_r^{(n)}(\rho_l^{(n)}(\alpha_A(a), \alpha_V(v)), \alpha_A(bc)) - \rho_l^{(n)}(\alpha_A^2(a), \rho_r^{(n)}(\alpha_V(v), bc))\}$$

$$=\circlearrowleft_{(a,b,c)}\{\rho_r(\rho_l^{(n)}(\alpha_A(a), \alpha_V(v)), \alpha_A^{n+1}(bc)) - \rho_l(\alpha_A^{n+2}(a), \rho_r^{(n)}(\alpha_V(v), bc))\}$$

$$=\circlearrowleft_{(a,b,c)}\{\rho_r(\rho_l(\alpha_A^{n+1}(a), \alpha_V(v)), \alpha_A^{n+1}(bc)) - \rho_l(\alpha_A^{n+2}(a), \rho_r(\alpha_V(v), \alpha_A^n(bc)))\}$$

$$=\circlearrowleft_{(a,b,c)}\{\rho_r(\rho_l(\alpha_A^{n+1}(a), \alpha_V(v)), \alpha_A(\alpha_A^n(bc)))$$
$$\quad - \rho_l(\alpha_A(\alpha_A^{n+1}(a)), \rho_r(\alpha_V(v), \alpha_A^n(bc)))\}$$

$$=\circlearrowleft_{(a,b,c)} as_{A,V}(\alpha_A^{n+1}(a), \alpha_V(v), \alpha_A^n(bc))$$

$$=\circlearrowleft_{(a,b,c)} as_{A,V}(\alpha_A^{n+1}(a), \alpha_V^{n+1}(v), \alpha_A^n(bc)) \quad \text{(by the hypothesis } \alpha_V = \alpha_V^{n+1})$$

$$= \alpha_V^n(\circlearrowleft_{(a,b,c)} as_{A,V}(\alpha_A(a), \alpha_V(v), bc)) \quad \text{(by (23) and the linearity of } \alpha_V^n)$$

$$= 0 \quad \text{(by (19) in } V).$$

Then we get (19) for $V^{(n)}$. Finally remarking that

$$as_{A,V^{(n)}}(\rho_r^n(v,a), \alpha_A(b), \alpha_A(c))$$

$$= as_{A,V^{(n)}}(v \cdot \alpha_A^n(a), \alpha_A(b), \alpha_A(c))$$

$$= \rho_r^n(\rho_r^n(v \cdot \alpha_A^n(a), \alpha_A(b), \alpha_A^2(c)) - \rho_r^n(\alpha_V(v) \cdot \alpha_A^{n+1}(a), \mu(\alpha_A(b), \alpha(_A(c)))$$

$$= \rho_r(\rho_r(v \cdot \alpha_A^n(a), \alpha_A^{n+1}(b), \alpha_A^{n+2}(c))$$
$$\quad - \rho_r(\alpha_V(v) \cdot \alpha^{n+1}(a), \mu(\alpha_A^{n+1}(b), \alpha_A^{n+1}(c)))$$

$$= \alpha_{A,V}(v \cdot \alpha_A^n(a), \alpha_A^{n+1}(b), \alpha_A^{n+1}(c)),$$

and similarly

$$as_{A,V^{(n)}}(\rho_r^n(v,c), \alpha_A(b), \alpha_A(a)) = as_{A,V}(v \cdot \alpha_A^n(c), \alpha_A^{n+1}(b), \alpha_A^{n+1}(a)),$$

$$as_{A,V^{(n)}}(ac, \alpha_A(b), \alpha_V(v)) = as_{A,V}(\alpha_A^n(a)\alpha_A^n(c), \alpha_A^{n+1}(b), \alpha_V(v))$$

(20) is proved for $V^{(n)}$ as it follows:

$$as_{A,V^{(n)}}(\rho_r^n(v,a), \alpha_A(b), \alpha_A(c)) + as_{A,V^{(n)}}(\rho_r^n(v,c), \alpha_A(b), \alpha_A(a))$$

$$+ as_{A,V^{(n)}}(ac, \alpha_A(b), \alpha_V(v))$$

$$= \alpha_V(v \cdot \alpha_A^n(a), \alpha_A^{n+1}(b), \alpha_A^{n+1}(c)) + as_{A,V}(v \cdot \alpha_A^n(c), \alpha_A^{n+1}(b), \alpha_A^{n+1}(a))$$

$$+ as_{A,V}(\alpha_A^n(a)\alpha_A^n(c), \alpha_A^{n+1}(b), \alpha_V(v))$$

$$= \alpha_V(v \cdot \alpha_A^n(a), \alpha_A(\alpha_A^n(b)), \alpha_A(\alpha_A^n(c)))$$
$$+ as_{A,V}(v \cdot \alpha_A^n(c), \alpha_A(\alpha_A^n(b)), \alpha_A(\alpha_A^n(a)))$$
$$+ as_{A,V}(\alpha_A^n(a)\alpha_A^n(c), \alpha_A(\alpha_A^n(b)), \alpha_V(v))$$
$$= 0 (\text{by (20) in } V).$$

We conclude that $V^{(n)}$ is a Hom-Jordan A-bimodule. ∎

EXAMPLE 4.15. Consider the Hom-Jordan algebra $\mathcal{A}^+$ of the Examples 2.10 and the subspace $V = \mathrm{span}(e_1, e_3)$ of A. Then (V, μ_V, α_V) is a Hom-ideal of $\mathcal{A}^+$ where $\mu_V = \mu_{A|_V}$ and $\alpha_V = \alpha_{A|_V}$. It follows that $(V, \rho_l, \rho_r, \alpha_V)$ is a Hom-Jordan $\mathcal{A}^+$-bimodule where ρ_l and ρ_r are defined as in Examples 4.7. We have $\alpha_V^2 = Id_V$, then by Proposition 4.14, the structure maps $\rho_l^{(2)} = \rho_l \circ (\alpha_A^2 \otimes Id_V)$ and $\rho_r^{(2)} = \rho_r \circ (Id_V \otimes \alpha_A^2)$ give the Hom-module (V, α_V) the structure of a Hom-Jordan $\mathcal{A}^+$-bimodule that we denote by $V^{(2)}$.

COROLLARY 4.16. *Let (A, μ, α_A) be a Hom-Jordan algebra and (V, α_V) be a Hom-Jordan A-bimodule with the structure maps ρ_l and ρ_r such that α_V is an involution. Then (V, α_V) is a Hom-Jordan A-bimodule with the structure maps $\rho_l^{(2)} = \rho_l \circ (\alpha_A^2 \otimes Id_V)$ and $\rho_r^{(2)} = \rho_r \circ (Id_V \otimes \alpha_A^2)$.*

EXAMPLE 4.17. Consider the Hom-Jordan algebra $\mathcal{B}^+$ of the Examples 2.10 and the subspace $V = \mathrm{span}(e_1, e_2)$ of B. Then (V, μ_V, α_V) is a Hom-ideal of $\mathcal{B}^+$ where $\mu_V = \mu_{B|_V}$ and $\alpha_V = \alpha_{B|_V}$. Therefore $(V, \rho_l, \rho_r, \alpha_V)$ is a Hom-Jordan $\mathcal{B}^+$-bimodule where ρ_l and ρ_r are defined as in Examples 4.7. Note that α_V is involutive, i.e., $\alpha_V^2 = Id_V$, then by Corollary 4.16, the structure maps $\rho_l^{(2)} = \rho_l \circ (\alpha_B^2 \otimes Id_V)$ and $\rho_r^{(2)} = \rho_r \circ (Id_V \otimes \alpha_B^2)$ give the Hom-module (V, α_V) the structure of a Hom-Jordan $\mathcal{B}^+$-bimodule.

The following result is similar to theorem 3.8. It says that Jordan bimodules can be deformed into Hom-Jordan bimodules via an endomorphism.

THEOREM 4.18. *Let (A, μ) be a Jordan algebra, V be a Jordan A-bimodule with the structure maps ρ_l and ρ_r, α_A be an endomorphism of the Jordan algebra A and α_V be a linear self-map of V such that $\alpha_V \circ \rho_l = \rho_l \circ (\alpha_A \otimes \alpha_V)$ and $\alpha_V \circ \rho_r = \rho_r \circ (\alpha_V \otimes \alpha_A)$. Write A_{α_A} for the Hom-Jordan algebra $(A, \mu_{\alpha_A}, \alpha_A)$ and V_{α_V} for the Hom-module (V, α_V). Then the maps:*

$$\tilde{\rho}_l = \alpha_V \circ \rho_l \qquad and \qquad \tilde{\rho}_r = \alpha_V \circ \rho_r$$

give the Hom-module V_{α_V} the structure of a Hom-Jordan A_{α_A}-bimodule.

Proof. It is easy to prove that the relation (16) for V_{α_V} holds and both maps $\tilde{\rho}_l$, $\tilde{\rho}_r$ are morphisms. Remarking that

$$as_{A,V_{\alpha_V}} = \alpha_V^2 \circ as_{A,V} \tag{26}$$

we first compute

$$
\begin{aligned}
\circlearrowleft_{(a,b,c)} as_{A,V_{\alpha_V}} &(\alpha_A(a), \alpha_V(v), \mu_{\alpha_A}(b,c)) \\
&= \circlearrowleft_{(a,b,c)} \alpha_V^2(as_{A,V}(\alpha_A(a), \alpha_V(v), \alpha_A(bc))) \quad \text{(by (26))} \\
&= \circlearrowleft_{(a,b,c)} \alpha_V^3((as_{A,V}(a,v,bc)) \quad \text{(by (23))} \\
&= \alpha_V^3(\circlearrowleft_{(a,b,c)} (as_{A,V}(a,v,bc)) \\
&= 0 \quad \text{(by (19) in V)}
\end{aligned}
$$

and then, we get (19) for V_{α_V}. Finally, we get

$$
\begin{aligned}
as_{A,V_{\alpha_V}}&(\tilde{\rho}_r(v,a), \alpha_A(b), \alpha_A(c)) + as_{A,V_{\alpha_V}}(\tilde{\rho}_r(v,c), \alpha_A(b), \alpha_A(a)) \\
&\qquad\qquad + as_{A,V_{\alpha_V}}(\mu_{\alpha_A}(a,c), \alpha_A(b), \alpha_V(v)) \\
&= \alpha_V^2(as_{A,V}(\tilde{\rho}_r(v,a), \alpha_A(b), \alpha_A(c))) + \alpha_V^2(as_{A,V}(\tilde{\rho}_r(v,c), \alpha_A(b), \alpha_A(a))) \\
&\quad + \alpha_V^2(as_{A,V}(\mu_{\alpha_A}(a,c), \alpha_A(b), \alpha_V(v))) \quad \text{(by (26))} \\
&= \alpha_V^2(as_{A,V}(\alpha_V(v \cdot a), \alpha_A(b), \alpha_A(c))) \\
&\quad + \alpha_V^2(as_{A,V}(\alpha_V(v \cdot c), \alpha_A(b), \alpha_A(a))) \\
&\quad + \alpha_V^2(as_{A,V}(\alpha_A(ac), \alpha_A(b), \alpha_V(v))) \\
&= \alpha_V^3(as_{A,V}(v \cdot a, b, c)) + \alpha_V^3(as_{A,V}(v \cdot c, b, a)) \\
&\quad + \alpha_V^3(as_{A,V}(ac, b, v)) \quad \text{(by 23)} \\
&= \alpha_V^3(as_{A,V}(v \cdot a, b, c) + as_{A,V}(v \cdot c, b, a) + as_{A,V}(ac, b, v)) \\
&= 0 \quad \text{(by (20) in V)}
\end{aligned}
$$

which is (20) for V_{α_V}. Therefore the Hom-module V_{α_V} has a Hom-Jordan A_{α_A}-bimodule structure. ∎

COROLLARY 4.19. *Let (A, μ) be a Jordan algebra, V be a Jordan A-bimodule with the structure maps ρ_l and ρ_r, α_A be an endomorphism of the Jordan algebra A and α_V be a linear self-map of V such that $\alpha_V \circ \rho_l = \rho_l \circ (\alpha_A \otimes \alpha_V)$ and $\alpha_V \circ \rho_r = \rho_r \circ (\alpha_V \otimes \alpha_A)$.*

Moreover, suppose that there exists $n \in \mathbb{N}$ such that $\alpha_V^n = Id_V$. Write A_{α_A} for the Hom-Jordan algebra $(A, \mu_{\alpha_A}, \alpha_A)$ and V_{α_V} for the Hom-module

(V, α_V). Then the maps:

$$\tilde{\rho}_l^{(n)} = \rho_l \circ (\alpha_A^{n+1} \otimes \alpha_V) \text{ and } \tilde{\rho}_r^{(n)} = \rho_r \circ (\alpha_V \otimes \alpha_A^{n+1}) \qquad (27)$$

give the Hom-module V_α the structure of a Hom-Jordan A_{α_A}-bimodule for each $n \in \mathbb{N}$.

Proof. The proof follows from Proposition 4.14 and Theorem 4.18. ∎

Similarly to Hom-alternative algebras, the split null extension, determined by the given bimodule over a Hom-Jordan algebra, is constructed as follows:

THEOREM 4.20. *Let (A, μ, α_A) be a Hom-Jordan algebra and (V, α_V) be a Hom-Jordan A-bimodule with the structure maps ρ_l and ρ_r. Then $(A \oplus V, \tilde{\mu}, \tilde{\alpha})$ is a Hom-Jordan algebra where*
$\tilde{\mu} : (A \oplus V)^{\otimes 2} \to A \oplus V, \tilde{\mu}(a+m, b+n) := ab + a \cdot n + m \cdot b$ *and* $\tilde{\alpha} : A \oplus V \to A \oplus V,$
$\tilde{\alpha}(a + m) := \alpha_A(a) + \alpha_V(m)$

Proof. First, the commutativity of $\tilde{\mu}$ follows from the one of μ. Next, the multiplicativity of $\tilde{\alpha}$ with respect to $\tilde{\mu}$ follows from the one of α with respect to μ and the fact that ρ_l and ρ_r are morphisms of Hom-modules. Finally, we prove the Hom-Jordan identity (3) for $E = A \oplus V$ as it follows

$as_E(\tilde{\mu}(x + m, x + m), \tilde{\alpha}(y + n), \tilde{\alpha}(x + m))$

$\quad = \tilde{\mu}(\tilde{\mu}(\tilde{\mu}(x + m, x + m), \tilde{\alpha}(y + n)), \tilde{\alpha}^2(x + m)) - \tilde{\mu}(\tilde{\alpha}(\tilde{\mu}(x + m, x + m)),$
$\qquad \tilde{\mu}(\tilde{\alpha}(y + n), \tilde{\alpha}(x + m)))$

$\quad = \tilde{\mu}(\tilde{\mu}(x^2 + x \cdot m + m \cdot x, \alpha_A(y) - \tilde{\mu}(\alpha_A(x^2) + \alpha_V(n)), \alpha_A^2(x) + \alpha_V^2(m))$
$\qquad + \alpha_V(x \cdot m) + \alpha_V(m \cdot x), \tilde{\mu}(\alpha_A(y) + \alpha_V(n), \alpha_A(x) + \alpha_V(m)))$

$\quad = \tilde{\mu}(x^2 \alpha_A(y) + x^2 \cdot \alpha_V(n) + (x \cdot m) \cdot \alpha_A(y) + (m \cdot x) \cdot \alpha_A(y), \alpha_A^2(x)$
$\qquad + \alpha_V^2(m)) - \tilde{\mu}(\alpha_A^2(x^2) + \alpha_V(x \cdot m) + \alpha_V(m \cdot x), \alpha_A(y)\alpha_A(x)$
$\qquad + \alpha_A(y) \cdot \alpha_V(m) + \alpha_V(n) \cdot \alpha_A(x))$

$\quad = (x^2 \alpha_A(y))\alpha_A^2(x) + (x^2 \alpha_A(y)) \cdot \alpha_V^2(m) + (x^2 \cdot \alpha_V(n)) \cdot \alpha_A^2(x)$
$\qquad + ((x \cdot m) \cdot \alpha_A(y)) \cdot \alpha_A^2(x) + ((m \cdot x) \cdot \alpha_A(y)) \cdot \alpha_A^2(x))$
$\qquad - \alpha_A(x^2)(\alpha_A(y)\alpha_A(x)) - \alpha_A(x^2) \cdot (\alpha_A(y) \cdot \alpha_V(m))$
$\qquad - \alpha_A(x^2) \cdot (\alpha_V(n) \cdot \alpha_A(x)) - \alpha_V(x \cdot m) \cdot (\alpha_A(y)\alpha_A(x))$
$\qquad - \alpha_V(m \cdot x) \cdot (\alpha_A(y)\alpha_A(x))$

$$= as_A(x^2, \alpha_A(y), \alpha_A(x)) + as_{A,V}(x^2, \alpha_A(y), \alpha_V(m))$$
$$+ as_{A,V}(x^2, \alpha_V(n), \alpha_A(x)) + as_{A,V}(x \cdot m, \alpha_A(y), \alpha_A(x))$$
$$+ as_{A,V}(m \cdot x, \alpha_A(y), \alpha_A(x))$$
$$= \underbrace{as_{A,V}(m \cdot x, \alpha_A(y), \alpha_A(x)) + as_{A,V}(m \cdot x, \alpha_A(y), \alpha_A(x))}_{0}$$
$$+ \underbrace{as_{A,V}(x^2, \alpha_V(n), \alpha_A(x)) + as_{A,V}(x^2, \alpha_A(y), \alpha_V(m))}_{0}$$
$$+ \underbrace{as_A(x^2, \alpha_A(y), \alpha_A(x))}_{0} = 0,$$

where the first 0 follows from (20), the second from (19) (see Remarks 4.6) and the last from the Hom-Jordan identity (3) in A. We conclude then that $(A \oplus V, \tilde{\mu}, \tilde{\alpha})$ is a Hom-Jordan algebra. ∎

Similarly as Hom-alternative algebra case, let give the following:

Remark 4.21. Consider the split null extension $A \oplus V$ determined by the Hom-Jordan bimodule (V, α_V) for the Hom-Jordan algebra (A, μ, α_A) in the previous theorem. Write elements $a + v$ of $A \oplus V$ as (a, v). Then there is an injective homomorphism of Hom-modules $i : V \to A \oplus V$ given by $i(v) = (0, v)$ and a surjective homomorphism of Hom-modules $\pi : A \oplus V \to A$ given by $\pi(a, v) = a$. Moreover, $i(V)$ is a Hom-ideal of $A \oplus V$ such that $A \oplus V / i(V) \cong A$. On the other hand, there is a morphism of Hom-algebras $\sigma : A \to A \oplus V$ given by $\sigma(a) = (a, 0)$ which is clearly a section of π. Hence, we obtain the abelian split exact sequence of Hom-Jordan algebras and (V, α_V) is a Hom-Jordan bimodule for A via π.

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