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On Generalized Lie Bialgebroids and Jacobi Groupoids

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Abstract: Generalized Lie bialgebroids are generalization of Lie bialgebroids and arises naturally from Jacobi manifolds. It is known that the base of a generalized Lie bialgebroid carries a Jacobi structure. In this paper, we introduce a notion of morphism between generalized Lie bialgebroids over a same base and prove that the induce Jacobi structure on the base is unique up to a morphism. Next we give a characterization of generalized Lie bialgebroids and use it to show that generalized Lie bialgebroids are infinitesimal form of Jacobi groupoids. We also introduce coisotropic subgroupoids of a Jacobi groupoid and these subgroupoids corresponds to, so called coisotropic subalgebroids of the corresponding generalized Lie bialgebroid.

Key words: Jacobi manifolds, coisotropic submanifolds, (generalized) Lie bialgebroids, Jacobi groupoids.

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1. Introduction

The notion of Lie bialgebroid was introduced by Mackenzie and Xu [11] as a generalization of Lie bialgebra and infinitesimal version of Poisson groupoid. Roughly, a Lie bialgebroid (A, A^*) over M is a Lie algebroid A over M such that its dual vector bundle A^* also carries a Lie algebroid structure which is compatible in a certain way with that of A. As an example, if (M, π) is a Poisson manifold, then (TM, T^*M) forms a Lie bialgebroid over M, where TM is the usual tangent Lie algebroid and T^*M is the cotangent Lie algebroid of the Poisson manifold M. A duality theorem for a Lie bialgebroid was shown in [11], that is, if (A, A^*) satisfy the criteria of a Lie bialgebroid, then (A^*, A) also satisfies a similar criteria. It was also proved in [11] that the base space of a Lie bialgebroid carries a natural Poisson structure. In [13], Kosmann-Schwarzbach gave a simple proof of the duality theorem for Lie bialgebroid and the fact that the base of a Lie bialgebroid has a Poisson structure.

Jacobi manifolds are generalization of Poisson manifolds. In [6], Iglesias and Marrero introduced a notion of generalized Lie bialgebroid (genaralization of Lie bialgebroid) in such a way that a Jacobi manifold associates a canonical generalized Lie bialgebroid structure. More precisely, a generalized Lie bialgebroid $\big((A,\phi_0),(A^*,X_0)\big)$ over M is a Lie algebroid A over M together with 1-cocycle $\phi_0 \in \Gamma A^*$ and such that the dual bundle A^* also carries a Lie algebroid structure with $X_0 \in \Gamma A$ be a 1-cocycle of it and satisfy some compatibility conditions like a Lie bialgebroid. Given a Jacobi manifold (M,Λ,E), if we consider the Lie algebroid $TM \times \mathbb{R} \to M$ with 1-cocycle $(0,1) \in \Gamma(T^*M \times \mathbb{R}) = \Gamma(T^*M) \oplus C^\infty(M)$ and the Lie algebroid on the dual 1-jet bundle $T^*M \times \mathbb{R} \to M$ with 1-cocycle $(-E,0) \in \Gamma(TM \times \mathbb{R}) = \Gamma(TM) \oplus C^\infty(M)$, then the pair $\big((TM \times \mathbb{R},(0,1)),(T^*M \times \mathbb{R},(-E,0))\big)$ is a generalized Lie bialgebroid over M [6]. Moreover it is shown in [6] that if $\big((A,\phi_0),(A^*,X_0)\big)$ is a generalized Lie bialgebroid over M, the base manifold M induces a Jacobi structure.

We remark that, Lie algebroid structures on a vector bundle $A \to M$ are in one-to-one correspondence with the linear Poisson structures on the dual bundle A^* [10]. This correspondence has been extended in [5] to the Jacobi setup by introducing a notion of linear Jacobi structure on a vector bundle. More precisely, they showed that given a Lie algebroid $A \to M$ with a 1-cocycle $\phi \in \Gamma A^*$, the dual bundle A^* carries a linear Jacobi structure and conversely, given a linear Jacobi structure on the dual A^* of a vector bundle A induces a Lie algebroid structure on A together with a 1-cocycle $\phi \in \Gamma A^*$.

The notion of morphism between Lie bialgebroids were introduced in [11]. In this paper, we define a notion of morphism between generalized Lie bialgebroids over a same base. If all the cocycles are zero, then it reduces to the morphism between Lie bialgebroids. We also showed that, if there is a morphism between two generalized Lie bialgebroids over a same base, then the induced Jacobi structures on the base coming from two generalized Lie bialgebroids are same (Section 3).

Poisson groupoids were introduced by Weinstein [15] as a unification of Poisson Lie groups and symplectic groupoids. In [11], the authors gave an equivalent definition of a Poisson groupoid as a Lie groupoid $G \rightrightarrows M$ with a Poisson structure Λ on the total space G such that the bundle map $\Lambda^\sharp : T^*G \to TG$ is a Lie groupoid morphism from the cotangent Lie groupoid $T^*G \to A^*G$ to the tangent Lie groupoid $TG \to TM$, where $AG \to M$ is the Lie algebroid of G. This motivates them to give an equivalent characterization of its infinitesimal object, that is Lie bialgebroid. They showed that a pair of Lie algebroids (A,A^*) in duality over M is a Lie bialgebroid if and only if the bundle map $\Lambda_A^\sharp \circ R_A : T^*A^* \to T^*A \to TA$ is a Lie algebroid morphism from the cotangent Lie algebroid $T^*A^* \to A^*$ of the linear Poisson manifold

A^*, to the tangent Lie algebroid $TA \to TM$ of A, where $R_A : T^*A^* \to T^*A$ is the canonical isomorphism defined in [11] and Λ_A being the induced linear Poisson structure on A. This description is rather complicated but useful to show that Lie bialgebroids are infinitesimal of Poisson groupoids.

Jacobi groupoids were introduced by Iglesias and Marrero [9] as a generalization of both Poisson groupois and contact groupoids. More precisely, a Jacobi groupoid is a Lie groupoid $G \rightrightarrows M$ together with a Jacobi structure (Λ, E) on G and a multiplicative function σ on G, such that the bundle map $(\Lambda, E)^\sharp : T^*G \times \mathbb{R} \to TG \times \mathbb{R}$ defined by $(\Lambda, E)^\sharp(\omega_g, \gamma) = (\Lambda^\sharp(\omega_g) + \gamma E(g), -\langle \omega_g, E(g)\rangle)$, is a Lie groupoid morphism between the twisted cotangent Lie groupoid $T^*G \times \mathbb{R} \rightrightarrows A^*G$ to the twisted tangent Lie groupoid $TG \times \mathbb{R} \rightrightarrows TM \times \mathbb{R}$ (cf. Definition 4.7). These twisting has been done by using the multiplicative function σ. Thus it is possible to give an equivalent characterization of a generalized Lie bialgebroid in terms of some Lie algebroid morphism. More precisely, we show that a pair of Lie algebroids with 1-cocycles $\big((A, \phi_0), (A^*, X_0)\big)$ in duality over M is a generalized Lie bialgebroid over M if and only if the map $(\Lambda_A, E_A)^\sharp \circ (R_A, -id) : T^*A^* \times \mathbb{R} \to T^*A \times \mathbb{R} \to TA \times \mathbb{R}$ is a Lie algebroid morphism from the 1-jet Lie algebroid $T^*A^* \times \mathbb{R} \to A^*$ of the linear Jacobi manifold A^*, to the twisted tangent Lie algebroid $TA \times \mathbb{R} \to TM \times \mathbb{R}$ of (A, ϕ_0), where (Λ_A, E_A) be the linear Jacobi structure on A (cf. Theorem 4.2). Note that an another characterization of a generalized Lie bialgebroid in terms of some Jacobi algebroid morphism was given in the thesis of D. Iglesias Ponte [4]. Then the Theorem 4.2 is being used to show that the infinitesimal form of Jacobi groupoids are generalized Lie bialgebroids (cf. Theorem 4.9) (Section 4).

Coisotropic subgroupoids of a Poisson groupoid were introduced in [16] as a generalization of coisotropic subgroups of a Poisson Lie group. The Lie algebroids of coisotropic subgroupoids of a Poisson groupoid are coisotropic subalgebroids of the corresponding Lie bialgebroid. Here we introduce coisotropic subgroupoids of a Jacobi groupoid and show that the Lie algebroid of a coisotropic subgroupoid appears as some notion of coisotropic subalgebroid of the corresponding generalized Lie bialgebroid (Section 5).

Notations. Given a Lie groupoid $G \rightrightarrows M$, by $\alpha, \beta : G \to M$, we denote the source and target maps and $\epsilon : M \to G$ the unit map. Two elements $g, h \in G$ are composable if $\alpha(g) = \beta(h)$, and denote by $G^{(2)} \subset G \times G$ the set of composable pairs. A morphism between two Lie groupoids $G_1 \rightrightarrows M_1$ and $G_2 \rightrightarrows M_2$ is a smooth map $F : G_1 \to G_2$ over $f : M_1 \to M_2$ which commutes with all structure maps.

Given a vector bundle $A \to M$, there is a cononical isomorphism $R_A : T^*A^* \to T^*A$ which is defined as follows. Suppose the vector bundle A is locally $A\big|_U = U \times V$, where $U \subseteq M$ is open, then the map R_A is locally defined by $R_A(\chi, \psi, Y) = (-\chi, Y, \psi)$, where $\chi \in T^*M$, $\psi \in V^*$, $Y \in V$ (see [11] for more details).

2. Preliminaries

In this section, we recall the definitions and basic facts about Jacobi manifolds, Lie algebroids and (generalized) Lie bialgebroids [3, 5, 6, 10, 11].

• Jacobi manifolds

Definition 2.1. Let M be a smooth manifold. A *Jacobi structure* on M is a pair (Λ, E), where Λ is a 2-vector field and E is a vector field on M satisfying

$$[\Lambda, \Lambda] = 2E \wedge \Lambda, \qquad \mathcal{L}_E\Lambda = [E, \Lambda] = 0,$$

where $[\ ,\ ]$ is the Schouten bracket on the space of multivector fields of M. The manifold M endowed with a Jacobi structure is called a *Jacobi manifold*.

If (M, Λ, E) is a Jacobi manifold, then one can define a bilinear, skew-symmetric bracket on the space of smooth functions by the following

$$\{f, g\} = \Lambda(\delta f, \delta g) + fE(g) - gE(f)$$

for all $f, g \in C^\infty(M)$. The bracket $\{\ ,\ \}$ satisfies the Jacobi identity and the property of a first order differential operator on each arguments, that is,

$$\{f, gh\} = g\{f, h\} + h\{f, g\} - gh\{f, 1\}$$

for all $f, g, h \in C^\infty(M)$. Conversely, any bilinear, skew-symmetric bracket on $C^\infty(M)$ which satisfies Jacobi identity and also first order differential operator on each argument defines a Jacobi structure on M.

Note that, if $E = 0$, then Λ defines a Poisson structure on M [14]. Apart from Poisson and symplectic manifolds, contact and locally conformal symplectic (l.c.s) manifolds are also examples of Jacobi manifolds [3, 9]. Given a Jacobi bracket $\{\ ,\ \}$ on M and any nowhere zero function $a \in C^\infty(M)$, one can define a new Jacobi bracket $\{\ ,\ \}^a$ by the following

$$\{f, g\}^a = \frac{1}{a}\{af, ag\} \qquad \forall f, g \in C^\infty(M).$$

A smooth map $\Phi : M \to N$ between two Jacobi manifolds is called a *Jacobi map* if $\{h\circ\Phi, h'\circ\Phi\}_M = \{h,h'\}_N\circ\Phi$, for any $h,h'\in C^\infty(N)$. For a nowhere zero function $a\in C^\infty(M)$, the pair (Φ,a) is called a *conformal Jacobi map* if Φ is a Jacobi map between $(M,\{\ ,\ \}_M^a)$ and $(N,\{\ ,\ \}_N)$.

Remark 2.2. (i) Given a Jacobi manifold (M,Λ,E), there is a vector bundle morphism $(\Lambda,E)^\sharp : T^*M\times\mathbb{R}\to TM\times\mathbb{R}$ given by

$$(\Lambda,E)^\sharp(\omega_m,\gamma) = \big(\Lambda^\sharp(\omega_m)+\gamma E(m), -\langle\omega_m, E(m)\rangle\big)$$

for $(\omega_m,\gamma)\in T_m^*M\times\mathbb{R}$, $m\in M$.
(ii) Let (Λ,E) be a Jacobi structure on M. Then the product manifold $M\times\mathbb{R}$ carries a Poisson structure

$$\widetilde{\Lambda} = e^{-t}\Big(\Lambda + \frac{\partial}{\partial t}\wedge E\Big),$$

where t is the usual coordinate on $\mathbb{R}$. The manifold $M\times\mathbb{R}$ together with the Poisson structure $\widetilde{\Lambda}$ is called the *Poissonization* of the Jacobi manifold (M,Λ,E).

The notion of coisotropic submanifolds of a Poisson manifold [15] is extended to the context of Jacobi manifolds.

Definition 2.3. ([3]) Let (M,Λ,E) be a Jacobi manifold. Then a submanifold $S\hookrightarrow M$ is called a *coisotropic submanifold* of M if

$$\Lambda^\sharp(T_xS)^0\subseteq T_xS$$

for all $x\in S$, where $\Lambda^\sharp : T^*M\to TM$ is the bundle map induced by the bivector field Λ, and $(T_xS)^0 = \{\alpha\in T_x^*M|\ \alpha(v)=0, \forall v\in T_xS\}$.

Similar to the Poisson case, one can prove the following result.

Proposition 2.4. *Let (M,Λ,E) be a Jacobi manifold with corresponding Jacobi bracket $\{\ ,\ \}$, and $C\hookrightarrow M$ be a closed submanifold. Then the followings are equivalent:*

(i) *C is a coisotropic submanifold;*
(ii) *the vanishing ideal $\mathcal{I}(C)=\{f\in C^\infty(M)\big|f|_C\equiv 0\}$ is a Lie subalgebra of $(C^\infty(M),\{\ ,\ \})$.*

• **Lie algebroids**

DEFINITION 2.5. A *Lie algebroid* $(A, [\ ,\ ], \rho)$ over a manifold M is a smooth vector bundle A over M together with a Lie bracket $[\ ,\ ]$ on the space ΓA of smooth sections of A and a bundle map $\rho : A \to TM$, called the *anchor*, such that

(i) the induced map $\rho : \Gamma A \to \mathfrak{X}(M)$ is a Lie algebra homomorphism;

(ii) for any $f \in C^\infty(M)$ and $X, Y \in \Gamma A$, the following condition holds

$$[X, fY] = f[X, Y] + (\rho(X)f)Y.$$

Given a Lie groupoid $G \rightrightarrows M$, its Lie algebroid consist of a vector bundle $AG \to M$ whose fiber at $x \in M$ consist of the tangent space $T_{\epsilon(x)}(\beta^{-1}(x))$. Then the space of sections ΓAG can be identified with the left invariant vector fields on G. Since the space of left invariant vector fields on G is closed under the Lie bracket, thus it induces a Lie bracket on ΓAG. The anchor is defined to be the differential of α restricted to AG.

Any Lie algebra is a Lie algebroid over a point, and the tangent bundle of any smooth manifold is a Lie algebroid with the usual Lie bracket on vector fields and identity as anchor. Given any Jacobi manifold, there is a canonical Lie algebroid associated to it, given by the following example.

EXAMPLE 2.6. ([6, 9]) Let (M, Λ, E) be a Jacobi manifold, then the 1-jet bundle $T^*M \times \mathbb{R}$ has a Lie algebroid structure $(T^*M \times \mathbb{R}, [\ ,\ ]_{(\Lambda,E)}, \rho_{(\Lambda,E)})$ over M, where the bracket and anchor are given by

$$\begin{aligned}[(\alpha, f), (\beta, g)]_{(\Lambda,E)} = \big(&\mathcal{L}_{\Lambda^\sharp(\alpha)}\beta - \mathcal{L}_{\Lambda^\sharp(\beta)}\alpha - \delta(\Lambda(\alpha, \beta)) + f\mathcal{L}_E\beta - g\mathcal{L}_E\alpha \\ &- \iota_E(\alpha \wedge \beta), \\ &\Lambda(\beta, \alpha) + \Lambda^\sharp(\alpha)(g) - \Lambda^\sharp(\beta)(f) + fE(g) - gE(f)\big)\end{aligned}$$

and

$$\rho_{(\Lambda,E)}(\alpha, f) = \Lambda^\sharp(\alpha) + fE$$

for all $(\alpha, f), (\beta, g) \in \Gamma(T^*M \times \mathbb{R}) = \Omega^1(M) \oplus C^\infty(M)$. When $E = 0$ (that is, when Λ is a Poisson structure on M), one recover to the cotangent Lie algebroid of M by projecting onto the first factor [14].

Given a Lie algebroid $(A, [\ ,\ ], \rho)$, the exterior algebra $\Gamma(\bigwedge^\bullet A)$ of multi-sections of A together with the generalized Schouten bracket, forms a Gerstenhaber algebra [10]. Moreover $\Gamma(\bigwedge^\bullet A^*)$ together with the Lie algebroid differential d forms a differential graded algebra. When $A = TM$ is the usual tangent bundle Lie algebroid, denote the differential of the Lie algebroid (that is, de-Rham differential of the manifold M) by δ.

It is known that, Lie algebroid structures on a vector bundle A are in one-to-one correspondence with linear Poisson structures on the dual bundle A^*. This correspondence has been extended to the Jacobi set up by introducing the notion of linear Jacobi structure on a vector bundle [5]. We now describe these in details.

Given a vector bundle, a Jacobi structure on the total space is called *linear Jacobi* structure, if the Jacobi bracket of two fibrewise linear functions is again linear, and the bracket of a linear function and the constant function 1 is a pull back function. Let $q : A \to M$ be a vector bundle over M with dual bundle $q^* : A^* \to M$. Then any section $X \in \Gamma A$ defines a fibrewise linear function l_X on the dual bundle A^*, namely

$$l_X(\alpha_x) = \alpha_x(X_x),$$

where $\alpha_x \in A_x^*$, $x \in M$. Conversely any fibrewise linear function on A^* is of this form. Then the space of linear functions and pull back functions (that is, of the form $f \circ q^*$, for $f \in C^\infty(M)$) generates $C^\infty(A^*)$.

Then we have the following theorem from [5].

Theorem 2.7. *Let $(A, [\ ,\ ], \rho)$ be a Lie algebroid over M and $\phi \in \Gamma A^*$ be a 1-cocycle. Consider the bracket $\{\ ,\ \}$ on $C^\infty(A^*)$ defined by*

$$\begin{aligned} \{l_X, l_Y\} &= l_{[X,Y]}, \\ \{l_X, f \circ q^*\} &= (\rho(X)f + \phi(X)f) \circ q^*, \\ \{f \circ q^*, g \circ q^*\} &= 0, \end{aligned}$$

for $X, Y \in \Gamma A$ and $f, g \in C^\infty(M)$. Then $\{\ ,\ \}$ defines a linear Jacobi structure on A^*. Moreover, this Jacobi structure is given by

$$\begin{aligned} \Lambda_{(A^*,\phi)} &= \Lambda_{A^*} + \Delta \wedge \phi^v, \\ E_{(A^*,\phi)} &= -\phi^v, \end{aligned}$$

where Λ_{A^*} is the linear Poisson structure on A^* induced from the Lie algebroid structure on A, Δ is the Liouville vector field on A^* and $\phi^v \in \mathfrak{X}(A^*)$ is the vertical lift of $\phi \in \Gamma A^*$.

The converse part of the above theorem also holds true (see [5] for more details). A pair (A, ϕ) of a Lie algebroid A and a 1-cocycle $\phi \in \Gamma A^*$ of it, is referred as a *Jacobi algebroid.*

Let $(A, [\ ,\ ], \rho)$ be a Lie algebroid over M and $\phi \in \Gamma A^*$ be a 1-cocycle. Then the vector bundle $\widetilde{A} = A \times \mathbb{R} \to M \times \mathbb{R}$ carries a Lie algebroid structure $([\ ,\ ]\tilde{}^{\phi}, \widetilde{\rho}^{\phi})$. For any $\widetilde{X}, \widetilde{Y} \in \Gamma\widetilde{A}$ considered as time dependent sections of A, the Lie bracket and anchor are given by

$$[\widetilde{X}, \widetilde{Y}]\tilde{}^{\phi} = [\widetilde{X}, \widetilde{Y}]\tilde{} + \phi(\widetilde{X})\frac{\partial \widetilde{Y}}{\partial t} - \phi(\widetilde{Y})\frac{\partial \widetilde{X}}{\partial t},$$

$$\widetilde{\rho}^{\phi}(\widetilde{X}) = \widetilde{\rho}(\widetilde{X}) + \phi(\widetilde{X})\frac{\partial}{\partial t},$$

where

$$[\widetilde{X}, \widetilde{Y}]\tilde{}(x,t) = [\widetilde{X}_t, \widetilde{Y}_t](x) \qquad \widetilde{\rho}(\widetilde{X})(x,t) = \rho(\widetilde{X}_t)(x),$$

and $\frac{\partial \widetilde{X}}{\partial t}$ denotes the derivatie of $\widetilde{X}$ with respect to time.

Now let $\Psi : \widetilde{A} \to \widetilde{A}$ be the isomorphism of vector bundles over the identity on $M \times \mathbb{R}$ defined by $\Psi(v,t) = (e^t v, t)$, for $(v,t) \in \widetilde{A} = A \times \mathbb{R}$. Then using Ψ and the Lie algebroid structure $([\ ,\ ]\tilde{}^{\phi}, \widetilde{\rho}^{\phi})$ on $\widetilde{A}$, one can define a new Lie algebroid structure $([\ ,\ ]\hat{}^{\phi}, \hat{\rho}^{\phi})$ on $\widetilde{A}$ such that the Lie algebroids $(\widetilde{A}, [\ ,\ ]\tilde{}^{\phi}, \widetilde{\rho}^{\phi})$ and $(\widetilde{A}, [\ ,\ ]\hat{}^{\phi}, \hat{\rho}^{\phi})$ are isomorphic. Namely, we have

$$[\widetilde{X}, \widetilde{Y}]\hat{}^{\phi} = e^{-t}\Big([\widetilde{X}, \widetilde{Y}]\tilde{} + \phi(\widetilde{X})\Big(\frac{\partial \widetilde{Y}}{\partial t} - \widetilde{Y}\Big) - \phi(\widetilde{Y})\Big(\frac{\partial \widetilde{X}}{\partial t} - \widetilde{X}\Big)\Big),$$

$$\hat{\rho}^{\phi}(\widetilde{X}) = e^{-t}\Big(\widetilde{\rho}(\widetilde{X}) + \phi(\widetilde{X})\frac{\partial}{\partial t}\Big),$$

for all $\widetilde{X}, \widetilde{Y} \in \Gamma\widetilde{A}$. Thus the total space of the dual bundle $\widetilde{A^*} = A^* \times \mathbb{R} \to M \times \mathbb{R}$ carries a linear Poisson structure $\Lambda_{A^* \times \mathbb{R}}$ and this Poisson structure is the Poissonization of the linear Jacobi structure $(\Lambda_{(A^*,\phi)}, E_{(A^*,\phi)})$ of A^* [5]. That is,

$$\Lambda_{A^* \times \mathbb{R}} = e^{-t}\Big(\Lambda_{(A^*,\phi)} + \frac{\partial}{\partial t} \wedge E_{(A^*,\phi)}\Big).$$

The notion of morphism between Lie algebroids was introduced in [2]. Here we recall an alternative definition from [11].

Definition 2.8. Let $A_1 \to M_1$ and $A_2 \to M_2$ be two Lie algebroids. Then a vector bundle morphism $F : A_1 \to A_2$ over $f : M_1 \to M_2$ is called a

Lie algebroid morphism if the graph of (F, f), that is,

$$\begin{aligned}\mathcal{C} := \big\{(\phi, \psi) \in A_1^* \times A_2^* \mid \\ \langle \phi, X\rangle = \langle \psi, F(X)\rangle, \text{ for all } X \in A_1 \text{ compatible with } \phi\big\}\end{aligned}$$

is a coisotropic submanifold of $A_1^* \times \overline{A_2^*}$, where A_1^* and A_2^* are equipped with the linear Poisson structures dual to the Lie algebroids A_1 and A_2, respectively.

If $M_1 = M_2$ and $f =$ identity, then F is a Lie algebroid morphism if and only if F preserves the Lie brackets and commute with anchors.

• Lie bialgebroids

Definition 2.9. ([11]) A *Lie bialgebroid* over M is a pair (A, A^*) of Lie algebroids in duality over M, such that the differential d_* on $\Gamma(\bigwedge^\bullet A)$ defined by the Lie algebroid structure of A^* and the Gerstenhaber bracket on $\Gamma(\bigwedge^\bullet A)$ defined by the Lie algebroid structure of A satisfies

$$d_*[X, Y] = [d_*X, Y] + [X, d_*Y],$$

for all $X, Y \in \Gamma A$.

Then there is a following characterization of a Lie bialgebroid [11].

Theorem 2.10. *Let A be a Lie algebroid over M such that its dual bundle A^* also carries a Lie algebroid structure. Consider the composition $\Pi = \Lambda_A^\sharp \circ R_A$,*

$$T^*A^* \longrightarrow T^*A \longrightarrow TA,$$

where Λ_A being the linear Poisson structure on A coming from the Lie algebroid A^*. Then (A, A^*) is a Lie bialgebroid if and only if

$$\begin{array}{ccc} T^*A^* & \xrightarrow{\Pi} & TA \\ \downarrow & & \downarrow \\ A^* & \xrightarrow[\rho_*]{} & TM \end{array}$$

is a Lie algebroid morphism, where $T^*A^* \to A^*$ is the cotangent Lie algebroid of the linear Poisson structure on A^* coming from the Lie algebroid A, and $TA \to TM$ is the tangent Lie algebroid of A.

• Generalized Lie bialgebroids

Given a Lie algebroid $(A, [\ ,\ ], \rho)$ over M with 1-cocycle $\phi \in \Gamma A^*$, there is an ϕ-deformed Lie algebroid representation

$$\begin{aligned} \rho^{\phi} : \Gamma A \times C^{\infty}(M) &\to C^{\infty}(M) \\ (X, f) &\mapsto (\rho^{\phi}(X))f = \rho(X)f + \phi(X)f. \end{aligned}$$

Thus one can define ϕ-deformed Lie algebroid differential d^{ϕ} which is given by

$$\begin{aligned} d^{\phi} : \Gamma(\bigwedge{}^{\bullet} A^*) &\to \Gamma(\bigwedge{}^{\bullet+1} A^*) \\ \alpha &\mapsto d\alpha + \phi \wedge \alpha. \end{aligned}$$

Then the ϕ-deformed Lie derivative is defined using Cartan formula

$$\mathcal{L}_X^{\phi} : \Gamma(\bigwedge{}^{\bullet} A^*) \to \Gamma(\bigwedge{}^{\bullet} A^*), \quad \alpha \mapsto d^{\phi}\iota_X \alpha + \iota_X d^{\phi} \alpha$$

for $X \in \Gamma A$. One can also define ϕ-deformed Schouten bracket on the multisections of A by the formula

$$[P, Q]^{\phi} = [P, Q] + (p-1) P \wedge (\iota_{\phi} Q) - (-1)^{p-1}(q-1)(\iota_{\phi} P) \wedge Q,$$

for $P \in \Gamma(\bigwedge^p A), Q \in \Gamma(\bigwedge^q A)$ (see [1, 6]).

Let $(A, [\ ,\ ], \rho)$ be a Lie algebroid over M and $\phi_0 \in \Gamma A^*$ be a 1-cocyle. Assume that the dual bundle A^* also carries a Lie algebroid structure $([\ ,\ ]_*, \rho_*)$ and $X_0 \in \Gamma A$ be its 1-cocyle. Let the Lie derivative of the Lie algebroid A (resp. A^*) is denoted by $\mathcal{L}$ (resp. $\mathcal{L}_*$).

Definition 2.11. ([6]) The pair $\big((A, \phi_0), (A^*, X_0)\big)$ is said to be a *generalized Lie bialgebroid* over M if the following conditions are hold

$$d_*^{X_0}[X, Y] = [d_*^{X_0} X, Y]^{\phi_0} + [X, d_*^{X_0} Y]^{\phi_0}, \tag{1}$$

$$\mathcal{L}_{*\phi_0}^{X_0} P + \mathcal{L}_{X_0}^{\phi_0} P = 0 \tag{2}$$

for all $X, Y \in \Gamma A$ and $P \in \Gamma(\bigwedge^p A)$.

Remark 2.12. (i) The condition (2) of the above definition is equivalent to

$$\phi_0(X_0) = 0,\ \rho(X_0) = -\rho_*(\phi_0),\ \mathcal{L}_{*\phi_0} X + [X_0, X] = 0, \tag{3}$$

for all $X \in \Gamma A$. These follows from condition (2) by applying $P = f \in C^{\infty}(M)$ and $P = X \in \Gamma A$.

(ii) When $\phi_0 = 0$ and $X_0 = 0$, one recover the definition of a Lie bialgebroid.

Example 2.13. Given any smooth manifold M, the bundle $TM \times \mathbb{R} \to M$ has a Lie algebroid structure whose bracket and anchor are given by

$$[(X,f),(Y,g)] = \big([X,Y], X(g) - Y(f)\big), \quad pr(X,f) = X,$$

for $(X,f),(Y,g) \in \mathfrak{X}(M) \oplus C^\infty(M)$. Moreover $(0,1) \in \Omega^1(M) \oplus C^\infty(M) = \Gamma(T^*M \times \mathbb{R})$ is a 1-cocycle of this Lie algebroid. If (M,Λ,E) is a Jacobi manifold, then the 1-jet bundle $T^*M \times \mathbb{R} \to M$ also carries a Lie algebroid structure (cf. Example 2.6) and one can check that, $(-E,0) \in \mathfrak{X}(M) \oplus C^\infty(M) = \Gamma(TM \times \mathbb{R})$ is a 1-cocycle of it. For a Jacobi manifold (M,Λ,E), the pair $\big((TM\times\mathbb{R},(0,1)),(T^*M\times\mathbb{R},(-E,0))\big)$ is a generalized Lie bialgebroid over M [6].

Another interesting class of examples of generalized Lie bialgebroids are provided by strict Jacobi-Nijenhuis manifolds [7].

The relation between Lie bialgebroid and generalized Lie bialgebroid is given by the following result.

Proposition 2.14. ([6]) *Let $(A,[\ ,\ ],\rho)$ be a Lie algebroid over M with $\phi_0 \in \Gamma A^*$ be a 1-cocycle. Suppose the dual bundle $A^* \to M$ also carries a Lie algebroid structure $([\ ,\ ]_*,\rho_*)$ and $X_0 \in \Gamma A$ be a 1-cocycle of it. Consider the Lie algebroids $\widetilde{A} = (A\times\mathbb{R}, [\ ,\ ]\tilde{}^{\phi_0}, \widetilde{\rho}^{\phi_0})$ and $\widetilde{A}^* = (A^*\times\mathbb{R}, [\ ,\ ]\hat{}_*^{X_0}, \hat{\rho_*}^{X_0})$ in duality over $M\times\mathbb{R}$. Then,*

(i) *if $\big((A,\phi_0),(A^*,X_0)\big)$ is a generalized Lie bialgebroid over M, the pair $(\widetilde{A},\widetilde{A}^*)$ is a Lie bialgebroid over $M\times\mathbb{R}$;*

(ii) *if $(\widetilde{A},\widetilde{A}^*)$ is a Lie bialgebroid over $M\times\mathbb{R}$, the pair $\big((A,\phi_0),(A^*,X_0)\big)$ is a generalized Lie bialgebroid over M.*

Thus using the duality of a Lie bialgebroid, one can conclude the duality of a generalized Lie bialgebroid.

Proposition 2.15. ([6]) *If $\big((A,\phi_0),(A^*,X_0)\big)$ is a generalized Lie bialgebroid over M, then so is the pair $\big((A^*,X_0),(A,\phi_0)\big)$.*

Remark 2.16. One can also directly prove the duality of a generalized Lie bialgebroid following the proof of Kosmann-Schwarzbach [13] for Lie bialgebroids in the presence of cocycles, although these two proofs of the duality of a generalized Lie bialgebroid can be shown to be equivalent in the presence of the Proposition 2.14.

Given a generalized Lie bialgebroid over M, it is proved in [6] that the base M carries a Jacobi structure.

Let $\big((A,\phi_0),(A^*,X_0)\big)$ be a generalized Lie bialgebroid over M. Define a bracket

$$\{\ ,\ \} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

by

$$\{f,g\} = \langle d^{\phi_0} f, d_*^{X_0} g\rangle. \tag{4}$$

Proposition 2.17. *The bracket defined above satisfies the following properties*

$$d^{\phi_0}\{f,g\} = [d^{\phi_0} f, d^{\phi_0} g]_*, \tag{5}$$

$$d_*^{X_0}\{f,g\} = -[d_*^{X_0} f, d_*^{X_0} g]. \tag{6}$$

Proof. Since $\{f,g\} = \langle d^{\phi_0} f, d_*^{X_0} g\rangle = \rho_*^{X_0}(d^{\phi_0} f) g = [d^{\phi_0} f, g]_*^{X_0}$, we have

$$d^{\phi_0}\{f,g\} = [d^{\phi_0} f, d^{\phi_0} g]_*^{X_0} = [d^{\phi_0} f, d^{\phi_0} g]_*.$$

Similarly, $\{f,g\} = \rho^{\phi_0}(d_*^{X_0} g) f = [d_*^{X_0} g, f]^{\phi_0}$, therefore

$$d_*^{X_0}\{f,g\} = [d_*^{X_0} g, d_*^{X_0} f]^{\phi_0} = -[d_*^{X_0} f, d_*^{X_0} g].$$

∎

Theorem 2.18. ([6]) *Let $\big((A,\phi_0),(A^*,X_0)\big)$ be a generalized Lie bialgebroid over M. Then the bracket above defines a Jacobi structure on M.*

Remark 2.19. (i) From (4), we have $\{f,g\} = \langle df, d_* g\rangle + f\rho_*(\phi_0) g + g\rho(X_0) f$, therefore the induced Jacobi bivector field Λ_M and the vector field E_M is given by

$$\Lambda_M(\delta f, \delta g) = \langle df, d_* g\rangle = \big(\rho_* \circ \rho^*(\delta f)\big)(g),$$

$$E_M = \rho_*(\phi_0) = -\rho(X_0).$$

(ii) Let (M,Λ,E) be a Jacobi manifold. If we consider the generalized Lie bialgebroid $\big((TM\times\mathbb{R},(0,1)),(T^*M\times\mathbb{R},(-E,0))\big)$ given in Example 2.6, the induced Jacobi structure on M coincide with the original Jacobi structure.

(iii) The dual generalized Lie bialgebroid $\big((A^*,X_0),(A,\phi_0)\big)$ over M induces the opposite Jacobi structure of the above.

3. Morphism between generalized Lie bialgebroids

In this section, we introduce a notion of morphism between generalized Lie bialgebroids over a same base and prove that the induced Jacobi structure on the base of a generalized Lie bialgebroid is unique up to a morphism.

• Jacobi algebroid maps

Definition 3.1. Let (A, ϕ) and (B, ψ) be two Jacobi algebroids over M. Then a bundle map $\Phi : A \to B$ over M is called a *Jacobi algebroid map* if Φ is a Lie algebroid map and $\Phi^*(\psi) = \phi$.

Proposition 3.2. *Let (A, ϕ) and (B, ψ) be two Jacobi algebroids over M. Then a bundle map $\Phi : A \to B$ over M is a Jacobi algebroid map if and only if $\Phi^* : B^* \to A^*$ is a Jacobi map, under the dual linear Jacobi structures.*

Proof. Let $\Phi : A \to B$ is a Jacobi algebroid map. Then for all $X, Y \in \Gamma A$, we have

$$\begin{aligned} \{l_X, l_Y\} \circ \Phi^* &= l_{[X,Y]} \circ \Phi^* = l_{\Phi[X,Y]} = l_{[\Phi(X),\Phi(Y)]} = \{l_{\Phi(X)}, l_{\Phi(Y)}\} \\ &= \{l_X \circ \Phi^*, l_Y \circ \Phi^*\}. \end{aligned}$$

For any $f \in C^\infty(M)$, we also have

$$\begin{aligned} \{l_X, f \circ q_A^*\} \circ \Phi^* &= \big(\rho_A(X)f + \phi(X)f\big) \circ q_A^* \circ \Phi^* \\ &= \big(\rho_B(\Phi(X))f + (\psi, \Phi(X))f\big) \circ q_B^* \\ &= \{l_{\Phi(X)}, f \circ q_B^*\} \\ &= \{l_X \circ \Phi^*, f \circ q_A^* \circ \Phi^*\}. \end{aligned}$$

Moreover for any $f, g \in C^\infty(M)$,

$$\{f \circ q_A^*, g \circ q_A^*\} \circ \Phi^* = \{f \circ q_B^*, g \circ q_B^*\} = \{f \circ q_A^* \circ \Phi^*, g \circ q_A^* \circ \Phi^*\},$$

since both sides are equals to zero. Therefore, $\Phi^* : B^* \to A^*$ is a Jacobi map. The converse part is similar. ∎

Proposition 3.3. *Let (A, ϕ) and (B, ψ) be two Jacobi algebroids over M. Then a bundle map $\Phi : A \to B$ over M is a Jacobi algebroid map if and only if*

$$\widetilde{\Phi} = \Phi \times id : \big(A \times \mathbb{R}, [\ ,\ ]\tilde{}^{\phi}, \widetilde{\rho_A}^{\phi}\big) \to \big(B \times \mathbb{R}, [\ ,\ ]\tilde{}^{\psi}, \widetilde{\rho_B}^{\psi}\big)$$

is a Lie algebroid morphism over $M \times \mathbb{R}$.

Proof. Suppose Φ is a Jacobi algebroid map. Then for any $\tilde{X}, \tilde{Y} \in \Gamma(\tilde{A}) = \Gamma(A \times \mathbb{R})$, we have

$$\begin{aligned}
\big[\widetilde{\Phi}(\widetilde{X}), \widetilde{\Phi}(\widetilde{Y})\big]^{\tilde{}\,\psi} &= \big[\widetilde{\Phi}(\widetilde{X}), \widetilde{\Phi}(\widetilde{Y})\big]^{\tilde{}} + \psi\big(\widetilde{\Phi}(\widetilde{X})\big)\frac{\partial\big(\widetilde{\Phi}(\widetilde{Y})\big)}{\partial t} - \psi\big(\widetilde{\Phi}(\widetilde{Y})\big)\frac{\partial\big(\widetilde{\Phi}(\widetilde{X})\big)}{\partial t}\\
&= \widetilde{\Phi}\big[\widetilde{X}, \widetilde{Y}\big]^{\tilde{}} + \big(\widetilde{\Phi}^*(\psi), \widetilde{X}\big)\widetilde{\Phi}\Big(\frac{\partial \widetilde{Y}}{\partial t}\Big) - \big(\widetilde{\Phi}^*(\psi), \widetilde{Y}\big)\widetilde{\Phi}\Big(\frac{\partial \widetilde{X}}{\partial t}\Big)\\
&= \widetilde{\Phi}\big([\widetilde{X}, \widetilde{Y}]^{\tilde{}\,\phi}\big).
\end{aligned}$$

Also for any $\tilde{X} \in \Gamma(\tilde{A})$,

$$\begin{aligned}
(\widetilde{\rho_B}^{\psi} \circ \widetilde{\Phi})(\widetilde{X}) &= \widetilde{\rho_B}^{\psi}\big(\widetilde{\Phi}(\widetilde{X})\big)\\
&= \widetilde{\rho_B}\big(\widetilde{\Phi}(\widetilde{X})\big) + \psi\big(\widetilde{\Phi}(\widetilde{X})\big)\frac{\partial}{\partial t}\\
&= \widetilde{\rho_B \circ \Phi}(\widetilde{X}) + \langle \psi, \widetilde{\Phi}(\widetilde{X})\rangle\frac{\partial}{\partial t}\\
&= \widetilde{\rho_A}(\widetilde{X}) + \langle \Phi^*\psi, \widetilde{X}\rangle\frac{\partial}{\partial t}\\
&= \widetilde{\rho_A}(\widetilde{X}) + \phi(\widetilde{X})\frac{\partial}{\partial t} = \widetilde{\rho_A}^{\phi}(\widetilde{X}).
\end{aligned}$$

Hence $\widetilde{\Phi}$ defines a Lie algebroid morphism. The converse part is similar. ∎

Similarly Φ is a Jacobi algebroid map if and only if $\widetilde{\Phi} = \Phi \times id$ is a Lie algebroid map from $(A \times \mathbb{R}, [\ ,\ ]^{\hat{}\,\phi}, \widehat{\rho_A}^{\phi})$ to $(B \times \mathbb{R}, [\ ,\ ]^{\hat{}\,\psi}, \widehat{\rho_B}^{\psi})$ over $M \times \mathbb{R}$.

It is known that given a Lie algebroid A, a section $\alpha \in \Gamma A^*$ is a one cocycle of the Lie algebroid if and only if the image of α in A^* is a coisotropic submanifold of A^* with respect to linear Poisson structure [12]. Next we give an analogues of this result to Jacobi set up.

Proposition 3.4. *Let A be a Lie algebroid over M and $\phi \in \Gamma A^*$ be a 1-cocycle. Then $\alpha \in \Gamma A^*$ is a ϕ-deformed one cocycle of the Lie algebroid (that is, $d^{\phi}\alpha = 0$) if and only if the image of α in A^* is a coisotropic submanifold of A^* with respect to linear Jacobi structure.*

Proof. Let $\alpha \in \Gamma A^*$. Then for any $X \in \Gamma A$, the function $l_X - \langle \alpha, X\rangle \circ q^*$ on A^* vanishes on the image of α. In fact, the space of functions on A^* which vanishes on the image of α is generated by such kind of functions. Now for any $X, Y \in \Gamma A$, we have

$$
\begin{aligned}
&\big\{l_X - \langle\alpha, X\rangle \circ q^*, l_Y - \langle\alpha, Y\rangle \circ q^*\big\}\\
={}&\{l_X, l_Y\} - \big\{\langle\alpha, X\rangle \circ q^*, l_Y\big\} - \big\{l_X, \langle\alpha, Y\rangle \circ q^*\big\} + \big\{\langle\alpha, X\rangle \circ q^*, \langle\alpha, Y\rangle \circ q^*\big\}\\
={}&l_{[X,Y]} + \big(\rho(Y)\langle\alpha, X\rangle + \phi(Y)\langle\alpha, X\rangle\big) \circ q^* - \big(\rho(X)\langle\alpha, Y\rangle + \phi(X)\langle\alpha, Y\rangle\big) \circ q^*.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\big\{l_X - \langle\alpha, X\rangle \circ q^*, l_Y - \langle\alpha, Y\rangle \circ q^*\big\}(\alpha(m))\\
={}&\langle\alpha, [X,Y]\rangle(m) + \big(\rho(Y)\langle\alpha, X\rangle\big)(m) + (\phi(Y))(m)\langle\alpha, X\rangle(m)\\
&- \big(\rho(X)\langle\alpha, Y\rangle\big)(m) - (\phi(X))(m)\langle\alpha, Y\rangle(m)\\
={}&-(d\alpha)(X,Y)(m) - (\phi \wedge \alpha)(X,Y)(m)\\
={}&-(d^\phi\alpha)(X,Y)(m).
\end{aligned}
$$

Therefore, from Proposition 2.4, it follows that the image of α is a coisotropic submanifold of A^* with respect to linear Jacobi structure if and only if $d^\phi\alpha = 0$. ∎

• Generalized Lie bialgebroid morphisms

DEFINITION 3.5. A *morphism* between two generalized Lie bialgebroids $\big((A, \phi_0), (A^*, X_0)\big)$ and $\big((B, \psi_0), (B^*, Y_0)\big)$ over M is a map $\Phi : A \to B$ of Lie algebroids such that the dual map $\Phi^* : B^* \to A^*$ is also a Lie algebroid map and they preserves the cocycles. That is,

$$\Phi(X_0) = Y_0, \qquad \Phi^*(\psi_0) = \phi_0.$$

PROPOSITION 3.6. *Let $\big((A, \phi_0), (A^*, X_0)\big)$ and $\big((B, \psi_0), (B^*, Y_0)\big)$ be two generalized Lie bialgebroids over M. Then a bundle map $\Phi : A \to B$ is a generalized Lie bialgebroid morphism if and only if the following conditions hold:*

(i) *$\Phi : A \to B$ is a Jacobi algebroid map from (A, ϕ_0) to (B, ψ_0);*

(ii) *$\Phi : A \to B$ is a Jacobi map under the linear Jacobi structures on A and B coming from the Jacobi algebroids (A^*, X_0) and (B^*, Y_0) respectively.*

PROPOSITION 3.7. *Let $\big((A, \phi_0), (A^*, X_0)\big)$ and $\big((B, \psi_0), (B^*, Y_0)\big)$ be two generalized Lie bialgebroids over M. Then a bundle map $\Phi : A \to B$ is a generalized Lie bialgebroid morphism if and only if $\widetilde{\Phi} = \Phi \times id$ is a Lie bialgebroid morphism from $(\widetilde{A}, \widetilde{A}^*)$ to $(\widetilde{B}, \widetilde{B}^*)$ over $M \times \mathbb{R}$.*

Proof. Suppose Φ is a generalized Lie bialgebroid morphism. Therefore Φ is a Jacobi algebroid map from (A, ϕ_0) to (B, ψ_0). Hence from Proposition 3.3, we have $\widetilde{\Phi} = \Phi \times id$ is a Lie algebroid map from $(A \times \mathbb{R}, [\ ,\ ]^{\tilde{}\phi_0}, \widetilde{\rho_A}^{\phi_0})$ to $(B \times \mathbb{R}, [\ ,\ ]^{\tilde{}\psi_0}, \widetilde{\rho_B}^{\psi_0})$. Moreover Φ^* is a Jacobi algebroid map (A^*, X_0) to (B^*, Y_0). Therefore $\Phi^* \times id = \widetilde{\Phi}^*$ is a Lie algebroid map from $(A^* \times \mathbb{R}, [\ ,\ ]^{\hat{}X_0}, \widehat{\rho_A}^{X_0})$ to $(B^* \times \mathbb{R}, [\ ,\ ]^{\hat{}Y_0}, \widehat{\rho_B}^{Y_0})$. Hence $\tilde{\Phi}$ is a Lie bialgebroid morphism from $(\widetilde{A}, \widetilde{A}^*)$ to $(\widetilde{B}, \widetilde{B}^*)$ over $M \times \mathbb{R}$. The converse part is similar. ∎

THEOREM 3.8. *Let $\big((A, \phi_0), (A^*, X_0)\big)$ be a generalized Lie bialgebroid over M and (Λ_M, E_M) denotes the induced Jacobi structure on M. Then the map*

$$\Phi_A : A \to TM \times \mathbb{R}$$

defined by $\Phi_A(X) = (\rho(X), \phi_0(X))$, for $X \in \Gamma A$, is a morphism between the generalized Lie bialgebroids $\big((A, \phi_0), (A^*, X_0)\big)$ and $\big((TM \times \mathbb{R}, (0, 1)), (T^*M \times \mathbb{R}, (-E_M, 0))\big)$, where ρ is the anchor of the Lie algebroid A.

Moreover, if $\big((A, \phi_0), (A^*, X_0)\big)$ and $\big((B, \psi_0), (B^*, Y_0)\big)$ are two generalized Lie bialgebroids over M and $\Psi : A \to B$ is a generalized Lie bialgebroid morphism, then the corresponding induced Jacobi structures on the base manifold M are same.

Proof. The map Φ_A is clearly a Lie algebroid map and $\Phi_A(X_0) = (\rho(X_0), 0) = (-E_M, 0)$. Note that the dual map $\Phi_A^* : T^*M \times \mathbb{R} \to A^*$ is such that

$$\begin{aligned}\Phi_A^*(\alpha, f)(X) &= \big\langle (\alpha, f), \Phi_A(X) \big\rangle = \big\langle (\alpha, f), (\rho(X), \phi_0(X)) \big\rangle \\ &= \alpha(\rho(X)) + f\phi_0(X),\end{aligned}$$

for any $X \in \Gamma A$. Therefore, $\Phi_A^*(\alpha, f) = \rho^*(\alpha) + f\phi_0$. Hence, $\Phi_A^*(0, 1) = \phi_0$.

It is also a direct calculation to show that

$$\Phi_A^* : \big(T^*M \times \mathbb{R}, [\ ,\ ]_{(\Lambda_M, E_M)}, \rho_{(\Lambda_M, E_M)}\big) \to (A^*, [\ ,\ ]_*, \rho_*)$$

preserves the Lie bracket. It also commutes with the anchors, as

$$\begin{aligned}\rho_* \circ \Phi_A^*(\alpha, f) &= \rho_*(\rho^*(\alpha) + f\phi_0) = \rho_*(\rho^*(\alpha)) + f\rho_*(\phi_0) = \Lambda_M^\sharp(\alpha) + fE_M \\ &= \rho_{(\Lambda_M, E_M)}(\alpha, f),\end{aligned}$$

where ρ_* is the anchor of the Lie algebroid A^*.

To prove the last part of the theorem, let the Lie algebroid differential of A and A^* (resp. B and B^*) be denoted by d_A and d_{A^*} (resp. d_B and d_{B^*}). Similarly the anchors are denoted by ρ_A and ρ_{A^*} (resp. ρ_B and ρ_{B^*}). Then,

$$\begin{aligned}\{f,g\}_{(A,A^*)} &= \langle d_A^{\phi_0} f, d_{A^*}^{X_0} g\rangle = \rho_{A^*}^{X_0}(d_A^{\phi_0} f)g = \rho_{A^*}^{X_0}\big(\Phi_A^*(\delta f, f)\big)g \\ &= \rho_{A^*}^{X_0}\big(\Psi^*\Phi_B^*(\delta f, f)\big)g = \rho_{A^*}^{X_0}\Psi^*\big(\Phi_B^*(\delta f, f)\big)g \\ &= \rho_{B^*}^{Y_0}(d_B^{\psi_0} f)g = \{f,g\}_{(B,B^*)}.\end{aligned}$$

Hence the proof. ∎

Therefore the induced Jacobi structure on the base of a generalized Lie bialgebroid is unique up to a morphism.

4. Generalized Lie bialgebroids and Jacobi groupoids

In this section, we give a characterization of generalized Lie bialgebroids and using it, we show that generalized Lie bialgebroids are infinitesimal form of Jacobi groupoids.

We begin with an example of twisted version of tangent Lie algebroid. Given a Lie algebroid $q : A \to M$, the bundle $Tq : TA \to TM$ carries a Lie algebroid structure, called *tangent Lie algebroid* of A. If $\phi \in \Gamma A^*$ is a 1-cocycle of the Lie algebroid A, then one can define a twist on the tangent Lie algebroid.

Example 4.1. Let $A \to M$ be a Lie algebroid and $\phi \in \Gamma A^*$ be a 1-cocycle. Thus ϕ defines a linear function on A. Its complete lift defines a function on TA, which is linear with respect to the vector bundle structure $TA \to TM$. Hence it defines a section $\overline{\phi}$ of the dual bundle $(TA)^* \to TM$. Moreover $\overline{\phi} \in \Gamma(TA^*)$ becomes a 1-cocycle of the tangent Lie algebroid $TA \to TM$ [4]. Thus there is a Lie algebroid structure on $\widetilde{TA} = TA \times \mathbb{R}$ over $TM \times \mathbb{R}$, whose sections are considered as dependent sections of the tangent Lie algebroid $TA \to TM$. The Lie bracket and anchor of the Lie algebroid $\widetilde{TA}$ are given by

$$[\widetilde{X}, \widetilde{Y}]\tilde{}^{\overline{\phi}} = [\widetilde{X}, \widetilde{Y}]\tilde{} + \overline{\phi}(\widetilde{X})\frac{\partial \widetilde{Y}}{\partial t} - \overline{\phi}(\widetilde{Y})\frac{\partial \widetilde{X}}{\partial t},$$

$$\widetilde{\rho}^{\overline{\phi}}(\widetilde{X}) = \widetilde{\rho_{TA}}(\widetilde{X}) + \overline{\phi}(\widetilde{X})\frac{\partial}{\partial t},$$

where

$$[\widetilde{X}, \widetilde{Y}]\tilde{}(x,t) = [\widetilde{X}_t, \widetilde{Y}_t](x), \quad \widetilde{\rho_{TA}}(\widetilde{X})(x,t) = \rho_{TA}(\widetilde{X}_t)(x), \quad \text{for } \widetilde{X}, \widetilde{Y} \in \Gamma(\widetilde{TA}).$$

The Lie algebroid $\widetilde{TA} = TA \times \mathbb{R} \to TM \times \mathbb{R}$ defined above is called the *twisted tangent Lie algebroid* of (A, ϕ).

• An alternative characterization of generalized Lie bialgebroids

Let $(A, [\ ,\ ], \rho)$ be a Lie algebroid over M and $\phi_0 \in \Gamma A^*$ be a 1-cocycle of it. Suppose the dual bundle A^* also carries a Lie algebroid structure $([\ ,\ ]_*, \rho_*)$ and $X_0 \in \Gamma A$ be its 1-cocycle. Let (Λ_A, E_A) be the linear Jacobi structure on A coming from the Lie algebroid A^* and its 1-cocycle X_0. Then we have the following characterization of a generalized Lie bialgebroid.

THEOREM 4.2. *The pair $\big((A, \phi_0), (A^*, X_0)\big)$ is a generalized Lie bialgebroid over M if and only if the composition $(\Lambda_A, E_A)^\sharp \circ (R_A, -id) : T^*A^* \times \mathbb{R} \to T^*A \times \mathbb{R} \to TA \times \mathbb{R}$*

$$\begin{array}{ccc} T^*(A^*) \times \mathbb{R} & \xrightarrow{(\Lambda_A, E_A)^\sharp \circ (R_A, -id)} & TA \times \mathbb{R} \\ \downarrow & & \downarrow \\ A^* & \xrightarrow[\left(\rho_*(\_), X_0(\_)\right)]{} & TM \times \mathbb{R} \end{array} \tag{7}$$

is a Lie algebroid morphism, where the domain $T^*(A^*) \times \mathbb{R} \longrightarrow A^*$ is the 1-jet Lie algebroid of the linear Jacobi manifold A^* coming from the pair (A, ϕ_0), and the range $TA \times \mathbb{R} \to TM \times \mathbb{R}$ is the twisted tangent Lie algebroid of (A, ϕ_0).

Proof. Consider the pair of Lie algebroids $\big((A \times \mathbb{R}, [\ ,\ ]^{\tilde{}\phi_0}, \widetilde{\rho}^{\phi_0}), (A^* \times \mathbb{R}, [\ ,\ ]_*^{\hat{}X_0}, \widehat{\rho_*}^{X_0})\big)$ in duality over $M \times \mathbb{R}$ and consider

$$\begin{array}{ccc} T^*(A^* \times \mathbb{R}) & \xrightarrow{(\pi_{A \times \mathbb{R}})^\sharp \circ R_{A \times \mathbb{R}}} & T(A \times \mathbb{R}) \\ \downarrow & & \downarrow \\ A^* \times \mathbb{R} & \longrightarrow & T(M \times \mathbb{R}) \end{array} \tag{8}$$

where the left hand side is the cotangent Lie algebroid of the linear Poisson structure of $A^* \times \mathbb{R}$ coming from the Lie algebroid $(A \times \mathbb{R}, [\ ,\ ]^{\tilde{}\phi_0}, \widetilde{\rho}^{\phi_0})$, the right hand side is the tangent Lie algebroid of $A \times \mathbb{R}$, and $\pi_{A \times \mathbb{R}}$ is the linear Poisson structure on $A \times \mathbb{R}$ coming from the Lie algebroid structure $(A^* \times$

$\mathbb{R}, [\ ,\ ]_*^{\hat{}X_0}, \widehat{\rho_*}^{X_0})$. Note that $\pi_{A\times\mathbb{R}}$ is the Poissonization of the linear Jacobi structure (Λ_A, E_A) of A. Thus

$$(\pi_{A\times\mathbb{R}})^\sharp(w_a + \gamma dt|_t) = e^{-t}\Big(\Lambda_A^\sharp(w_a) + \gamma E_A(a), -w_a(E_a)\frac{\partial}{\partial t}\big|_t\Big)$$

for $w_a + \gamma dt|_t \in T^*_{(a,t)}(A\times\mathbb{R})$, $a \in A$.

Then,

$$\Big\{\big((X,\lambda),(Y,\mu)\big)\Big|X\in T_\phi A^*, Y\in T_\psi A^*; \lambda,\mu\in\mathbb{R}\Big\} \subseteq \big(TA^*\times\mathbb{R}\big)\times\overline{(TA^*\times\mathbb{R})}$$

is in the graph of (7) if and only if for all $(\Phi,\zeta)\in T^*_\phi(A^*)\times\mathbb{R}$, we have

$$\begin{aligned}\big\langle(\Phi,\zeta),(X,\lambda)\big\rangle &= \big\langle(\Lambda_A,E_A)^\sharp\circ(R_A,-id)(\Phi,\zeta),(Y,\mu)\big\rangle\\ &= \big\langle(\Lambda_A,E_A)^\sharp(R_A(\Phi),-\zeta),(Y,\mu)\big\rangle\\ &= \big\langle\big(\Lambda_A^\sharp(R_A\Phi)-\zeta E_A, -\langle R_A\Phi,E_A\rangle\big),(Y,\mu)\big\rangle\\ &= \big\langle Y,\Lambda_A^\sharp(R_A\Phi)-\zeta E_A\big\rangle - \mu\langle R_A\Phi,E_A\rangle.\end{aligned}$$

On the other hand

$$\Big\{\Big(\Big(X,\lambda\frac{d}{dt}\big|_t\Big),\Big(Z,\xi\frac{d}{dt}\big|_t\Big)\Big)\Big|X\in T_\phi A^*, Y\in T_\psi A^*; \lambda,\mu\in\mathbb{R}\Big\}$$
$$\subseteq T(A^*\times\mathbb{R})\times\overline{T(A^*\times\mathbb{R})}$$

is in the graph of (8) if and only if for all $(\Phi,\zeta dt|_t)\in T^*_{(\phi,t)}(A^*\times\mathbb{R})$, we have

$$\begin{aligned}&\Big\langle\big(\Phi,\zeta dt|_t\big),\big(X,\lambda\frac{d}{dt}\big|_t\big)\Big\rangle\\ =&\Big\langle\big(\pi_{A\times\mathbb{R}}\big)^\sharp\circ R_{A\times\mathbb{R}}\big(\Phi,\zeta dt|_t\big),\big(Z,\xi\frac{d}{dt}\big|_t\big)\Big\rangle\\ =&\Big\langle\big(\pi_{A\times\mathbb{R}}\big)^\sharp\big(R_A\Phi,-\zeta dt|_t\big),\big(Z,\xi\frac{d}{dt}\big|_t\big)\Big\rangle\\ =&\Big\langle e^{-t}\big(\Lambda_A^\sharp(R_A\Phi)-\zeta E_A,-\langle R_A\Phi,E_A\rangle\frac{d}{dt}\big|_t\big),\big(Z,\xi\frac{d}{dt}\big|_t\big)\Big\rangle\\ =&\,e^{-t}\big\langle Z,\Lambda_A^\sharp(R_A\Phi)-\zeta E_A\big\rangle - e^{-t}\xi\langle R_A\Phi,E_A\rangle.\end{aligned}$$

Therefore, $\big\{\big((X,\lambda),(Y,\mu)\big)\big|X\in T_\phi A^*, Y\in T_\psi A^*;\lambda,\mu\in\mathbb{R}\big\}$ is in the graph of (7) if and only if $\big\{\big((X,\lambda\frac{d}{dt}\big|_t),(e^tY,e^t\mu\frac{d}{dt}\big|_t)\big)\big|X\in T_\phi A^*, Y\in T_\psi A^*;\lambda,\mu\in\mathbb{R}\big\}$ is in the graph of (8). Moreover using the corresponding dual linear Poisson structures, one can show that

$$\Big\{\big((X,\lambda),(Y,\mu)\big)\Big|X\in T_\phi A^*, Y\in T_\psi A^*; \lambda,\mu\in\mathbb{R}\Big\} \subseteq \big(TA^*\times\mathbb{R}\big)\times\overline{(TA^*\times\mathbb{R})}$$

is a coisotropic submanifold if and only if

$$\left\{ \left(\left(X, \lambda \frac{d}{dt}|_t\right), \left(e^t Y, e^t \mu \frac{d}{dt}|_t\right) \right) \Big| X \in T_\phi A^*, Y \in T_\psi A^*; \lambda, \mu \in \mathbb{R} \right\} \subseteq T(A^* \times \mathbb{R}) \times \overline{T(A^* \times \mathbb{R})}$$

is a coisotropic submanifold.

Thus we have that (7) is a Lie algebroid morphism if and only if (8) is a Lie algebroid morphism. Hence the result follows from Proposition 2.14 and Theorem 2.10. ∎

• Jacobi groupoids and generalized Lie bialgebroids

Given a Lie groupoid $G \rightrightarrows M$, its tangent Lie groupoid is given by $TG \rightrightarrows TM$, whose structure maps are $T\alpha, T\beta, T\epsilon$ and the composition is denoted by $\oplus_{TG}$.

Definition 4.3. Let $G \rightrightarrows M$ be a Lie groupoid, $\eta \in \Omega^1(G)$ be a contact 1-form on G and $\sigma : G \to \mathbb{R}$ be a multiplicative function on G. Then the triple $(G \rightrightarrows M, \eta, \sigma)$ is called a *contact groupoid* if

$$\eta_{gh}(X_g \oplus_{TG} Y_h) = \eta_g(X_g) + e^{\sigma(g)} \eta_h(Y_h)$$

for all $(X_g, Y_h) \in (TG)^{(2)}$.

Example 4.4. ([9]) Given a Lie groupoid $G \rightrightarrows M$ with Lie algebroid $AG \to M$, it is known that the cotangent bundle T^*G has a Lie groupoid structure over A^*G. Denote the structure maps of this groupoid by $\tilde{\alpha}, \tilde{\beta}, \tilde{\epsilon}$ and the composition by $\oplus_{T^*G}$. Now if $\sigma : G \to \mathbb{R}$ be a multiplicative function on G, then one can twist the cotangent Lie groupoid and get a Lie groupoid structure on $T^*G \times \mathbb{R}$ over A^*G whose structure maps are given by

$$\tilde{\alpha}_\sigma(\omega_g, \gamma) = e^{-\sigma(g)} \tilde{\alpha}(\omega_g)$$
$$\tilde{\beta}_\sigma(\nu_h, \zeta) = \tilde{\beta}(\nu_h) - \zeta(\delta\sigma)_{A_{\beta(h)}G}$$
$$(\omega_g, \gamma) \oplus_{T^*G \times \mathbb{R}} (\nu_h, \zeta) = \big((\omega_g + e^{\sigma(g)} \zeta (\delta\sigma)_g) \oplus_{T^*G} (e^{\sigma(g)} \nu_h), \gamma + e^{\sigma(g)} \zeta\big).$$

Then if we consider the canonical contact 1-form η_G on $T^*G \times \mathbb{R}$ and the multiplicative function $\sigma \circ pr$, where $pr : T^*G \times \mathbb{R} \to G$ denotes the projection onto G, the triple $(T^*G \times \mathbb{R} \rightrightarrows A^*G, \eta_G, \sigma \circ pr)$ is a contact groupoid.

The Lie algebroid of a contact groupoid is given by the following [9].

THEOREM 4.5. *Let $(G \rightrightarrows M, \eta, \sigma)$ be a contact groupoid. Then the base M admits a unique Jacobi structure (Λ_0, E_0) such that (α, e^σ) is a conformal Jacobi map and β is an anti-Jacobi map. Moreover the Lie algebroid of $G \rightrightarrows M$ is isomorphic to the 1-jet Lie algebroid of the Jacobi manifold (M, Λ_0, E_0).*

Let $G \rightrightarrows M$ be a Lie groupoid with Lie algebroid $AG \to M$, and $\sigma : G \to \mathbb{R}$ be a multiplicative function on G. Then σ induces a 1-cocycle $\phi_0 \in \Gamma(A^*G)$ of the Lie algebroid AG, defined by

$$\phi_0|_x(X_x) = X_x(\sigma), \quad \text{for } X_x \in A_xG,\ x \in M.$$

If we consider the contact groupoid $(T^*G \times \mathbb{R} \rightrightarrows A^*G, \eta_G, \sigma \circ pr)$ over A^*G, then the induced Jacobi structure on A^*G is same as the linear Jacobi structure on A^*G coming from the Lie algebroid AG and its 1-cocycle ϕ_0 [9]. Thus the Lie algebroid of the twisted cotangent groupoid $T^*G \times \mathbb{R} \to A^*G$ is isomorphic to the 1-jet Lie algebroid $T^*(A^*G) \times \mathbb{R} \to A^*G$, where A^*G is the linear Jacobi manifold. Moreover this isomorphism s is given by $(j'_G)^{-1}$ followed by $(R_{AG}, -id)$, that is, $s = (j'_G)^{-1} \circ (R_{AG}, -id)$, where j'_G is the isomorphism between the Lie algebroids $A(T^*G \times \mathbb{R})$ and $T^*(AG) \times \mathbb{R}$ over A^*G.

Given a multiplicative function on a Lie groupoid, one can also twist the tangent Lie groupoid and get a Lie groupoid structure on $TG \times \mathbb{R}$ over $TM \times \mathbb{R}$.

EXAMPLE 4.6. ([9]) Let $\sigma : G \to \mathbb{R}$ be a multiplicative function on G, then there is a Lie groupoid structure on $TG \times \mathbb{R}$ over $TM \times \mathbb{R}$ whose structure maps are given by

$$\begin{aligned}
&(T\alpha)_\sigma(X_g, \lambda) = \big((T\alpha)(X_g), X_g(\sigma) + \lambda\big)\\
&(T\beta)_\sigma(Y_h, \mu) = \big((T\beta)(Y_h), \mu\big)\\
&(X_g, \lambda) \oplus_{TG \times \mathbb{R}} (Y_h, \mu) = (X_g \oplus_{TG} Y_h, \lambda).
\end{aligned}$$

The Lie algebroid of this Lie groupoid $TG \times \mathbb{R} \rightrightarrows TM \times \mathbb{R}$ is given by $TAG \times \mathbb{R} \to TM \times \mathbb{R}$, the twisted tangent Lie algebroid of (AG, ϕ_0). Let j_G denote the isomorphism between $TAG \times \mathbb{R}$ and $A(TG \times \mathbb{R})$ over $TM \times \mathbb{R}$, generalizing the isomorphism between TAG and ATG over TM.

DEFINITION 4.7. ([9]) (Jacobi groupoid) Let $G \rightrightarrows M$ be a Lie groupoid with a Jacobi structure (Λ, E) on G and $\sigma : G \to \mathbb{R}$ be a multiplicative function. Then $(G \rightrightarrows M, \Lambda, E, \sigma)$ is called a *Jacobi groupoid* if the bundle map $(\Lambda, E)^\sharp : T^*G \times \mathbb{R} \to TG \times \mathbb{R}$ is a Lie groupoid morphism

$$\begin{tikzcd} T^*G \times \mathbb{R} \arrow[r, "(\Lambda,E)^\sharp"] \arrow[d, shift left] \arrow[d, shift right] & TG \times \mathbb{R} \arrow[d, shift left] \arrow[d, shift right] \\ A^*G \arrow[r] & TM \times \mathbb{R} \end{tikzcd}$$

from the twisted cotangent Lie groupoid given in Example 4.4 to the twisted tangent Lie groupoid given in Example 4.6.

Example 4.8. Contact groupoids are examples of Jacobi groupoids [9]. Jacobi Lie groups [8] are just Jacobi groupoids over a point [9].

Suppose $(G \rightrightarrows M, \Lambda, E, \sigma)$ be a Jacobi groupoid. Then the dual bundle $A^*G \cong (TM)^0$ also carries a Lie algebroid structure whose bracket and anchor are given by

$$[\alpha, \beta]_*(x) = \big(\pi_1[(\tilde{\alpha}, 0), (\tilde{\beta}, 0)]_{(\Lambda,E)}\big)(x), \quad \rho_*(\omega_x) = \Lambda^\sharp(\omega_x), \quad \text{for } x \in M,$$

where $\alpha, \beta \in \Gamma A^*G \cong \Gamma(TM)^0$; $\tilde{\alpha}, \tilde{\beta}$ be their arbitrary extension to 1-forms on G, and $T^*G \times \mathbb{R} \to T^*G$ being the projection onto the first factor [9]. Moreover the vector field E induces a 1-cocycle $X_0 \in \Gamma AG$ of the Lie algebroid A^*G and is defined by

$$X_0|_x(\omega_x) = -\langle \omega_x, E(x) \rangle, \quad \text{for } \omega_x \in A^*_xG,\ x \in M.$$

Let (Λ_{AG}, E_{AG}) be the linear Jacobi structure on AG coming from the Lie algebroid A^*G and its 1-cocycle X_0.

Theorem 4.9. *Let $(G \rightrightarrows M, \Lambda, E, \sigma)$ be a Jacobi groupoid with Lie algebroid AG. Then $\big((AG, \phi_0), (A^*G, X_0)\big)$ is a generalized Lie bialgebroid over M.*

Proof. Since $(\Lambda, E)^\sharp : T^*G \times \mathbb{R} \longrightarrow TG \times \mathbb{R}$ is a morphism of Lie groupoids, thus applying the Lie functor, we get a morphism $A\big((\Lambda, E)^\sharp\big) : A(T^*G \times \mathbb{R}) \longrightarrow A(TG \times \mathbb{R})$

$$\begin{tikzcd} A(T^*G \times \mathbb{R}) \arrow[r, "A\left((\Lambda,E)^\sharp\right)"] \arrow[d] & A(TG \times \mathbb{R}) \arrow[d] \\ A^*G \arrow[r] & TM \times \mathbb{R} \end{tikzcd}$$

between corresponding Lie algebroids.

Then by an argument similar to [11] shows that the diagram

$$\begin{tikzcd}
A(T^*G\times\mathbb{R}) \arrow[r, "A\left((\Lambda,E)^\sharp\right)"] \arrow[d, "j'_G"'] & A(TG\times\mathbb{R}) \\
T^*(AG)\times\mathbb{R} \arrow[r, "(\Lambda_{AG},E_{AG})^\sharp"'] & T(AG)\times\mathbb{R} \arrow[u, "j_G"']
\end{tikzcd}$$

commutes, where $j_G : TAG\times\mathbb{R}\to A(TG\times\mathbb{R})$ is the isomorphism of Lie algebroids over $TM\times\mathbb{R}$, and $j'_G : A(T^*G\times\mathbb{R})\to T^*(AG)\times\mathbb{R}$ is the isomorphism of Lie algebroids over A^*G.

Thus it follows that,

$$\begin{tikzcd}
 & A(T^*G\times\mathbb{R}) \arrow[r, "A\left((\Lambda,E)^\sharp\right)"] \arrow[d, "j'_G"'] & A(TG\times\mathbb{R}) \\
 & T^*(AG)\times\mathbb{R} \arrow[r, "(\Lambda_{AG},E_{AG})^\sharp"] & T(AG)\times\mathbb{R} \arrow[u, "j_G"'] \\
T^*(A^*G)\times\mathbb{R} \arrow[uur, bend left, "s"] \arrow[ur, "(R_{AG},-id)"'] \arrow[urr, bend right, "(\Lambda_{AG},E_{AG})^\sharp\circ(R_{AG},-id)"'] & &
\end{tikzcd}$$

also commutes, as

$$\begin{aligned}
A\big((\Lambda,E)^\sharp\big)\circ s &= A\big((\Lambda,E)^\sharp\big)\circ(j'_G)^{-1}\circ(R_{AG},-id)\\
&= j_G\circ(\Lambda_{AG},E_{AG})^\sharp\circ(R_{AG},-id).
\end{aligned}$$

Note that, $s=(j'_G)^{-1}\circ(R_{AG},-id)$ is an isomorphism of Lie algebroids over A^*G, and also j_G is an isomorphism of Lie algebroids over $TM\times\mathbb{R}$. From the diagram below,

$$\begin{tikzcd}
 & A(T^*G\times\mathbb{R}) \arrow[rr, "A\left((\Lambda,E)^\sharp\right)"] \arrow[dl] & & A(TG\times\mathbb{R}) \arrow[dr] & \\
A^*G \arrow[rrrr, "\left(\rho_*(_),X_0(_)\right)"'] & & & & TM\times\mathbb{R} \\
 & T^*(A^*G)\times\mathbb{R} \arrow[ul] \arrow[uu, "s"] \arrow[rr, "(\Lambda_{AG},E_{AG})^\sharp\circ(R_{AG},-id)"'] & & T(AG)\times\mathbb{R} \arrow[uu, "j_G"] \arrow[ur] &
\end{tikzcd}$$

since the top row is a morphism between Lie algebroids over $\big(\rho_*(\_), X_0(\_)\big)$: $A^*G \to TM \times \mathbb{R}$, the bottom row is also a morphism between Lie algebroids over the map $\big(\rho_*(\_), X_0(\_)\big) : A^*G \to TM \times \mathbb{R}$. Thus the result follows from Theorem 4.2. ∎

5. Coisotropic subgroupoids of Jacobi groupoid

In this section we introduce the notion of coisotropic subgroupoids of a Jacobi groupoid which is a straight forward generalization of coisotropic subgroupoids of a Poisson groupoid [16]. We also study their infinitesimal counterpart.

• Coisotropic subgroupoids

Definition 5.1. Let $(G \rightrightarrows M, \Lambda, E, \sigma)$ be a Jacobi groupoid. Then a subgroupoid $H \rightrightarrows N$ of $G \rightrightarrows M$ is called a *coisotropic subgroupoid* if H is a coisotropic submanifold of G.

Example 5.2. (i) Any Poisson groupoid $(G \rightrightarrows M, \Lambda)$ can be considered as a Jacobi groupoid with $E = 0$ and $\sigma = 0$. Then coisotropic subgroupoids of $(G \rightrightarrows M, \Lambda)$ are the coisotropic subgroupoids of $(G \rightrightarrows M, \Lambda, 0, 0)$.

(ii) Let $(G \rightrightarrows M, \Lambda, E, \sigma)$ be a Jacobi groupoid. Then M is a coisotropic submanifold of G and carries an induced Jacobi structure [9]. If $N \hookrightarrow M$ be a coisotropic submanifold, the subgroupoid $G|_N := \alpha^{-1}(N) \cap \beta^{-1}(N)$ is a coisotropic subgroupoid of $(G \rightrightarrows M, \Lambda, E, \sigma)$.

Note that, the infinitesimal object corresponding to a Jacobi groupoid $(G \rightrightarrows M, \Lambda, E, \sigma)$ is the generalized Lie bialgebroid $\big((AG, \phi_0), (A^*G, X_0)\big)$. Therefore it is natural to ask how the Lie algebroid of a coisotropic subgroupoid $H \rightrightarrows N$ is related to the generalized Lie bialgebroid $\big((AG, \phi_0),$ $(A^*G, X_0)\big)$. To answer this question, we introduce coisotropic subalgebroids of a generalized Lie bialgebroid and show that infinitesimal form of coisotropic subgroupoids of a Jacobi groupoid appear as coisotropic subalgebroids of the corresponding generalized Lie bialgebroid $\big((AG, \phi_0), (A^*G, X_0)\big)$.

Definition 5.3. Let $\big((A, \phi_0), (A^*, X_0)\big)$ be a generalized Lie bialgebroid over M. Then a Lie subalgebroid $B \to N$ of $A \to M$ is called a *coisotropic subalgebroid* of $\big((A, \phi_0), (A^*, X_0)\big)$ if $B \hookrightarrow A$ is a coisotropic submanifold, where A is equipped with the linear Jacobi structure coming from (A^*, X_0).

PROPOSITION 5.4. *Let $A \to M$ be a Lie algebroid and $\phi_0 \in \Gamma A^*$ be a 1-cocycle. Then a subbundle $B \to N$ of $A \to M$ is a Lie subalgebroid (and hence $\phi_0|_N \in \Gamma B^*$ is a 1-cocycle of it) if and only if B^0 is a coisotropic submanifold of A^*, where A^* is equipped with the linear Jacobi structure and $B^0_x = \{\gamma \in A^*_x \big|\ \gamma(v) = 0,\ \forall v \in B_x\}$, $x \in N$.*

Proof. First suppose that, B is a Lie subalgebroid of $A \to M$. Note that for any $X \in \Gamma A$, the function l_X is a linear function on A^*. Among the functions l_X for $X \in \Gamma A$, those which vanishes on B^0 are precisely those for which $X\big|_N \in \Gamma B$. Let $X, Y \in \Gamma A$ be such that $X\big|_N, Y\big|_N \in \Gamma B$. Then l_X, l_Y are linear functions on A^* vanishes on B^0. We have the Jacobi bracket

$$\{l_X, l_Y\} = l_{[X,Y]}.$$

Since $B \to N$ is a Lie subalgeboid of $A \to M$, we have $[X, Y]\big|_N \in \Gamma B$. Therefore the function $\{l_X, l_Y\}$ also vanishes on B^0. Among the pull back functions on A^*, those which vanishes on B^0 are of the form $f \circ q^*$, for some $f \in C^\infty(M)$ with $f\big|_N \equiv 0$. Therefore for any $X \in \Gamma A$ and $f \in C^\infty(M)$ with $X\big|_N \in \Gamma B$, $f\big|_N \equiv 0$, we have

$$\{l_X, f \circ q^*\} = \big(\rho(X)f + \phi(X)f\big) \circ q^*.$$

Since $\rho(X)\big|_N \in TN$ and $f\big|_N \equiv 0$ $\big($that is, $(\delta f)|_N \in (TN)^0\big)$, the function $\rho(X)f + \phi(X)f = \langle \rho(X), \delta f\rangle + \phi(X)f$ is vanishes on N, and hence, $\{l_X, f \circ q^*\}$ is vanishes on B^0. Thus by Proposition 2.4, we have B^0 is a coisotropic submanifold of A^*. ∎

Thus from the above proposition and since $(B^0)^0 = B$, we have the following.

PROPOSITION 5.5. *If $B \to N$ be a coisotropic subalgebroid of $\big((A, \phi_0), (A^*, X_0)\big)$, then $B^0 \to N$ is a coisotropic subalgebroid of $\big((A^*, X_0), (A, \phi_0)\big)$.*

It is known that (cf. Proposition 2.18), the base of a generalized Lie bialgebroid carries an induced Jacobi structure. The next result shows that the base of a coisotropic subalgebroid is a coisotropic submanifold with respect to this induced Jacobi structure.

PROPOSITION 5.6. *Let $\big((A, \phi_0), (A^*, X_0)\big)$ be a generalized Lie bialgebroid over M and $B \to N$ be a coisotropic subalgebroid of $\big((A, \phi_0), (A^*, X_0)\big)$. Then N is a coisotropic submanifold of M.*

Proof. If (Λ_M, E_M) denote the induced Jacobi structure on M, then from the Remark 2.19, we have

$$\Lambda_M^\sharp = \rho_* \circ \rho^*,$$

where ρ and ρ_* denote the anchors of the Lie algebroids A and A^* respectively. We first prove that $\rho^*(TN)^0 \subseteq B^0$. This is true because, $\langle \rho^*\xi_x, v\rangle = \langle \xi_x, \rho(v)\rangle = 0$, for $\xi_x \in (TN)_x^0$ and $v \in B_x$.

Thus we have

$$\Lambda_M^\sharp(TN)^0 = \rho_* \circ \rho^*(TN)^0 \subseteq \rho_*(B^0) \subseteq TN,$$

in the last inclusion we have used that B^0 is a Lie subalgebroid of A^*. Therefore, N is a coisotropic submanifold of M. ∎

• Infinitesimal form of Coisotropic subgroupoids

PROPOSITION 5.7. *Let $(G \rightrightarrows M, \Lambda, E, \sigma)$ be a Jacobi groupoid with generalized Lie bialgebroid $\big((AG, \phi_0), (A^*G, X_0)\big)$. Let $H \rightrightarrows N$ be a coisotropic subgroupoid with Lie algebroid $AH \to N$. Then $AH \to N$ is a coisotropic subalgebroid of $\big((AG, \phi_0), (A^*G, X_0)\big)$.*

Proof. Since $H \rightrightarrows N$ is a Lie subgroupoid of $G \rightrightarrows M$, therefore $AH \to N$ is a Lie subalgebroid of $AG \to M$. Next we claim that the anchor $\rho_* = \Lambda^\sharp\big|_{(TM)^0}$ of the Lie algebroid A^*G maps $(AH)^0$ to TN. Observe that, for any $x \in N$, $(AH)_x^0 = (TM)_x^0 \cap (TH)_x^0$ and $T_xN = T_xM \cap T_xH$. Therefore,

$$\rho_*(AH)^0 = \Lambda^\sharp(TM)^0 \cap \Lambda^\sharp(TH)^0 \subseteq TM \cap TH = TN,$$

here we have used the fact that both M and H are coisotropic submanifolds of G.

Let $\theta, \vartheta \in \Gamma A^*G \cong (TM)^0$ be such that $\theta\big|_N, \vartheta\big|_N \in (AH)^0$. Let $\widetilde{\theta}, \widetilde{\vartheta}$ be their respective extensions to 1-forms on G which are conormal to H. Then the 1-form $\pi_1[(\widetilde{\theta}, 0), (\widetilde{\vartheta}, 0)]_{(\Lambda,E)}$ on G is conormal to both M and H, as M and H are both coisotropic submanifolds of G.
Therefore,

$$\big(\pi_1[(\widetilde{\theta}, 0), (\widetilde{\vartheta}, 0)]_{(\Lambda,E)}\big)\big|_N \in (TM)^0 \cap (TH)^0 = (AH)^0.$$

Hence $(AH)^0 \to N$ defines a Lie subalgebroid of $A^*G \to M$. Therefore, by Proposition 5.4, it follows that $AH \hookrightarrow AG$ is a coisotropic submanifold. Thus, $AH \to N$ is a coisotropic subalgebroid of $\big((AG, \phi_0), (A^*G, X_0)\big)$. ∎

Corollary 5.8. *Let $(G \rightrightarrows M, \Lambda, E, \sigma)$ be a Jacobi groupoid and $H \rightrightarrows N$ be a coisotropic subgroupoid of it. Then N is a coisotropic submanifold of M.*

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References

[1] Y. Hagiwara, Nambu-Jacobi structures and Jacobi algebroids, *J. Phys. A: Math. Gen.* **37** (26) (2004), 6713 – 6725.

[2] P. J. Higgins, K. Mackenzie, Algebraic constructions in the category of Lie algebroids, *J. Algebra* **129** (1) (1990), 194 – 230.

[3] R. Ibáñez, M. de León, J. C. Marrero, D. Martín de Diego, Co-isotropic and Legendre-Lagrangian submanifolds and conformal Jacobi morphisms, *J. Phys. A: Math. Gen.* **30** (15) (1997), 5427 – 5444.

[4] D. Iglesias, "Grupos y Grupoides de Lie y Estructuras de Jacobi", Ph.D Thesis, Universidad de la Laguna, 2003.

[5] D. Iglesias, J. C. Marrero, Some linear Jacobi structures on vector bundles, *C. R. Acad. Sci., Paris I, Math.* **331** (2) (2000), 125 – 130.

[6] D. Iglesias and J. C. Marrero, Generalized Lie bialgebroids and Jacobi structure, *J. Geom. Phys.* **40** (2) (2001), 176 – 199.

[7] D. Iglesias, J. C. Marrero, Generalized Lie Bialgebroids and Strong Jacobi-Nijenhuis Structures, *Extracta Math.* **17** (2) (2002), 259 – 271.

[8] D. Iglesias, J. C. Marrero, Generalized Lie bialgebras and Jacobi structures on Lie groups, *Israel J. Math.* **133** (2003), 285 – 320.

[9] D. Iglesias, J. C. Marrero, Jacobi groupoids and generalized Lie bialgebroids, *J. Geom. Phys.* **48** (2-3) (2003), 385 – 425.

[10] K. C. H. Mackenzie, "Lie Groupoids and Lie Algebroids in Differential Geometry", London Mathematical Society Lecture Note Series 124, Cambridge University Press, Cambridge, 2005.

[11] K. C. H. Mackenzie, P. Xu, Lie bialgebroids and Poisson groupoids, *Duke Math. J.* **73** (2) (1994), 415 – 452.

[12] K. C. H. Mackenzie, P. Xu, Classical lifting process and multiplicative vector fields, *Quart. J. Math.* **49** (193) (1998), 59 – 85.

[13] Y. Kosmann-Schwarzbach, Exact Gerstenhaber algebras and Lie bialgebroids, *Acta Appl. Math.* **41** (1-3) (1995), 153 – 165.

[14] I. Vaisman, "Lectures on the Geometry of Poisson Manifolds", Progress in Mathematics 118, Birkhäuser Verlag, Basel, 1994.

[15] A. Weinstein, Coisotropic calculus and Poisson groupoids, *J. Math. Soc. Japan* **40** (4) (1988), 705-727.

[16] P. Xu, On Poisson groupoids, *Inter. J. Math.* **6** (1) (1995), 101 – 124.