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Genus zero of projective symplectic groups

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Abstract: A transitive subgroup $G \leq S_N$ is called a genus zero group if there exist non identity elements $x_1, \ldots, x_r \in G$ satisfying $G = \langle x_1, x_2, \ldots, x_r \rangle$, $\prod_{i=1}^{r} x_i = 1$ and $\sum_{i=1}^{r} \operatorname{ind} x_i = 2N - 2$. The Hurwitz space $\mathcal{H}_r^{in}(G)$ is the space of genus zero coverings of the Riemann sphere $\mathbb{P}^1$ with r branch points and the monodromy group G.
In this paper, we assume that G is a finite group with $\mathrm{PSp}(4, q) \leq G \leq \mathrm{Aut}(\mathrm{PSp}(4, q))$ and G acts on the projective points of 3-dimensional projective geometry $\mathrm{PG}(3, q)$, q is a prime power. We show that G possesses no genus zero group if $q > 5$. Furthermore, we study the connectedness of the Hurwitz space $\mathcal{H}_r^{in}(G)$ for a given group G and $q \leq 5$.

Key words: symplectic group, fixed point, genus zero group.

MSC (2020): 20B15, 20C33.

1. Introduction

A one dimensional compact manifold is called Riemann surface. Topologically, such surfaces are either spheres or tori which have been glued together. The number of holes so joined is called the genus. Let $F\colon X \to \mathbb{P}^1$ be a meromorphic function from a compact connected Riemann surface X of genus g into the Riemann sphere $\mathbb{P}^1$. For every meromorphic function there is a positive integer N such that all points have exactly N preimages. So every compact Riemann surface can be made into the branched covering of $\mathbb{P}^1$. It is known that one of the basic strategies of the whole subject of algebraic topology is to find methods to reduce topological problem about continuous maps and spaces into pure algebraic problems about homomorphisms and groups by using the fundamental group. The points p are called the branch points of F if $|F^{-1}(p)| < N$. It is well known that the set of branch points is finite and it will be denoted by $B = \{p_1, \ldots, p_r\}$. For $q \in \mathbb{P}^1 \setminus B$, the fundamental group $\pi_1(\mathbb{P}^1 \setminus B, q)$ is a free group which is generated by all homotopy classes of loops γ_i winding once around the point p_i. These loops of generators γ_i

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are subject to the single relation that $\gamma_1 \cdot \ldots \cdot \gamma_r = 1$ in $\pi_1(\mathbb{P}^1 \setminus B, q)$. The explicit and well known construction of Hurwitz shows that a Riemann surface X with N branching coverings of $\mathbb{P}^1$ is defined in the following way: consider the preimage $F^{-1}(q) = \{x_1, \ldots, x_N\}$, every loop in γ in $\mathbb{P}^1 \setminus B$ can be lifted to N paths $\tilde{\gamma_1}, \ldots, \tilde{\gamma_N}$ where $\tilde{\gamma_i}$ is the unique path lift of γ and $\tilde{\gamma_i}(0) = x_i$ for every i. The endpoints $\tilde{\gamma_i}(1)$ also lie over q. That is $\tilde{\gamma_i}(1) = x_{\sigma(i)}$ in $F^{-1}(q)$ where σ is a permutation of the indices $\{1, \ldots, N\}$ and it depends only on γ. Thus it gives a group homomorphism $\phi : \pi_1(\mathbb{P}^1 \setminus B, q) \to S_N$. The image of ϕ is called the monodromy group of F and denoted by $G = Mon(X, F)$. Since X is connected, then G is a transitive subgroup of S_N. Thus a group homomorphism is determined by choosing N permutations $x_i = \phi(\gamma_i)$, $i = 1, \ldots, r$ and satisfying the relations

$$G = \langle x_1, x_2, \ldots, x_r \rangle, \tag{1}$$

$$\prod_{i=1}^{r} x_i = 1, \quad x_i \in G^{\#} = G \setminus \{1\}, \quad i = 1, \ldots, r, \tag{2}$$

$$\sum_{i=1}^{r} \operatorname{ind} x_i = 2(N + g - 1), \tag{3}$$

where $\operatorname{ind} x = N - \operatorname{orb}(x)$, $\operatorname{orb}(x)$ is the number of orbits of the group generated by x on Ω where $|\Omega| = N$. Equation (3) is called the Riemann Hurwitz formula. A transitive subgroup $G \leq S_N$ is called a genus g group if there exist $x_1, \ldots, x_r \in G$ satisfying (1), (2) and (3) and then we call $(x_1, \ldots, x_r)$ the genus g system of G. If the action of G on Ω is primitive, we call G a primitive genus g group and $(x_1, \ldots, x_r)$ a primitive genus g system.

A group G is said to be almost simple if it contains a non-abelian simple group S and $S \leq G \leq \operatorname{Aut}(S)$. In [4], Kong worked on almost simple groups whose socle is a projective special linear group. Also, she gave a complete list for some almost simple groups of Lie rank 2 up to ramification type in her PhD thesis for genus 0, 1 and 2 system. Furthermore, she showed that the almost simple groups with socle $\mathrm{PSL}(3, q)$ do not possess genus low tuples if $q \geq 16$. In [6], Mohammed Salih gave the classification of some almost simple groups with socle $\mathrm{PSL}(3, q)$ up to braid action and diagonal conjugation.

The symplectic group $\mathrm{Sp}(n, q)$ is the group of all elements of $\mathrm{GL}(n, q)$ preserving a non-degenerate alternating form; the non degenerate leads to n being even. The projective symplectic group $\mathrm{PSp}(n, q)$ is obtained by from $\mathrm{Sp}(n, q)$ on factoring it by the subgroup of scalar matrices it contains (which has order at most 2) [1]. In this paper we consider a finite group G with

$\mathrm{PSp}(4,q) \leq G \leq \mathrm{Aut}(\mathrm{PSp}(4,q))$ and G acts on the projective points of 3-dimensional projective geometry $\mathrm{PG}(3,q)$, q is a prime power.

We will now describe the work carried out in this paper. In the second section we review some basic concepts and results will be used later. In the third section, we provide some basic facts for computing fixed points and generating tuples. Finally, we show that G possesses no genus 0 group if $\mathrm{PSp}(4,q) \leq G \leq \mathrm{Aut}(\mathrm{PSp}(4,q))$ and G acts on the projective points of 3-dimensional projective geometry $\mathrm{PG}(3,q)$, q is a prime power and $q > 5$. Furthermore, we study the connectedness of the Hurwitz space G if $q \leq 5$.

2. Preliminary

We begin by introducing some definitions and stating a few results which will be needed later. Assume that G is a finite permutation group of degree N. The signature of the r-tuple $x = (x_1, \ldots, x_r)$ is the r-tuple $d = (d_1, \ldots, d_r)$ where $d_i = o(x_i)$. We assume that $d_i \leq d_j$ if $i \leq j$, because of the braid action on x. The following result will tell us the tuple x can not generate G, where $G = \mathrm{PGL}(4,q)$ or $\mathrm{PSL}(4,q)$ if (ii), (iii) and (iv) below hold. So, setting $A(d) = \sum_{i=1}^{r} \frac{d_i - 1}{d_i}$, we have $A(d) \geq \frac{85}{42}$.

Proposition 2.1. ([3]) *Assume that a group G acts transitively and faithfully on Ω and $|\Omega| = N$. Let $r \geq 2$, $G = \langle x_1, \ldots, x_r \rangle$, $\prod_{i=1}^{r} x_i = 1$ and $o(x_i) = d_i > 1$, $i = 1, \ldots, r$. Then one of the following holds:*

(i) $\sum_{i=1}^{r} \frac{d_i - 1}{d_i} \geq \frac{85}{42}$;

(ii) $r = 4$, $d_i = 2$ *for each* $i = 1$ *and* $G'' = 1$;

(iii) $r = 3$ *and (up to permutation)* $(d_1, d_2, d_3) =$

  (a) $(3,3,3)$, $(2,3,6)$ *or* $(2,4,4)$ *and* $G'' = 1$;

  (b) $(2,2,d)$ *and* G *is dihedral;*

  (c) $(2,3,3)$ *and* $G \cong A_4$;

  (d) $(2,3,4)$ *and* $G \cong S_4$;

  (e) $(2,3,5)$ *and* $G \cong A_5$;

(iv) $r = 2$ *and* G *is cyclic.*

For a permutation x of the finite set Ω, let $\mathrm{Fix}(x)$ denote the fixed points of x on Ω and $f(x) = |\mathrm{Fix}(x)|$ is the number of fixed points of x. Note that the conjugate elements have the same number of fixed points. The following result provides a useful connection between fixed points and indices.

Lemma 2.2. ([3]) *If x is a permutation of order d on a set of size N, then* $\operatorname{ind} x = N - \frac{1}{d}\sum_{y\in\langle x\rangle} f(y)$ *where $\langle x\rangle$ is the cyclic group generated by x.*

The fixed point ratio of x is defined by $\mathrm{fpr}(x) = \frac{f(x)}{N}$. The codimension of the largest eigenspace of a linear transformation $\bar{g}$ in $\mathrm{GL}(n,q)$ is denoted by $v(\bar{g})$. Ω denotes the set of the projective points of projective geometry $\mathrm{PG}(n-1,q)$ that is the set of 1-dimensional subspaces of vector space over a finite field $\mathrm{GF}(q)$. In this paper we take $|\Omega| = \frac{q^n-1}{q-1}$ and $n = 4$, so we have $|\Omega| = q^3 + q^2 + q + 1$.

The center of $\mathrm{GL}(n,q)$ is the set of all scalar matrices and denoted by $Z(\mathrm{GL}(n,q))$. The projective general linear group and the projective special linear group are defined by

$$\mathrm{PGL}(n,q) = \frac{\mathrm{GL}(n,q)}{Z(\mathrm{GL}(n,q))} \qquad \text{and} \qquad \mathrm{PSL}(n,q) = \frac{\mathrm{SL}(n,q)}{Z(\mathrm{SL}(n,q))}$$

respectively, where $Z(\mathrm{SL}(n,q)) = \mathrm{SL}(n,q) \cap Z(\mathrm{GL}(n,q))$. They act primitively on Ω.

Let $\langle v\rangle \in \Omega$ be a fixed point of $g \in \mathrm{PGL}(n,q)$ and let $\bar{g}$ be an element in the preimage of g in $\mathrm{GL}(n,q)$ that fixes $\langle v\rangle$. The fixed points of g are the 1-spaces spanned by eigenvectors of $\bar{g}$. So we classify non identity elements in $\mathrm{PGL}(4,q)$ by their fixed points as follows:

Table 1: Number of Fixed points

$v(\bar{g})$	Type of eigenspaces of $\bar{g} \in GL(n,q)$	Number of fixed points of $g \in PGL(4,q)$
4	no eigenspace	0
3	one 1-dimensional eigenspace	1
3	two 1-dimensional eigenspaces	2
3	three 1-dimensional eigenspaces	3
2	one 2-dimensional eigenspace	$q+1$
1	one 3-dimensional eigenspace	q^2+q+1
2	one 1-dimensional and one 2-dimensional eigenspaces	$q+2$
2	one 2-dimensional and one 2-dimensional eigenspaces	$2q+2$
1	one 1-dimensional and one 3-dimensional eigenspaces	q^2+q+2
2	one 1-dimensional, one 1-dimensional and one 2-dimensional eigenspaces	$q+3$

According to Table 1, we have two cases. If $v(\bar{g}) = 1$, then g fixes q^2+q+1 or q^2+q+2 points. Otherwise, it fixes at most $2q+2$ points. From this fact, we will show that there are no genus zero systems for $\mathrm{PSL}(4,q)$ and $\mathrm{PGL}(4,q)$ when $q > 37$.

The following result is an interesting tool to compute β in the next section.

Lemma 2.3. (Scott Bound, [7]) *Let $G \leq \mathrm{GL}(n,q)$. If a triple $x = (x_1, x_2, x_3)$ satisfies $G = \langle x_1, x_2, x_3\rangle$ and $x_1x_2x_3 = 1$, then $v(x_i) + v(x_j) \geq n$ where $i \neq j$ and $1 \leq i, j \leq 3$. In particular if $n \geq 3$ and $i \neq j$, then $v(x_i) \geq 2$ or $v(x_j) \geq 2$.*

Lemma 2.4. ([5]) *If $\frac{1}{N}\sum_{i=1}^{r}\sum_{j=1}^{d_i-1}\frac{f(x^j)}{d_i} < A(d)-2$, then d is not a genus zero system.*

3. Existence of genus zero system

Now, we are going to apply Lemma 2.4, to exclude all signatures which do not satisfy the Riemann Hurwitz formula. As a result, we will obtain Theorem 3.1.

Let F be the set of elements with q^2+q+1 or q^2+q+2 fixed points in $\mathrm{PGL}(4,q)$. So we have

$$\mathrm{fpr}(x) \leq \begin{cases} \dfrac{q^2+q+2}{N} & \text{if } x \in F, \\ \dfrac{2q+2}{N} & \text{if } x \notin F. \end{cases}$$

Assume that $\alpha = \frac{q^2+q+2}{N}$ and $\gamma = \frac{2q+2}{N}$. Combining the Riemann Hurwitz formula as done in [4], we obtain the following inequality

$$A(d) \leq \frac{2+\epsilon+\beta(\alpha-\gamma)}{1-\gamma} \tag{4}$$

where $\epsilon = \frac{-2}{N}$ and $\beta = \sum_{i=1}^{r}\frac{|\langle x_i\rangle^{\#}\cap F|}{d_i}$. If $\alpha = \gamma$ in inequality (4), then we obtain the following

$$A(d) \leq \frac{2+\epsilon}{1-\alpha} \tag{5}$$

and we have $\mathrm{fpr}(x) \leq \alpha$. Now bound β for the tuple $x = (x_1, \ldots, x_r)$.

Following [4], if $x_i \in F$, then every power of x_i that are non-identity are also in F, because if any point fixed by x_i, then it is also fixed by x_i^l. Therefore, $f(x_i) \leq f(x_i^l)$. In this situation, there are $d_i - 1$ elements in F in $\langle x_i\rangle$. If $x_i \notin F$, then there are $\phi(d_i)$ generators in $\langle x_i\rangle$ where ϕ is the Euler's function.

All of these generators are not in F either, so there are at most $d_i - \phi(d_i) - 1$ elements in F in $\langle x_i \rangle$. We obtain

$$\beta \leqslant \sum_{x_i \in F} \frac{d_i - 1}{d_i} + \sum_{x_i \notin F} \frac{d_i - \phi(d_i) - 1}{d_i}.$$

Notice that Lemma 2.3 will tell us for the tuple of length 3, that at most one element lies in F.

In $\mathrm{PGL}(4,q)$ and $\mathrm{PSL}(4,q)$, $\alpha = \frac{q^2+q+2}{q^3+q^2+q+1}$ and $\gamma = \frac{2q+2}{q^3+q^2+q+1}$, $\epsilon = \frac{-2}{q^3+q^2+q+1}$ for every genus 0 tuples.

Let $q \geq 16$, then using inequality (5), we have $A(d) \leq \frac{32}{15}$. If $r \geq 4$, then $A(d) \geq A((2,2,2,3)) = \frac{13}{6}$. But $\frac{32}{15} < \frac{13}{6}$. So the number of branch points r must be 3.

Now we are looking for signatures which satisfy the inequality $\frac{85}{42} \leq A(d) \leq \frac{32}{15}$. This leads d only can be $(2,3,d_3)$ with $7 \leq d_3 \leq 30$, $(2,4,d_3)$ with $5 \leq d_3 \leq 8$, $(2,5,5)$, $(2,5,6)$, $(3,3,4)$, $(3,3,5)$.

Now, we compute β for all signatures which satisfy $\frac{85}{42} \leq A(d) \leq \frac{32}{15}$.

$d =$	$\beta \leqslant$
$(2,3,n)$ with $7 \leqslant n \leqslant 30$	$\frac{41}{30}$
$(2,4,n)$ with $5 \leqslant n \leqslant 8$	$\frac{5}{4}$
$(2,5,n)$ with $5 \leqslant n \leqslant 6$	$\frac{13}{10}$
$(3,3,n)$ with $4 \leqslant n \leqslant 5$	$\frac{11}{12}$

In the above table the maximum β is $\frac{41}{30}$. Now set $\beta \leq \frac{41}{30}$ and $q \geq 16$. We substitute them in inequality (4) and we obtain that $A(d) \leq \frac{9064}{4335}$. From this, we find all signatures d, which are the following:

$d =$	$\beta \leqslant$
$(2,3,n)$ with $7 \leqslant n \leqslant 13$	$\frac{5}{4}$
$(2,4,n)$ with $4 \leqslant n \leqslant 6$	$\frac{5}{4}$
$(3,3,4)$	$\frac{11}{12}$

Again, we choose the maximum β in the above table which is $\beta \leq \frac{5}{4}$. So we put $\beta \leq \frac{5}{4}$ and $q \geq 16$ in inequality (4) and hence $A(d) \leq \frac{3012}{1445}$. Therefore, all signatures are $(2,3,d_3)$ with $7 \leq d_3 \leq 12$, $(2,4,5)$,$(2,4,6)$, $(3,3,4)$.

Finally, for each signature d we can compute β and $A(d)$ and put in inequality (4). So we can solve it and obtaining the values of q.

THEOREM 3.1. *If* $\mathrm{PGL}(4,q)$ *or* $\mathrm{PSL}(4,q)$ *possesses genus zero system, then one of the following holds:*

(i) $q \leq 13$;

(ii) d *and* q *as shown in the following table*

d	β	$A(d)$	q
(2,3,7)	6/7	85/42	16, 17, 19, 23, 25, 27, 29, 31, 32, 37
(2,3,8)	25/24	49/24	16, 17, 19, 23, 25
(2,3,9)	8/9	37/18	16, 17
(2,3,10)	7/6	31/15	16, 17
(2,3,12)	5/4	25/12	16
(2,4,5)	21/20	41/20	16, 17, 19
(2,4,6)	5/4	25/12	16
(3,3,4)	5/4	25/12	16

The next results are devoted to compute indices of elements of order 2, 3, 4 and 5 in $\mathrm{PSL}(4,q)$. Let e_d be an element of order d in G.

LEMMA 3.2. *In* $G = \mathrm{PSL}(4,q)$:

(i) *If* $2 \nmid q$, *then* $f(e_2) = 0$, $\operatorname{ind} e_2 = \frac{N}{2}$ *or* $f(e_2) = 2q+2$, $\operatorname{ind} e_2 = \frac{N-(2q+2)}{2}$.

(ii) *If* $2 \mid q$, *then* $f(e_2) = q^2+q+1$, $\operatorname{ind} e_2 = \frac{N-(q^2+q+1)}{2}$ *or* $f(e_2) = q+1$, $\operatorname{ind} e_2 = \frac{N-(q+1)}{2}$.

Proof. There are at most two conjugacy classes of involutions in G. For each such class, we give a representative e_2. Let Z be the center of $\mathrm{SL}(4,q)$.

(i) Suppose that q is even. Note that, since $Z = \{I\}$, we can identify G with $\mathrm{SL}(4,q)$. Take the involution

$$e_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

whose only eigenvalue is 1. The corresponding eigenspace is

$$E_1 = \left\{ (v_2, v_2, v_3, v_4)^T \,:\, v_2, v_3, v_4 \in GF(q) \right\}.$$

Since it has dimension 3, from Table 1 we achieve that e_2 has q^2+q+1 fixed points. Therefore, $\operatorname{ind} e_2 = \frac{N-(q^2+q+1)}{2}$. As representative of the other class, we can take

$$e_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

whose only eigenvalue is 1. The associated eigenspace, which has dimension 2, is

$$E_1 = \{(v_2, v_2, v_4, v_4)^T \,:\, v_2, v_4 \in GF(q)\}.$$

From Table 1, we obtain that e_2 has $q+1$ fixed points, whence $\operatorname{ind} e_2 = \frac{N-(q+1)}{2}$.

(ii) Suppose that q is odd. The matrix

$$e_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is a non central element of $\mathrm{SL}(4,q)$: its projective image in G has order 2. The eigenvalues of e_2 are 1 and -1. The associated eigenspaces are

$$E_1 = \{(0, 0, v_3, v_4)^T \,:\, v_3, v_4 \in GF(q)\},$$
$$E_{-1} = \{(v_1, v_2, 0, 0)^T \,:\, v_1, v_2 \in GF(q)\}.$$

They both have dimension 2: from Table 1, we get that e_2 has $2q+2$ fixed points. Hence, $\operatorname{ind} e_2 = \frac{N-(2q+2)}{2}$.

(iii) Suppose that $q \equiv 3 \pmod 4$ (so $Z = \{\pm I\}$). In this case, we have another conjugacy class of involutions. Take

$$e_2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and note that $g^2 = -I \in Z$. Hence, the projective image of e_2 in $\mathrm{PSL}(4,q)$ has order 2. The characteristic polynomial of e_2 is $(x^2+1)^2$, which has no root in $GF(q)$. From Table 1, we deduce that $\operatorname{ind} e_2 = \frac{N}{2}$. ∎

In $G = \mathrm{PSL}(4,q)$, if $3 \mid q-5$, then there are two conjugacy classes of elements of order 3. Otherwise there are four conjugacy classes of elements of order 3.

LEMMA 3.3. *In* $G = \mathrm{PSL}(4,q)$*:*

(i) *If* $q \equiv 2 \pmod 3$*, then* $f(e_3) = 0,\ q+1$, $\operatorname{ind} e_3 = \frac{2N}{3},\ \frac{2}{3}(N-q-1)$.

(ii) *If* $q \equiv 1 \pmod 3$*, then* $f(e_3) \in \{2q+2, q+3, q^2+q+2\}$, $\operatorname{ind} e_3 \in \left\{\frac{2}{3}(N-2q-2),\ \frac{2}{3}(N-q-3),\ \frac{2}{3}(N-q^2-q-2)\right\}$.

(iii) *If* $q \equiv 0 \pmod 3$*, then* $f(e_3) \in \{q+1, q^2+q+1\}$, $\operatorname{ind} e_3 \in \left\{\frac{2}{3}(N-q-1),\ \frac{2}{3}(N-q^2-q-1)\right\}$.

Proof. Suppose element e_3 has prime order 3 in G. Then all powers of e_3 except the identity have the same fixed points. Now $\operatorname{ind} e_3 = \frac{2}{3}(q^3+q^2+q+1-f(e_3))$ and $\operatorname{ind} e_3$ is an integer.

(i) Since 3 divides $(q^3+q^2+q+1-f(e_3))$, this gives $f(e_3) \in \{0, 3, q+1, 2q+2\}$. Next, we will show that $2q+2$ and 3 can not exist. We check only the first $2q+2$. Suppose that v is an eigenvector of $\bar{e}_3$ then $\bar{e}_3 v = \lambda v$ for some nonzero number in $GF(q)$. So $v = Iv = (\bar{e}_3)^3 v = \lambda^3 v$. So $\lambda^3 = 1$. But $3 \nmid q-1$, there is no element of order 3 in $GF(q)$, we obtain $\lambda = 1$. So all eigenvector of $\bar{e}_3$ belong to eigenvalue 1. Suppose that $\bar{e}_3$ fixes $2q+2$ points, $\bar{e}_3$ has two 2-dimensional eigenspaces. Both of them belong to 1. We get $\bar{e}_3$ is the identity. This is a contradiction. In similar way, proving 3 can not exist.

(ii) Since 3 divides $(q^3+q^2+q+1-f(e_3))$, then $f(e_3) \in \{1, q+3, 2q+2, q^2+q+2\}$. Next we will show that 1 can not exist. Since $3 \mid q-1$, then $\bar{e}_3$ is conjugate to one of the following:

$$\begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}, \quad \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \beta & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\alpha = \beta^{-1}$ is a fixed element of order 3. This implies that $f(e_3) \in \{q+3, 2q+2, q^2+q+2\}$. Therefore, $\operatorname{ind} e_3 \in \{\frac{2}{3}(N-2q-2), \frac{2}{3}(N-q-3), \frac{2}{3}(N-q^2-q-2)\}$.

(iii) Since 3 divides $(q^3+q^2+q+1-f(e_3))$, so $f(e_3) \in \{1, q+1, q^2+q+1\}$. In similar way, proving 1 can not exist. So $\operatorname{ind} e_3 \in \{\frac{2}{3}(N-q-1), \frac{2}{3}(N-q^2-q-1)\}$. ∎

Lemma 3.4. *In* PSL$(4,q)$*:*

(i) *If* $q \equiv 1 \pmod 4$, *then* $f(e_4) = f(e_4^2) = 0$ *and* $\operatorname{ind} e_4 = \frac{3N}{4}$, *or* $f(e_4) = q+1$, $f(e_4^2) = 2q+2$ *and* $\operatorname{ind} e_4 = \frac{3N-(4q+4)}{4}$.

(ii) *If* $q \equiv 0 \pmod 4$, *then* $f(e_4) = q+1$, $f(e_4^2) = q^2+q+1$ *and* $\operatorname{ind} e_4 = \frac{3N-\left(q^2+3q+3\right)}{4}$, *or* $f(e_4) = 1$, $f(e_4^2) = q+1$ *and* $\operatorname{ind} e_4 = \frac{3N-(q+3)}{4}$.

(iii) *If* $q \equiv 3 \pmod 4$, *then* $f(e_4) \in \left\{0, 2q+2, q+3, q^2+q+2, q+3\right\}$, $f(e_4^2) \in \{\underbrace{2q+2}_{3\text{-}times}, \underbrace{q^2+q+2}_{2\text{-}times}\}$ *and* $\operatorname{ind} e_4 \in \left\{\frac{3N-(6q+6)}{4}, \frac{3N-(2q+2)}{4}, \frac{3N-(4q+8)}{4}, \frac{3N-3\left(q^2+q+2\right)}{4}, \frac{3N-\left(q^2+3q+8\right)}{4}\right\}$.

Proof. The proof is similar as Lemma 3.3. ∎

Lemma 3.5. *In* PSL$(4,q)$*:*

(i) *If* $q \equiv 1 \pmod 5$, *then* $f(e_5) \in \left\{q+3, 2q+2, q^2+q+2\right\}$ *and* $\operatorname{ind} e_5 = \frac{4N-4f(e_5)}{5}$.

(ii) *If* $q \equiv 2 \pmod 5$ *or* $q \equiv 3 \pmod 5$, *then* $f(e_5) = 0$ *and* $\operatorname{ind} e_5 = \frac{4N}{5}$.

(iii) *If* $q \equiv 4 \pmod 5$, *then* $f(e_5) \in \{0, q+1\}$ *and* $\operatorname{ind} e_5 = \frac{4N}{5}, \frac{4N-4(q+1)}{5}$.

(iv) *If* $q \equiv 0 \pmod 5$, *than* $f(e_5) \in \left\{1, q+1, q^2+q+1\right\}$ *and* $\operatorname{ind} e_5 = \frac{4N-4f(e_5)}{5}$.

Proof. The proof is similar as Lemma 3.3. ∎

Table 2: Indices of some elements in PSL$(4,q)$

q	16	17	19	23	25
$\operatorname{ind} e_6$	3498, 3578, 3588, 3626				
$\operatorname{ind} e_8$		4536, 4548, 4556, 4560, 4552	6324	11130, 11106, 11118	14222, 14214, 14226
$\operatorname{ind} e_9$	3822	4640, 4624, 4636			
$\operatorname{ind} e_{10}$	3884, 3916, 3788, 3896				
$\operatorname{ind} e_{12}$	3930				

Proposition 3.6. *In* $\mathrm{PSL}(4,q)$*, there is no generating tuple of genus zero if* $16 \leq q \leq 37$.

Proof. From Theorem 3.1, we have to deal with seven possible signatures in the different groups $\mathrm{PSL}(4,q)$. Since $7 \nmid |\mathrm{PSL}(4,q)|$ where $q = 17$, 19, 31, there is no signature $(2,3,7)$ in $\mathrm{PSL}(4,q)$ and $8 \nmid |\mathrm{PSL}(4,16)|$, there is no signature $(2,3,8)$ in $\mathrm{PSL}(4,16)$. Also, $10 \nmid |\mathrm{PSL}(4,17)|$, there is no signature $(2,3,10)$ in $\mathrm{PSL}(4,17)$. If $q = 16$, 23, 25, 32, 37, then $f(e_7) = 1$ and $\operatorname{ind} e_7 = \frac{6N-6}{7}$. If $q = 27$, then $f(e_7) = 0$, $q+1$ and $\operatorname{ind} e_7 = \frac{6N}{7}, \frac{6N-6(q+1)}{7}$. If $q = 29$, then $f(e_7) = 2q+2$, $q+3$, q^2+q+2 and $\operatorname{ind} e_7 = \frac{6N-6(2q+2)}{7}, \frac{6N-6(q+3)}{7}, \frac{6N-6(q^2+q+2)}{7}$. We can compute the indices of elements of order 2 and 3 by Lemma 3.2 and Lemma 3.3. The sum of the indices of the signature $(2,3,7)$ does not fit the Riemann Huwrtiz formula. By using Lemma 3.2, Lemma 3.3, Lemma 3.4 and Lemma 3.5, we can compute the indices of elements of orders 2, 3, 4 and 5. On the other hand, from Table 2, we can get the indices of the elements of the other orders. Therefore, the sum of the indices of the given signatures do not fit the Riemann Hurwitz formula. This completes the proof. ∎

Lemma 3.7. *The groups* $\mathrm{PSL}(4,q)$ *do not possess genus zero system, if* $q = 7, 9$.

Proof. The corresponding tuples of the following signatures satisfy the Riemann Hurwitz formula $(2,3,d)$, $d \in \{7,8,9,14,16,19,24,28,42\}$ and $(3,3,d)$, $d \in \{7,8,9,14\}$. However none of them generate the group $\mathrm{PSL}(4,7)$ that is, do not satisfy (3). Also, the associated tuple of the signature $(2,4,6)$ fits the Riemann Hurwitz formula. It is not satisfied (3). ∎

The following GAP codes can be used to show that there is no tuples satisfying the Riemann Hurwitz formula:

```
cc:=List(ConjugacyClasses(group),Representative);;
N:=DegreeAction(group);;
ind:=List(cc,x->N-Length(Orbits(Group(x),[1..N])));;
ss:=Elements(ind);;
s:=Difference(ss,[ss[1]]);;
poss:=RestrictedPartitions(2N-2,s);
```

Lemma 3.8. *The groups* $\mathrm{PSL}(4,q)$ *do not possess genus zero system if* $q = 8, 11, 13$.

Proof. The proof is a straightforward computation. ∎

THEOREM 3.9. *If G is the projective symplectic group with* $\mathrm{PSp}(4,q) \leq G \leq \mathrm{Aut}(\mathrm{PSp}(4,q))$*,* $q > 5$*, then G does not possess genus zero system.*

Proof. Since $\mathrm{PSp}(4,q)$ is a subgroup of $\mathrm{PSL}(4,q)$, then from Theorem 3.1, Proposition 3.6, Lemma 3.7 and Lemma 3.8, we show that the group $\mathrm{PSL}(4,q)$ does not possess genus zero system. So is $\mathrm{PSp}(4,q)$, as desired. ∎

4. CONNECTED COMPONENTS OF THE HURWITZ SPACE

The details of the following can be found in [6]. The computation shows that there are exactly 165 braid orbits of G. The degree and the number of the branch points are given in Table 3. Furthermore, we discuss the connectedness of the Hurwitz space for these groups.

Our main result is Theorem 4.1, which gives the complete classification of primitive genus 0 systems of G.

THEOREM 4.1. *Up to isomorphism, there exist exactly* 5 *primitive genus zero groups G with* $\mathrm{PSp}(4,q) \leq G \leq \mathrm{Aut}(PSp(4,q))$ *for* $q \leq 5$*. The corresponding primitive genus zero groups are enumerated in Tables* 5*,* 6 *and* 4*.*

This will be done by both the proof in algebraic topology and calculations of GAP (Groups, Algorithms, Programming) software [2].

PROPOSITION 4.2. *If* $G = \mathrm{PSp}(4,4).2$ *and* $|\Omega| = 85A$*, then* $\mathcal{H}_r^{in}(G,C)$ *is connected.*

Proof. Since we have just one braid orbit for all types C and the Nielsen classes $\mathcal{N}(C)$ are the disjoint union of braid orbits. From [6, Proposition 2.4], we obtain that the Hurwitz space $\mathcal{H}_r^{in}(G,C)$ is connected. ∎

The proof of the following proposition is similar as Proposition 4.2.

PROPOSITION 4.3. *If* $G = \mathrm{PSp}(4,3)$ *is the projective symplectic group and* $r > 3$*, then* $\mathcal{H}_r^{in}(G,C)$ *is connected.*

PROPOSITION 4.4. *If* $G = \mathrm{PSp}(4,4)$ *and* $|\Omega| = 120A$*, then* $\mathcal{H}_r^{in}(G,C)$ *is disconnected.*

Proof. Since we have more than one braid orbits for some types C and the Nielsen classes $\mathcal{N}(C)$ are the disjoint union of braid orbit. We obtain from [6, Proposition 2.4] that the Hurwitz space $\mathcal{H}_r^{in}(G, C)$ is disconnected. ∎

The proof of the following proposition is similar as Proposition 4.4.

Proposition 4.5. *If* $G = \mathrm{PSp}(4,5)$ *and* $|\Omega| = 156$, *then* $\mathcal{H}_r^{in}(G, C)$ *is disconnected.*

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Appendix

Table 3: Genus Zero Groups: Number of Components

			Number of connected components			
Degree	Number of Group up to Isomorphism	Number of Ramification Types	with $r = 3$	with $r = 4$	with $r = 5$	total
27	2	49	7	15	1	23
36	2	18	19	5	-	24
$40A$	2	20	39	4	-	43
$40B$	2	15	26	2	-	28
45	2	11	18	2	-	20
$120A$	1	1	4	-	-	4
$85A$	1	1	1	-	-	1
$156A$	1	2	22	-	-	22
Total	13	117	136	28	1	165

Table 4: Genus Zero Systems for Projective symplectic Groups

Degree	group	ramification type	N.O	L.O	ramification type	N.O	L.O
$120A$	$\mathrm{PSp}(4,4)$	(2A,4B,5E)	4	1			
$85A$	$\mathrm{PSp}(4,4).2$	(2C,4B,15A)	1	1			
$156A$	$\mathrm{PSp}(4,5)$	(2B,4B,5B)	11	1	(2B,4B,5A)	6	1

Table 5: Genus Zero Systems for PSp(4, 3)

Degree	ramification type	N.O	L.O	ramification type	N.O	L.O
27	(2A,5A,6B)	1	1	(2A,5A,6A)	1	1
	(2A,6F,9B)	3	1	(2A,6F,9A)	3	1
	(2A,6F,12B)	1	1	(2A,6F,12A)	1	1
	(2A,6D,9B)	1	1	(2A,6D,12B)	1	1
	(2A,6C,9A)	1	1	(2A,6C,12A)	1	1
	(2A,4B,9B)	1	1	(2A,4B,9A)	1	1
	(2A,4A,9B)	3	1	(2A,4A,9A)	3	1
	(2A,4A,12B)	3	1	(2A,4A,12A)	3	1
	(2A,2A,2A,6B)	1	9	(2A,2A,2A,6A)	1	9
36	(2B,4B,9B)	3	1	(2B,4B,9A)	3	1
	(2B,4B,12B)	3	1	(2B,4B,12A)	3	1
	(2B,6B,5A)	1	1	(2B,6B,9B)	1	1
	(2B,4A,9B)	1	1	(2B,4A,9A)	1	1
	(2B,2B,2B,6B)	1	9	(2B,2B,2B,6A)	1	9
$40A$	(2B,6B,5A)	1	1	(2B,6B,9B)	1	1
	(2B,6A,5A)	1	1	(2B,6A,9B)	1	1
	(2B,4A,5A)	1	1	(2B,4A,9B)	1	1
$40B$	(2B,4A,9B)	1	1	(2B,4A,9A)	1	1
	(2B,6C,5A)	1	1	(2B,6B,5A)	1	1
	(2B,5A,12B)	1	1	(2B,5A,12A)	1	1
	(2B,5A,9B)	1	1	(2B,5A,9A)	1	1
45	(2B,4A,9A)	1	1	(2B,4A,9B)	1	1
	(2B,6B,5A)	1	1	(2B,6A,5A)	1	1

Table 6: Genus Zero Systems for PSp(4, 3) : 2

Degree	ramification type	N.O	L.O	ramification type	N.O	L.O
27	(2C,4A,10A)	1	1	(2C,5A,6A)	2	1
	(2A,12B,9A)	1	1	(2D,6A,8A)	2	1
	(2D,6A,10A)	2	1	(2C,4D,12A)	2	1
	(2C,6G,12B)	2	1	(2C,4D,9A)	3	1
	(2C,6G,10A)	2	1	(2C,6F,12A)	2	1
	(2C,6F,9A)	3	1	(2C,4C,12B)	4	1
	(2C,4C,8A)	6	1	(2C,4C,10A)	5	1
	(2C,6E,12B)	3	1	(2C,6E,8A)	4	1
	(2C,6E,10A)	4	1			
	(2D,2D,2C,6A)	1	24	(2D,2C,2C,6G)	1	48
	(2D,2C,2C,4C)	1	112	(2D,2C,2C,6E)	1	78
	(2D,2C,2C,4A)	1	20	(2A,2A,4A,6B)	1	1
	(2C,2C,2C,4D)	1	96	(2C,2C,2C,6F)	1	108
	(2A,2D,2C,12A)	1	12	(2A,2D,2C,9A)	1	18
	(2A,2C,2C,12B)	1	24	(2A,2C,2C,8A)	1	32
	(2A,2C,2C,10A)	1	30	(2A,2D,2C,2C,2C)	1	648
36	(2C,6F,5A)	2	1	(2C,4B,10A)	5	1
	(2C,4B,8A)	6	1	(2C,4B,12B)	4	1
	(2C,4A,10A)	1	1	(2B,6F,10A)	2	1
	(2B,6F,8A)	2	1	(2C,2D,2D,4C)	1	112
	(2C,2D,2D,4A)	1	20	(2C,2C,2D,6E)	1	24
$40A$	(2D,4D,8A)	6	1	(2D,6E,12A)	1	1
	(2D,6E,10A)	1	1	(2D,6D,8A)	2	1
	(2D,6A,9A)	6	1	(2D,6A,5A)	4	1
	(2D,6A,5A)	7	1	(2D,4B,9A)	3	1
	(2D,4B,12B)	2	1	(2D,4A,10A)	1	1
	(2C,2C,2D,6A)	1	24	(2C,2C,2D,4A)	1	20
	(2C,2C,2C,4B)	1	96	(2C,2C,2C,6B)	1	234
$40B$	(2D,4C,8A)	6	1	(2D,6E,5A)	7	1
	(2D,6D,8A)	2	1	(2D,6C,5A)	2	1
	(2D,4A,10A)	1	1			
	(2D,2D,2C,4A)	1	20	(2A,2D,2D,5A)	1	35
45	(2B,4D,8A)	6	1	(2B,4B,10A)	1	1
	(2B,4A,9A)	3	1	(2B,4A,12A)	2	1
	(2B,6D,5A)	2	1			
	(2B,2B,2D,4B)	1	20	(2B,2B,2B,4A)	1	96