EXTRACTA MATHEMATICAE

Article in press
Available online December 14, 2022

A note on isomorphisms of quantum systems

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Received July 4, 2022
Accepted November 9, 2022

Presented by Martin Mathieu

Abstract: We consider the question as to whether a quantum system is uniquely determined by all values of all its observables. For this, we consider linearly nuclear GB*-algebras over W*-algebras as models of quantum systems.

Key words: quantum system, observables, GB*-algebra, Jordan homomorphism.

MSC (2020): 46H05, 46H15, 46H35, 47L10, 47L30, 81P05, 81P16.

1. Introduction

The main objective of this paper is to determine whether all values of all observables in a quantum system are sufficent to determine the quantum system uniquely. To answer this question, we first have to find a suitable mathematical framework in which to reformulate the question.

In the well known formalism of Haag and Kastler, a quantum system takes on the following form: The observables of the system are self-adjoint elements of a $*$-algebra A with identity element 1, and the states of the system are positive linear functionals ϕ of A for which $\phi(1) = 1$. This is well in agreement with the Hilbert space formalism, where the observables are linear operators on a Hilbert space H, and all states are unit vectors in H. Since observables are generally unbounded linear operators on a Hilbert space (such as position and momentum operators, which are unbounded linear operators on the Hilbert space $L^2(\mathbb{R})$), one requires the $*$-algebra A above to at least partly consist of unbounded linear operators on some Hilbert space. The question is then what $*$-algebra of unbounded linear operators one must take to house the observables of the quantum system under consideration. A candidate can be found among the elements in the class of GB*-algebras, which are locally

* This work is based on the research supported wholly by the National Research Foundation of South Africa (Grant Number 132194).

ISSN: 0213-8743 (print), 2605-5686 (online)

convex $*$-algebras serving as generalizations of C*-algebras, and were first studied by G.R. Allan in [2], and later by P.G. Dixon in [6] to include non locally convex $*$-algebras (see Section 2 for the definition of a GB*-algebra). Every GB*-algebra $A[\tau]$ contains a C*-algebra $A[B_0]$ which is dense in A (see Section 2).

In [13], the author motivated why one can model a quantum system as a GB*-algebra $A[\tau]$ which is nuclear as a locally convex space (referred to as a *linearly nuclear* GB*-algebra for here on). In addition to this, it would be useful to have that $A[B_0]$ is a W*-algebra (i.e., a von Neumann algebra): Since $A[\tau]$ is also assumed to be locally convex, A can be faithfully represented as a $*$-algebra B of closed densely defined linear operators on a Hilbert space (see [6, Theorem 7.11] or [10, Theorem 6.3.5]). If we denote this $*$-isomorphism by $\pi : A \to B$, then $\pi(A[B_0]) = B_b$, where B_b is the $*$-algebra of all bounded linear operators in B, and is a von Neumann algebra (this follows from [6, Theorem 7.11], or [10, Theorem 6.3.5]). Let $x \in A$ be self-adjoint. Then $\pi(x)$ is a self-adjoint element of B and $\pi(x) = \int_{\sigma(\pi(x))} \lambda d\ P_\lambda$.

Now $(1 + y^*y)^{-1} \in B_b$ for all $y \in B$. By [7, Proposition 2.4], it follows that all $y \in B$ are affiliated with B_b. Therefore, $P_\lambda \in B_b$ for all $\lambda \in \sigma(\pi(x))$. The spectral projections, P_λ, $\lambda \in \sigma(\pi(x))$, are important for determining the probability of a particle in a certain set (see [8, Postulate 4, p. 13]).

So far, the observables of a quantum mechanical system are self-adjoint elements of a locally convex $*$-algebra $A[\tau]$ (more specifically, in our case, a linearly nuclear GB*-algebra with $A[B_0]$ a W*-algebra). In fact, one can sharpen this by noting that if $x, y \in A$ are self-adjoint (i.e., observables), then $x \circ y = \frac{1}{2}(xy + yx)$ is again self-adjoint, i.e., an observable. In 1932, J. von Neumann and collaborators proposed that a Jordan algebra be used to house the observables of a quantum system (see [5, Introduction]). A linear mapping $\phi : A \to A$ with $\phi(x \circ y) = \phi(x) \circ \phi(y)$ for all $x, y \in A_s$, where A_s denotes the set of self-adjoint elements of A, is called a *Jordan homomorphism*. We note here that A_s is a Jordan algebra with respect to the operation $\circ$ above. If, in addition, ϕ is a bijection, then ϕ is called a *Jordan isomorphism*, i.e., an isomorphism of Jordan algebras. A Jordan isomorphism is therefore an isomorphism of quantum systems (see [5, Introduction]). Observe that a linear map ϕ is a Jordan homomorphism if and only if $\phi(x^2) = \phi(x)^2$ for all $x \in A$.

We already know that all possible values of an observable, when considered as a self-adjoint unbounded linear operator on a Hilbert space, are in the spectrum of the observable. An interesting question is therefore if a quantum system is uniquely determined by the values/measurements of its observables.

To answer this question in our setting of a linearly nuclear GB*-algebra $A[\tau]$ with $A[B_0]$ a W*-algebra (an abstract algebra of unbounded linear operators), one requires a notion of spectrum of an element which is an analogue of the notion of spectrum of a self-adjoint unbounded linear operator on a Hilbert space. The required notion is the Allan spectrum of an element of a locally convex algebra. The values/measurements of a self-adjoint element $x \in A$ (i.e., an observable) are therefore in $\sigma_A(x)$, the Allan spectrum of x, as defined in Definition 2.3 below in Section 2. If no confusion arises, we write $\sigma(x)$ instead of $\sigma_A(x)$.

The above question can be reformulated as follows: Let $A[\tau]$ be a linearly nuclear GB*-algebra with $A[B_0]$ a W*-algebra. Let $\phi : A \to A$ be a bijective self-adjoint linear map such that $\sigma(\phi(x)) = \sigma(x)$ for all $x \in A_s$, where A_s is the set of all self-adjoint elements of A. Is ϕ a Jordan isomorphism?

Below, in Corollary 3.6, we answer this question affirmatively for the case where $A[\tau]$ has the additional property of being a Fréchet algebra, i.e., a complete and metrizable algebra. We do not require the GB*-algebra to be linearly nuclear in this result.

The above result is similar to results which are partial answers to a special case of an unanswered question of I. Kaplansky: If A and B are Banach algebras with identity, and $\phi : A \to B$ is a bijective linear map such that $\mathrm{Sp}_B(\phi(x)) = \mathrm{Sp}_A(x)$ for all $x \in A$, is it true that ϕ is a Jordan isomorphism? Here, $\mathrm{Sp}_A(x)$ refers to the spectrum of x, which is the set $\{\lambda \in \mathbb{C} : \lambda 1 - x \text{ is not invertible in } A\}$. The answer to this question remains unresolved for C*-algebras, but it has been shown, by B. Aupetit, to have an affirmative answer if A and B are von Neumann algebras (see [3, Theorem 1.3]). For the physical problem under consideration, we have to replace the spectrum of x in Kaplansky's question with the Allan spectrum of x, as explained above. We refer the reader to [4] for an excellent introduction to Kaplansky's problem.

Section 2 of this paper contains all the background material required to understand the discussion in Section 3, where the main result is presented.

2. Preliminaries

In this section, we give all background material on generalized GB*-algebras (GB*-algebras, for short) which is required to understand the main results of this paper. GB*-algebras were introduced in the late sixties by G.R. Allan in [2], and taken further, in the early seventies, by P.G. Dixon in

[6, 7]. Recently, the author, along with M. Fragoulopoulou, A. Inoue and I. Zarakas, published a monograph on GB*-algebras [10] containing much of the developed theory on this topic. Almost all concepts and results in this section are due Allan and Dixon, and can be found in [1, 2, 6]. We will, however, use [10] as a reference.

A *topological algebra* is an algebra which is a topological vector space and in which multiplication is separately continuous. If a topological algebra is equipped with a continuous involution, then it is called a *topological* $*$*-algebra*. A *locally convex* $*$*-algebra* is a topological $*$-algebra which is locally convex as a topological vector space. We say that a topological algebra is a *Fréchet algebra* if it is complete and metrizable.

Definition 2.1. ([10, Definition 3.3.1]) Let $A[\tau]$ be a unital topological $*$-algebra and let $\mathcal{B}^*$ denote a collection of subsets B of A with the following properties:

(i) B is absolutely convex, closed and bounded;

(ii) $1 \in B$, $B^2 \subset B$ and $B^* = B$.

For every $B \in \mathcal{B}^*$, denote by $A[B]$ the linear span of B, which is a normed algebra under the gauge function $\|\cdot\|_B$ of B. If $A[B]$ is complete for every $B \in \mathcal{B}^*$, then $A[\tau]$ is called *pseudo-complete*.

An element $x \in A$ is called *bounded*, if for some nonzero complex number λ, the set $\{(\lambda x)^n : n = 1, 2, 3, \dots\}$ is bounded in A. We denote by A_0 the set of all bounded elements in A.

A unital topological $*$-algebra $A[\tau]$ is called *symmetric* if, for every $x \in A$, the element $(1 + x^*x)^{-1}$ exists and belongs to A_0.

Definition 2.2. ([10, Definition 3.3.2]) A symmetric pseudo-complete locally convex $*$-algebra $A[\tau]$, such that the collection $\mathcal{B}^*$ has a greatest member, denoted by B_0, is called a *GB*-algebra* over B_0.

Every C*-algebra is a GB*-algebra. An example of a GB*-algebra, which generally need not be a C*-algebra, is a pro-C*-algebra. By a *pro-C*-algebra*, we mean a complete topological $*$-algebra $A[\tau]$ for which the topology τ is defined by a directed family of C*-seminorms.

Another example of a GB*-algebra which is not a pro-C*-algebra is the locally convex $*$-algebra $L^\omega([0,1]) = \cap_{p\geq 1} L^p([0,1])$ defined by the family of seminorms $\{\|\cdot\|_p : p \geq 1\}$, where $\|\cdot\|_p$ is the L^p-norm on $L^p([0,1])$ for all $p \geq 1$.

If A is commutative, then $A_0 = A[B_0]$ [10, Lemma 3.3.7(ii)]. In general, A_0 is not a $*$-subalgebra of A, and $A[B_0]$ contains all normal elements of A_0, i.e., all $x \in A$ such that $xx^* = x^*x$ [10, Lemma 3.3.7(i)].

Definition 2.3. ([10, Definition 2.3.1]) Let $A[\tau]$ be topological algebra with identity element 1 and $x \in A$. The set $\sigma_A(x)$ is the subset of $\mathbb{C}^*$, the one-point compactification of $\mathbb{C}$, defined as follows:

(i) if $\lambda \neq \infty$, then $\lambda \in \sigma_A(x)$ if $\lambda 1 - x$ has no bounded inverse in A;

(ii) $\infty \in \sigma_A(x)$ if and only if $x \notin A_0$.

We define $\rho_A(x)$ to be $\mathbb{C}^* \setminus \sigma_A(x)$.

If there is no risk of confusion, then we write $\sigma(x)$ to denote $\sigma_A(x)$.

Proposition 2.4. ([10, Theorem 3.3.9, Theorem 4.2.11]) *If $A[\tau]$ is a GB*-algebra, then the Banach $*$-algebra $A[B_0]$ is a C*-algebra, which is sequentially dense in A. Moreover, $(1 + x^*x)^{-1} \in A[B_0]$ for every $x \in A$ and B_0 is the unit ball of $A[B_0]$.*

The next proposition has to do with extensions of characters of the commutative C*-algebra $A[B_0]$ to the GB*-algebra A, which could be infinite valued.

Proposition 2.5. ([10, Proposition 2.5.4]) *Let $A[\tau]$ be a commutative pseudocomplete locally convex $*$-algebra with identity. Then, for any character ϕ on A_0, there exists a $\mathbb{C}^*$-valued function ϕ' on A having the following properties:*

(i) *ϕ' is an extension of ϕ;*

(ii) *$\phi'(\lambda x) = \lambda \phi'(x)$ for all $\lambda \in \mathbb{C}$ (with the convention that $0.\infty = 0$);*

(iii) *$\phi'(x + y) = \phi'(x) + \phi'(y)$ for all $x, y \in A$ for which $\phi'(x)$ and $\phi'(y)$ are not both ∞;*

(iv) *$\phi'(xy) = \phi'(x)\phi'(y)$ for all $x, y \in A$ for which $\phi'(x)$ and $\phi'(y)$ are not both $0, \infty$ in some order;*

(v) *$\phi'(x^*) = \overline{\phi'(x)}$ for all $x \in A$ (with the convention that $\overline{\infty} = \infty$).*

3. The main result

The following example is an example of a linearly nuclear GB*-algebra over a W*-algebra, which is not a C*-algebra.

Example 3.1. Consider a family $\{H_\alpha : \alpha \in \Lambda\}$ of finite dimensional Hilbert spaces. Then, for every $\alpha \in \Lambda$, we have that $B(H_\alpha)$ is a finite dimensional C*-algebra, and hence a linearly nuclear space, with respect to the operator norm $\|\cdot\|_\alpha$. Let $A = \Pi_\alpha B(H_\alpha)$. Then A is a pro-C*-algebra in the product topology τ, when all $B(H_\alpha)$ are equipped with their operators norms $\|\cdot\|_\alpha$ [9, Chapter 2]. Furthermore, $A[\tau]$ is linearly nuclear since it is a product of linearly nuclear spaces. Observe that $x\xi = (x_\alpha(\xi_\alpha))_\alpha$ for all $\xi = (\xi_\alpha)_\alpha \in H$, where H is the direct sum of the Hilbert spaces H_α. Note that H is itself a Hilbert space. Now

$$\begin{aligned} A[B_0] &= \big\{x = (x_\alpha)_\alpha \in A : \sup_\alpha \|x_\alpha\|_\alpha < \infty\big\} \\ &= \oplus_\alpha B(H_\alpha), \end{aligned}$$

and this is a von Neumann algebra with respect to the norm $\sup_\alpha \|x_\alpha\|_\alpha$.

Lemma 3.2. *If x is a self-adjoint element of a GB*-algebra $A[\tau]$, then x is a projection if and only if $\sigma(x) \subseteq \{0, 1\}$.*

Proof. Let $x \in A$ be a projection and let B be a maximal commutative $*$-subalgebra of A containing x. Then $\sigma_B(x) = \sigma_A(x)$ (see [10, Proposition 2.3.2]) and B is a GB*-algebra over the C*-algebra $B_b = A[B_0] \cap B$ (see [6]). Let M_0 denote the character space of the commutative C*-algebra B_b. Then, by Proposition 2.5 and [10, Corollary 3.4.10], it follows that

$$\begin{aligned} \sigma_B(x) &= \big\{\widehat{x}(\phi) = \phi'(x) : \phi \in M_0\big\} \\ &= \big\{\phi(x) : \phi \in M_0\big\} \\ &\subseteq \{0, 1\}. \end{aligned}$$

The second equality above follows from the fact that $x \in A[B_0]$, due to the fact that x is a projection, and therefore $x \in B_b$. Therefore $\sigma_A(x) \subseteq \{0, 1\}$.

Now assume that $\sigma_A(x) \subseteq \{0, 1\}$. Let B be a maximal commutative $*$-subalgebra of A containing x. Then $\sigma_B(x) = \sigma_A(x)$. Like above, we have that

$$\big\{\widehat{x}(\phi) = \phi'(x) : \phi \in M_0\big\} = \sigma_B(x) = \sigma_A(x) \subseteq \{0, 1\}$$

for all characters ϕ on $A[B_0]$. Therefore $\widehat{x}$ is an idempotent function. Since $x \mapsto \widehat{x}$ is an algebra $*$-isomorphism [10, Theorem 3.4.9], we get that x is an idempotent element of A. Therefore x is a projection because x is self-adjoint. ∎

If A and B are $*$-algebras and $\phi : A \to B$ a linear map such that $\phi(x^2) = \phi(x)^2$ for all self-adjoint elements x in A, then ϕ is a Jordan homomorphism [3, page 922]. We require this in the proof of Proposition 3.3 below.

PROPOSITION 3.3. *Let $A[\tau]$ be a GB*-algebra with $A[B_0]$ a W*-algebra, and let B be a topological $*$-algebra. Suppose further that the multiplications on A and B are jointly continuous. If $\phi : A \to B$ is a continuous linear mapping which maps projections to projections, then ϕ is a Jordan homomorphism.*

Proof. Let s be a self-adjoint element in $A[B_0]$. By the spectral theorem, and the fact that $A[B_0]$ is a W*-algebra, there is a sequence (s_n) of finite linear combinations of orthogonal projections in $A[B_0]$ such that $s_n \to s$ in norm [11, Theorem 5.2.2], and hence also with respect to the topology τ on A, since the restriction of the topology τ to $A[B_0]$ is weaker than the norm topology of $A[B_0]$. Therefore $\phi(s_n^2) = \phi(s_n)^2$ for every n. Hence, since ϕ is continuous, and since the multiplications on A and B are jointly continuous, it follows that

$$\phi(s^2) = \phi\Big(\lim_{n\to\infty} s_n^2\Big) = \phi\Big(\lim_{n\to\infty} s_n\Big)^2 = \phi(s)^2.$$

This holds for any self-adjoint element $s \in A[B_0]$. By the paragraph following Lemma 3.2, $\phi|_{A[B_0]}$ is a Jordan homomorphism.

Let $x \in A$. Then there is a sequence (x_n) in $A[B_0]$ such that $x_n \to x$. Since ϕ is continuous, $A[B_0]$ is dense in A, and the multiplications on A and B are jointly continuous, it follows that $\phi(x^2) = \phi(x)^2$. This holds for every $x \in A$, and therefore ϕ is a Jordan homomorphism. ∎

We say that an element x in a GB*-algebra $A[\tau]$ is *positive* if there exists $y \in A$ such that $x = y^*y$. The following proposition is required to prove Theorem 3.5 below.

PROPOSITION 3.4. ([12, PROPOSITION 7]) *Let $A[\tau_1]$ and $B[\tau_2]$ be Fréchet GB*-algebras. If $\phi : A \to B$ is a linear mapping which maps positive elements of A to positive elements of B, then ϕ is continuous.*

Theorem 3.5. *Let $A[\tau]$ be a Fréchet GB*-algebra with $A[B_0]$ a W*-algebra, and let $\phi : A \to A$ be a self-adjoint linear map such that $\sigma(\phi(x)) \subseteq \sigma(x)$ for all $x \in A_s$, where A_s is the set of all self-adjoint elements of A. Then ϕ is a Jordan isomorphism.*

Proof. By hypothesis and [10, Proposition 6.2.1], it follows that if $x \in A$ is a positive element, then $\sigma(\phi(x)) \subset \sigma(x) \subseteq [0, \infty]$, and therefore $\phi(x)$ is a positive element in A. Therefore ϕ maps positive elements of A to positive elements of A. By Proposition 3.4 and the fact that A is a Fréchet GB*-algebra, it follows that ϕ is continuous.

We now show that if $p \in A$ is a projection, then $\phi(p)$ is also a projection in A. If $p \in A$ is a projection, then p and $\phi(p)$ are self-adjoint elements in A. Therefore, by Lemma 3.2, $\sigma(p) \subseteq \{0, 1\}$. Since $\sigma(\phi(p)) \subseteq \sigma(p)$, we get that $\sigma(\phi(p)) \subseteq \{0, 1\}$. By Lemma 3.2 again, $\phi(p)$ is a projection.

Since $A[B_0]$ is a W*-algebra and the multiplication on A is jointly continuous (because A is a Fréchet algebra), it follows from Proposition 3.3 that ϕ is a Jordan homomorphism. ∎

The following corollary is the desired result of this section, and affirms that all quantum mechanical isomorphisms, in the context of Fréchet GB*-algebras, are Jordan isomorphisms.

Corollary 3.6. *Let $A[\tau]$ be a Fréchet GB*-algebra with $A[B_0]$ a W*-algebra, and let $\phi : A \to A$ be a bijective self-adjoint linear map such that $\sigma(\phi(x)) = \sigma(x)$ for all $x \in A_s$, where A_s is the set of all self-adjoint elements of A. Then ϕ is a Jordan isomorphism.*

In [3], B. Aupetit proved that any bijective linear map $\phi : A \to B$ between von Neumann algebras A and B, satisfying $\mathrm{Sp}_B(\phi(x)) = \mathrm{Sp}_A(x)$ for all $x \in A$, is a Jordan homomorphism. Observe that ϕ need not be self-adjoint. The proof of Aupetit's result in [3] is complicated and relies on a deep spectral characterization of idempotents in a semi-simple Banach algebra (see [3, Theorem 1.1]). If we additionally assume that ϕ is self-adjoint, then one has a much simpler proof of his result, namely, the proof of Corollary 3.6 for the case where $A[\tau]$ is a von Neumann algebra.

Acknowledgements

The author wishes to thank the referee for his/her meticulous reading of the manuscript, which significantly enhanced its quality.

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