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Adjoints of Generalized Composition Operators with Linear Fractional Symbol

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Abstract: Given a positive integer n and $\varphi : \mathbb{U} \to \mathbb{U}$, an analytic self-map of the open unit disc in the complex plane, the generalized composition operator $C_\varphi^{(n)}$ is defined by $C_\varphi^{(n)} f = f^{(n)} \circ \varphi$ for f belonging to some Hilbert space of analytic functions on $\mathbb{U}$. In this paper, we investigate some properties of generalized composition operators on the weighted Hardy spaces. Then we obtain adjoints of generalized composition operators with linear fractional symbol acting on the Hardy, Bergman and Dirichlet spaces.

Key words: Generalized composition operator, Adjoint, Weighted Hardy space.

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1. Introduction

Let $\mathbb{U}$ denote the open unit disc of the complex plane. For each sequence $\beta = \{\beta_n\}$ of positive numbers, the weighted Hardy space $H^2(\beta)$ consists of analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on $\mathbb{U}$ for which the norm

$$\|f\|_\beta = \left(\sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 \right)^{\frac{1}{2}}$$

is finite. Notice that the above norm is induced by the following inner product

$$\left\langle \sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} b_n z^n \right\rangle_\beta = \sum_{n=0}^{\infty} a_n \overline{b_n} \beta_n^2,$$

and that the monomials z^n form a complete orthogonal system for $H^2(\beta)$. Consequently, the polynomials are dense in $H^2(\beta)$. Observe that particular instances of the sequence $\beta = \{\beta_n\}$ yield well known Hilbert spaces of analytic functions. Indeed, $\beta_n = 1$ corresponds to the Hardy space $H^2(\mathbb{U})$. If $\beta_0 = 1$, $\beta_n = n^{1/2}$ for $n \geq 1$, the norm obtained is the one in the Dirichlet space $\mathcal{D}$ and if $\beta_n = (n+1)^{-1/2}$, we get the Bergman space $A^2(\mathbb{U})$. The inner product of two functions f and g in mentioned spaces may also be computed by integration. For the Hardy space, $H^2(\mathbb{U})$,

$$\langle f, g\rangle_{H^2(\mathbb{U})} = \int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\frac{d\theta}{2\pi},$$

where f and g are defined a.e. on $\partial\mathbb{U}$ via radial limits (see [4]). In case of the Bergman space,

$$\langle f, g\rangle_{A^2(\mathbb{U})} = \int_{\mathbb{U}} f(z)\overline{g(z)}dA(z),$$

where dA is the normalized area measure on $\mathbb{U}$ and for the Dirichlet space, the inner product is given by

$$\langle f, g\rangle_{\mathcal{D}} = f(0)\overline{g(0)} + \int_{\mathbb{U}} f'(z)\overline{g'(z)}dA(z).$$

If u is analytic on the open unit disc $\mathbb{U}$ and φ is an analytic map of the unit disc into itself, the weighted composition operator on $H^2(\beta)$ with symbols u and φ is the operator $(W_{u,\varphi}f)(z) = u(z)f(\varphi(z))$ for f in $H^2(\beta)$. When $u(z) \equiv 1$ we call the operator a composition operator and denote it by C_φ. The multiplication operator M_u corresponds to the case $\varphi(z) = z$ and is given by $M_u f(z) = u(z)f(z)$. For general information in this context one can see the excellent monographs [3], [13] and [15].

In recent years the concept of composition and weighted composition operator has been generalized in the literate. The generalized weighted composition operator $D^n_{\varphi,u}$ for nonnegative integer n, which introduced by Zhu [16] (see also [14]), is defined by $(D^n_{\varphi,u}f)(z) = u(z)f^{(n)}(\varphi(z))$. We denote the generalized composition operator by $C^{(n)}_\varphi$. Motivation to study this type of operators apart from their own importance is that they appear in the adjoint of composition operators on the space of analytic functions with derivative in the Hardy space [12].

One of the most fundamental questions related to composition and weighted composition operators and their generalizations is how to obtain a reasonable representation for their adjoints. The problem of computing the adjoint of a composition operator induced by linear fractional symbol on the Hardy space was solved by Cowen [2]. Hurst [8] using an analogous argument obtained the solution in the weighted Bergman space $A^2_\alpha(\mathbb{U})$. Mentioned demonstrations was by composition of another composition operator and two Toeplitz operators. In 2003, Gallardo and Montes [5] computed the adjoint of a composition operator with linear fractional symbol acting on the Dirichlet space by a different method from those used by Cowen and Hurst. Hammond, Moorhouse and Robbins [7] solved the case for rationally induced composition operators on the Hardy space, $H^2(\mathbb{U})$. Bourdon and Shapiro [1] obtained the Hammond-Moorhouse-Robbins formula in a straightforward algebraic fashion. For more investigation we refer the interested reader to [10] and [11]. Goshabulaghi and Vaezi in [6] gave the adjoint formula for rationally induced composition operators on the Dirichlet and Bergman spaces.

In this paper we investigate some properties of generalized composition operators on the weighted Hardy spaces. Then we obtain the adjoint of linear fractionally induced generalized composition operators acting on the Hardy, Bergman and Dirichlet spaces.

2. Generalized composition operators on $H^2(\beta)$

Every weighted Hardy space $H^2(\beta)$ contains a family of reproducing kernels $\{K_w : w \in \mathbb{U}\}$; that is, $\langle f, K_w\rangle = f(w)$ for every $f \in H^2(\beta)$. This property extends to higher derivatives of elements of $H^2(\beta)$. Indeed, for any $w \in \mathbb{U}$ there exists $K_{w,n} \in H^2(\beta)$ such that for any $f \in H^2(\beta)$, $f^{(n)}(w) = \langle f, K_{w,n}\rangle_\beta$. We call $K_{w,n}$, the generalized reproducing kernel function.

Theorem 2.1. [3, Theorem 2.16] *The generalized reproducing kernel function $K_{w,n}$ for the weighted Hardy space $H^2(\beta)$ is given by*

$$K_{w,n}(z) = \sum_{m=n}^{\infty} \frac{m(m-1)\cdots(m-(n-1))}{\beta_m^2} \bar{w}^{m-n} z^m.$$

Simple computations based on theorem 2.1 gives the following proposition.

Proposition 2.2. *The generalized reproducing kernel function $K_{w,n}$ at the point $z \in \mathbb{U}$ on the Hardy, Bergman and Dirichlet spces is given by*

$$\frac{n!z^n}{(1-\bar{w}z)^{n+1}}, \quad \frac{(n+1)!z^n}{(1-\bar{w}z)^{n+2}}, \quad \frac{(n-1)!z^n}{(1-\bar{w}z)^{n}},$$

respectively.

As a consequence of Littlewood's Subordination Principle every composition operator on the Hardy space is bounded. By [9, Proposition 3.4] this fact also holds for every composition operator on the Bergman space. The case for generalized composition operators is rather different. As we will see, if $\|\varphi\|_\infty < 1$, then $C_\varphi^{(n)}$ even belongs to the class of Hilbert-Schmidt operators on the aforementioned spaces. In spite of this, there exist examples of φ in which $\|\varphi\|_\infty = 1$ and $C_\varphi^{(n)}$ is unbounded.

Proposition 2.3. *Let $H^2(\beta)$ satisfies $\lim_{m\to\infty} \frac{\beta_{m-n}}{\beta_m} \neq 0$ and $\varphi(z) = az$ with $|a| = 1$. Then $C_\varphi^{(n)} : H^2(\beta) \to H^2(\beta)$ is unbounded.*

Proof. For $m \geq n$, define $f_m(z) = \frac{1}{m(m-1)\cdots(m-(n-1))\beta_m} z^m$. Then $\|f_m\|_\beta = \frac{1}{m(m-1)\cdots(m-(n-1))}$ and so $\{f_m\}_{m=n}^\infty$ converges to zero on $H^2(\beta)$. But $(C_\varphi^{(n)} f_m)(z) = \frac{1}{\beta_m} a^{m-n} z^{m-n}$ and accordingly, $\|C_\varphi^{(n)} f_m\| = \frac{\beta_{m-n}}{\beta_m}$ and hence $C_\varphi^{(n)} f_m$ does not converges to zero as we expect. Consequently, $C_\varphi^{(n)}$ is unbounded on $H^2(\beta)$. ∎

By Proposition 2.3 for $\varphi(z) = az$ with $|a| = 1$, $C_\varphi^{(n)}$ is unbounded on the Hardy, Bergman and Dirichlet spaces.

Theorem 2.4. *Let φ be an analytic self map of $\mathbb{U}$ such that for any positive integer $k \geq n$, $\|\varphi^k\|_\beta \leq c^k$ for some constant $0 < c < 1$. Then $C_\varphi^{(n)}$ is a Hilbert-Schmidt operator on $H^2(\beta)$.*

Proof. Let $\{e_m\}_{m=0}^\infty$ be defined by $e_m(z) = \frac{1}{\beta_m} z^m$. Then $\{e_m\}_{m=0}^\infty$ forms an orthonormal basis for $H^2(\beta)$ and for each $m \geq n$,

$$e_m^{(n)}(z) = \frac{m(m-1)\cdots(m-(n-1))}{\beta_m} z^{m-n}.$$

Accordingly,

$$\begin{aligned} \sum_{m=0}^\infty \|C_\varphi^{(n)} e_m\|_\beta^2 &= \sum_{m=0}^\infty \|e_m^{(n)} \circ \varphi\|_\beta^2 \\ &= \sum_{m=n}^\infty \frac{m^2(m-1)^2\cdots(m-(n-1))^2}{\beta_m^2} \|\varphi^{m-n}\|_\beta^2 \\ &\leq \sum_{m=n}^\infty \frac{m^2(m-1)^2\cdots(m-(n-1))^2}{\beta_m^2} c^{2(m-n)} = \|K_{c,n}\|_\beta^2 < \infty, \end{aligned}$$

which leads to $C_\varphi^{(n)}$ being Hilbert-Schmidt on $H^2(\beta)$. ∎

In view of Proposition 2.3, for $\varphi(z) = az$ with $|a| = 1$, $C_\varphi^{(n)}$ is unbounded on the Hardy and Bergman spaces. If φ is an analytic self map of $\mathbb{U}$ and $\|\varphi\|_\infty < 1$, the situation is very different than when $\|\varphi\|_\infty = 1$, as the following corollary shows.

Corollary 2.5. *Let φ be an analytic self map of $\mathbb{U}$ and $\|\varphi\|_\infty < 1$. Then $C_\varphi^{(n)}$ is Hilbert-Schmidt on the Hardy and Bergman spaces.*

Theorem 2.6. *Let φ be an analytic self map of $\mathbb{U}$ such that $\|\varphi\|_\infty < 1$. Then $C_\varphi^{(n)}$ is Hilbert-Schmidt on the Dirichlet space $\mathcal{D}$.*

Proof. Let $\{e_m\}_{m=0}^\infty$ be defined by $e_m(z) = \frac{1}{\sqrt{m}} z^m$. Then $\{e_m\}_{m=0}^\infty$ forms an orthonormal basis for $\mathcal{D}$ and for each $m \geq n+1$,

$$e_m^{(n)}(z) = \frac{m(m-1)\cdots(m-(n-1))}{\sqrt{m}} z^{m-n}.$$

It is clear that $\varphi^{m-n} \in \mathcal{D}$. Moreover, for each $m \geq n+1$ we have

$$\|\varphi'\varphi^{m-n-1}\|_{A^2}^2 \leq \|\varphi'\|_{A^2}^2 \|\varphi\|_\infty^{2(m-n-1)} \leq \|\varphi\|_{\mathcal{D}}^2 \|\varphi\|_\infty^{2(m-n-1)},$$

and hence

$$\begin{aligned} \|\varphi^{m-n}\|_{\mathcal{D}}^2 &= |\varphi(0)|^{2(m-n)} + (m-n)^2 \|\varphi'\varphi^{m-n-1}\|_{A^2}^2 \\ &\leq |\varphi(0)|^{2(m-n)} + (m-n)^2 \|\varphi\|_{\mathcal{D}}^2 \|\varphi\|_\infty^{2(m-n-1)}. \end{aligned}$$

Accordingly, using the root test we see that

$$\begin{aligned} \sum_{m=n+1}^\infty \|C_\varphi^{(n)} e_m\|_{\mathcal{D}}^2 &= \sum_{m=n+1}^\infty \|e_m^{(n)} \circ \varphi\|_{\mathcal{D}}^2 \\ &= \sum_{m=n+1}^\infty \frac{m^2(m-1)^2\cdots(m-(n-1))^2}{m} \|\varphi^{m-n}\|_{\mathcal{D}}^2 \\ &\leq \sum_{m=n+1}^\infty m(m-1)^2\cdots(m-(n-1))^2 |\varphi(0)|^{2(m-n)} \\ &\quad + \sum_{m=n+1}^\infty m(m-1)^2\cdots(m-n)^2 \|\varphi\|_{\mathcal{D}}^2 \|\varphi\|_\infty^{2(m-n-1)} < \infty, \end{aligned}$$

which leads to $C_\varphi^{(n)}$ being Hilbert-Schmidt on $\mathcal{D}$. ∎

3. Adjoints of generalized composition operator with linear fractional symbol

Corollary 2.5 and Theorem 2.6 of Section 2 guaranties that a wide class of generalized composition operators are bounded on the Hardy, Bergman and Dirichlet spaces. Specially, there always exist bounded linear fractionally induced generalized composition operators on the mentioned spaces and therefore computing the adjoint of generalized composition operators with linear fractional symbol makes sense. Now let $\varphi(z)=\frac{az+b}{cz+d}$ be a linear fractional self map of $\mathbb{U}$. It was shown by Cowen [3] that corresponding to φ, $\sigma(z)=\frac{\bar{a}z-\bar{c}}{-\bar{b}z+\bar{d}}$ is a self map of $\mathbb{U}$. In the sequel, for $n\geq 1$, we compute the adjoint of linear fractionally induced generalized composition operator $C_\varphi^{(n)}$ on the Hardy, Bergman and Dirichlet spaces.

Theorem 3.1. *Let $C_\varphi^{(n)}$ be a bounded generalized composition operator on $H^2(\mathbb{U})$ with linear fractional symbol. Then for each $f\in H^2(\mathbb{U})$ and $z\in\mathbb{U}$,*

$$C_\varphi^{(n)^*}f(z)=f(0)g_0(z)+D^n_{\sigma,u}M_vf(z),$$

where $u(z)=\frac{z^n}{(-\bar{b}z+\bar{d})^{n+1}}$, $v(z)=\frac{(\bar{d}z+\bar{c})^{n+1}}{z}$ and

$$g_0(z)=-\frac{z^n}{(-\bar{b}z+\bar{d})^{n+1}}\left(\frac{(\bar{d}z+\bar{c})^{n+1}}{z}\right)^{(n)}(\sigma(z))+\frac{n!\bar{d}^{n+1}z^n}{(-\bar{b}z+\bar{d})^{n+1}}.$$

Proof. A simple computation shows that

$$1-\overline{\varphi(w)}z=(-\bar{b}z+\bar{d})\left(\frac{1-\sigma(z)\bar{w}}{\bar{c}\bar{w}+\bar{d}}\right). \tag{3.1}$$

Moreover for every $g\in H^2(\mathbb{U})$, Proposition 2.2 implies that

$$g^{(n)}(w)=\langle g,K_{w,n}\rangle=\int_0^{2\pi}\frac{n!g(e^{i\theta})e^{-in\theta}}{(1-we^{-i\theta})^{n+1}}\frac{d\theta}{2\pi}.$$

Hence for $f\in H^2(\mathbb{U})$ with $f(0)=0$ and $z\neq 0$ we have

$$
\begin{aligned}
C_\varphi^{(n)^*} f(z) &= \langle C_\varphi^{(n)^*} f, K_z\rangle = \langle f, C_\varphi^{(n)} K_z\rangle = \langle f, K_z^{(n)} \circ \varphi\rangle \\
&= \langle f, n!\bar{z}^n (K_z \circ \varphi)^{n+1}\rangle \\
&= n! z^n \int_0^{2\pi} \frac{f(e^{i\theta})}{(1-\overline{\varphi(e^{i\theta})}z)^{n+1}} \frac{d\theta}{2\pi} \\
&= n! z^n \int_0^{2\pi} \frac{f(e^{i\theta})}{(-\bar{b}z+\bar{d})^{n+1}\left(\frac{1-\sigma(z)e^{-i\theta}}{\bar{c}e^{-i\theta}+\bar{d}}\right)^{n+1}} \frac{d\theta}{2\pi} \\
&= \frac{n! z^n}{(-\bar{b}z+\bar{d})^{n+1}} \int_0^{2\pi} \frac{(\bar{c}e^{-i\theta}+\bar{d})^{n+1} f(e^{i\theta})}{(1-\sigma(z)e^{-i\theta})^{n+1}} \frac{d\theta}{2\pi} \\
&= \frac{z^n}{(-\bar{b}z+\bar{d})^{n+1}} \int_0^{2\pi} \frac{n! \frac{(\bar{d}e^{i\theta}+\bar{c})^{n+1}}{e^{i\theta}} f(e^{i\theta}) e^{-in\theta}}{(1-\sigma(z)e^{-i\theta})^{n+1}} \frac{d\theta}{2\pi} \\
&= \frac{z^n}{(-\bar{b}z+\bar{d})^{n+1}} \langle h, K_{\sigma(z),n}\rangle = \frac{z^n}{(-\bar{b}z+\bar{d})^{n+1}} h^{(n)}(\sigma(z)),
\end{aligned} \tag{3.2}
$$

where $h(z) = \frac{(\bar{d}z+\bar{c})^{n+1}}{z} f(z)$ for $z \neq 0$ and $h(0) = \bar{c}^{n+1} f'(0)$. Notice that letting $g_1(z) = (\bar{d}z+\bar{c})^{n+1}$ we have $g_1 \in H^\infty(\mathbb{U})$ and hence $h \in H^2(\mathbb{U})$. Now let $f \in H^2(\mathbb{U})$ be arbitrary. Then for $z \neq 0$,

$$
\begin{aligned}
(C_\varphi^{(n)^*} f(0))(z) &= \langle C_\varphi^{(n)^*} f(0), K_z\rangle = \langle f(0), C_\varphi^{(n)} K_z\rangle = \langle f(0), K_z^{(n)} \circ \varphi\rangle \\
&= \langle f(0), n!\bar{z}^n (K_z \circ \varphi)^{n+1}\rangle = n! f(0) z^n \overline{(K_z \circ \varphi)(0)}^{n+1} \\
&= n! f(0) z^n (K_{\varphi(0)}(z))^{n+1} = \frac{n! f(0) z^n}{(1-\overline{\varphi(0)}z)^{n+1}} \\
&= \frac{n! \bar{d}^{n+1} f(0) z^n}{(-\bar{b}z+\bar{d})^{n+1}}.
\end{aligned} \tag{3.3}
$$

On the other hand, by (3.2)

$$
(C_\varphi^{(n)^*}(f-f(0)))(z) = \frac{z^n}{(-\bar{b}z+\bar{d})^{n+1}} h_0^{(n)}(\sigma(z)), \tag{3.4}
$$

where $h_0(z) = \frac{(\bar{d}z+\bar{c})^{n+1}}{z}(f-f(0)))(z)$. Combining equalities (3.3) and (3.4) results

$$C_\varphi^{(n)^*}f(z) = f(0)\left(\frac{n!\bar{d}^{n+1}z^n}{(-\bar{b}z+\bar{d})^{n+1}} - \frac{z^n}{(-\bar{b}z+\bar{d})^{n+1}}\left(\frac{(\bar{d}z+\bar{c})^{n+1}}{z}\right)^{(n)}(\sigma(z))\right)$$
$$+\frac{z^n}{(-\bar{b}z+\bar{d})^{n+1}}\left(\frac{(\bar{d}z+\bar{c})^{n+1}}{z}f(z)\right)^{(n)}(\sigma(z))$$
$$= f(0)g_0(z) + D^n_{\sigma,u}M_v f(z).$$

Now, by analyticity the statement of theorem holds for arbitrary $z \in \mathbb{U}$. ∎

THEOREM 3.2. *Let $C_\varphi^{(n)}$ and C_σ^{n+1} be bounded generalized composition operators on $A^2(\mathbb{U})$ with linear fractional symbol. Then for each $f \in A^2(\mathbb{U})$ and $z \in \mathbb{U}$,*

$$C_\varphi^{(n)^*}f(z) = f(0)g_0(z) + D^{n+1}_{\sigma,u}M_vQf(z),$$

where

$$g_0(z) = \frac{z^n}{(-\bar{b}z+\bar{d})^{n+2}}\left((n+1)!\bar{d}^{n+2} - \left(\frac{(\bar{d}z+\bar{c})^{n+2}}{z}\right)^{(n+1)}(\sigma(z))\right),$$

$u(z) = \frac{z^n}{(-\bar{b}z+\bar{d})^{n+2}}$, $v(z) = \frac{(\bar{d}z+\bar{c})^{n+2}}{z^2}$ and $Qf = F$ is the antiderivative of f with $F(0) = 0$.

Proof. For arbitrary $g \in A^2(\mathbb{U})$, Proposition 2.2 implies that

$$g^{(n)}(w) = \langle g, K_{w,n}\rangle = \int_{\mathbb{U}} \frac{(n+1)!g(t)\bar{t}^n}{(1-\bar{t}w)^{n+2}}dA(t).$$

Furthermore $K_z(t) = \frac{1}{(1-\bar{z}t)^2}$ and hence $K_z^{(n)}(t) = \frac{(n+1)!\bar{z}^n}{(1-\bar{z}t)^{n+2}}$. Therefore

$$(K_z^{(n)} \circ \varphi)(t) = \frac{(n+1)!\bar{z}^n}{(1-\bar{z}\varphi(t))^{n+2}}.$$

Let $g(w) = \frac{(cw+d)^{n+2}}{(1-\overline{\sigma(z)}w)^{n+2}}$. Then $g \in H^2(\mathbb{U})$ and by Lemma 2 of [10] for any f contained in the Dirichlet space, and hence for any polynomial f,

$$\langle f', g\rangle_{A^2} = \langle f, \iota g\rangle_{H^2}, \tag{3.5}$$

where ι is the identity map $\iota(z) = z$. Now, using (3.1) and (3.5), for any polynomial f with $f(0) = 0$ we have

$$\begin{aligned}
C_\varphi^{(n)^*} f(z) &= \langle C_\varphi^{(n)^*} f, K_z\rangle = \langle f, C_\varphi^{(n)} K_z\rangle = \langle f, K_z^{(n)} \circ \varphi\rangle \\
&= \langle f, (n+1)!\bar{z}^n (K_z \circ \varphi)^{n+2}\rangle = \int_{\mathbb{U}} \frac{(n+1)! f(w) z^n}{(1-\overline{\varphi(w)}z)^{n+2}} dA(w) \\
&= (n+1)! z^n \int_{\mathbb{U}} \frac{f(w)}{(-\bar{b}z+\bar{d})^{n+2}\left(\frac{1-\sigma(z)\bar{w}}{\bar{c}\bar{w}+\bar{d}}\right)^{n+2}} dA(w) \\
&= \frac{(n+1)! z^n}{(-\bar{b}z+\bar{d})^{n+2}} \int_{\mathbb{U}} \frac{(\bar{c}\bar{w}+\bar{d})^{n+2} f(w)}{(1-\sigma(z)\bar{w})^{n+2}} dA(w) \\
&= \frac{(n+1)! z^n}{(-\bar{b}z+\bar{d})^{n+2}} \langle f, g\rangle_{A^2} = \frac{(n+1)! z^n}{(-\bar{b}z+\bar{d})^{n+2}} \langle F, \iota g\rangle_{H^2} \\
&= \frac{(n+1)! z^n}{(-\bar{b}z+\bar{d})^{n+2}} \int_0^{2\pi} \frac{F(e^{i\theta}) e^{-i\theta} (\bar{c}e^{-i\theta}+\bar{d})^{n+2}}{(1-\sigma(z)e^{-i\theta})^{n+2}} \frac{d\theta}{2\pi} \\
&= \frac{(n+1)! z^n}{(-\bar{b}z+\bar{d})^{n+2}} \int_0^{2\pi} \frac{F(e^{i\theta}) \frac{(\bar{d}e^{i\theta}+\bar{c})^{n+2}}{e^{2i\theta}} e^{-i(n+1)\theta}}{(1-\sigma(z)e^{-i\theta})^{n+2}} \frac{d\theta}{2\pi} \\
&= \frac{z^n}{(-\bar{b}z+\bar{d})^{n+2}} \left(\frac{(\bar{d}z+\bar{c})^{n+2} F(z)}{z^2}\right)^{(n+1)} (\sigma(z)).
\end{aligned} \tag{3.6}$$

The last equality follows from generalized reproducing kernel property of $H^2(\mathbb{U})$. Let $A_0^2(\mathbb{U}) = \{f \in A^2(\mathbb{U}) : f(0) = 0\}$ and define the operator $T_0 : A_0^2(\mathbb{U}) \to A_0^2(\mathbb{U})$ by $Tf(z) = \frac{F(z)}{z^2}$ for $z \neq 0$ and $Tf(0) = \frac{1}{2} f'(0)$. Then T_0 is bounded on $A_0^2(\mathbb{U})$. Therefor, by continuity (3.6) holds for any $f \in A^2(\mathbb{U})$ with $f(0) = 0$. Moreover for arbitrary $f \in A^2(\mathbb{U})$,

$$\begin{aligned}
(C_\varphi^{(n)^*} f(0))(z) &= \langle f(0), K_z^{(n)} \circ \varphi\rangle = f(0)\overline{(K_z^{(n)} \circ \varphi)(0)} = \frac{f(0)(n+1)! z^n}{(1-\overline{\varphi(0)}z)^{n+2}} \\
&= \frac{f(0)(n+1)! \bar{d}^{n+2} z^n}{(-\bar{b}z+\bar{d})^{n+2}}.
\end{aligned} \tag{3.7}$$

It is clear that the antiderivative of $f - f(0)$ at z is $F(z) - f(0)z$, hence by

(3.6),

$$(C_\varphi^{(n)^*}(f - f(0)))(z)$$
$$= \frac{z^n}{(-\bar{b}z + \bar{d})^{n+2}} \left(\frac{(\bar{d}z + \bar{c})^{n+2}}{z^2} (F(z) - f(0)z) \right)^{(n+1)} (\sigma(z)). \tag{3.8}$$

Combine (3.7) and (3.8) and obtain

$$\begin{aligned} C_\varphi^{(n)^*} f(z) &= \frac{f(0)z^n}{(-\bar{b}z + \bar{d})^{n+2}} \left((n+1)!\bar{d}^{n+2} - (\frac{(\bar{d}z + \bar{c})^{n+2}}{z})^{(n+1)}(\sigma(z)) \right) \\ &\quad + \frac{z^n}{(-\bar{b}z + \bar{d})^{n+2}} \left(\frac{(\bar{d}z + \bar{c})^{n+2}}{z^2} F(z) \right)^{(n+1)} (\sigma(z)) \\ &= f(0)g_0(z) + D_{\sigma,u}^{n+1} M_v Q f(z). \end{aligned}$$

Now, by analyticity the statement of theorem holds for arbitrary $z \in \mathbb{U}$. ∎

Theorem 3.3. *Let $C_\varphi^{(n)}$ be a bounded generalized composition operator on $\mathcal{D}$ with linear fractional symbol. Then for $f \in \mathcal{D}$ and $z \in \mathbb{U}$,*

$$C_\varphi^{(n)^*} f(z) = f(0)g_0(z) + D_{\sigma,u}^n M_v f(z),$$

where $g_0(z) = \frac{(n-1)!\bar{d}^n z^n}{(-\bar{b}z+\bar{d})^n}$, $u(z) = \frac{(\bar{a}\bar{d}-\bar{b}\bar{c})z^{n+1}}{(-\bar{b}z+\bar{d})^{n+1}}$ and $v(z) = (\bar{d}z + \bar{c})^{n-1}$.

Proof. On the Dirichlet space $\mathcal{D}$, $K_z(t) = 1 + \ln \frac{1}{1-\bar{z}t}$ and hence $K_z^{(n)}(t) = \frac{(n-1)!\bar{z}^n}{(1-\bar{z}t)^n}$. Therefore using (3.1) we have

$$(K_z^{(n)} \circ \varphi)(w) = \frac{(n-1)!\bar{z}^n}{(1 - \bar{z}\varphi(w))^n} = \frac{(n-1)!\bar{z}^n}{(-b\bar{z} + d)^n} \left(\frac{cw + d}{1 - \overline{\sigma(z)}w} \right)^n,$$

which implies

$$(K_z^{(n)} \circ \varphi)'(w) = \frac{n!(ad - bc)\bar{z}^{n+1}}{(-b\bar{z} + d)^{n+1}} . \frac{(cw + d)^{n-1}}{(1 - \overline{\sigma(z)}w)^{n+1}}. \tag{3.9}$$

Now, abusing of notation, generalized reproducing kernel property of $H^2(\mathbb{U})$ implies that

$$\begin{aligned}\langle f(w), \frac{w(cw+d)^{n-1}}{(1-\overline{\sigma(z)}w)^{n+1}}\rangle_{H^2} &= \int_0^{2\pi} \frac{f(e^{i\theta})e^{-i\theta}(\bar{c}e^{-i\theta}+\bar{d})^{n-1}}{(1-\sigma(z)e^{-i\theta})^{n+1}}\frac{d\theta}{2\pi} \\ &= \int_0^{2\pi} \frac{f(e^{i\theta})(\bar{d}e^{i\theta}+\bar{c})^{n-1}e^{-in\theta}}{(1-\sigma(z)e^{-i\theta})^{n+1}}\frac{d\theta}{2\pi} \\ &= \frac{\left((\bar{d}z+\bar{c})^{n-1}f(z)\right)^{(n)}(\sigma(z))}{n!},\end{aligned}$$

since $(K_z^{(n)}\circ\varphi)' \in H^2(\mathbb{U})$, using (3.9) we see that

$$\begin{aligned} C_\varphi^{(n)^*}f(z) &= \langle C_\varphi^{(n)^*}f, K_z\rangle_{\mathcal{D}} = \langle f, C_\varphi^{(n)}K_z\rangle_{\mathcal{D}} = \langle f, K_z^{(n)}\circ\varphi\rangle_{\mathcal{D}} \\ &= f(0)\overline{K_z^{(n)}(\varphi(0))} + \langle f', (K_z^{(n)}\circ\varphi)'\rangle_{A^2} \\ &= f(0)\overline{K_z^{(n)}(\varphi(0))} + \langle f(w), w(K_z^{(n)}\circ\varphi)'(w)\rangle_{H^2} \\ &= f(0)\frac{(n-1)!\bar{d}^{\,n} z^n}{(-\bar{b}z+\bar{d})^n} + \frac{n!(\bar{a}\bar{d}-\bar{b}\bar{c})z^{n+1}}{(-\bar{b}z+\bar{d})^{n+1}}\langle f(w), \frac{w(cw+d)^{n-1}}{(1-\overline{\sigma(z)}w)^{n+1}}\rangle_{H^2} \\ &= f(0)\frac{(n-1)!\bar{d}^{\,n} z^n}{(-\bar{b}z+\bar{d})^n} \\ &\quad + \frac{n!(\bar{a}\bar{d}-\bar{b}\bar{c})z^{n+1}}{(-\bar{b}z+\bar{d})^{n+1}}.\frac{\left((\bar{d}z+\bar{c})^{n-1}f(z)\right)^{(n)}(\sigma(z))}{n!} \\ &= f(0)\frac{(n-1)!\bar{d}^{\,n} z^n}{(-\bar{b}z+\bar{d})^n} + \frac{(\bar{a}\bar{d}-\bar{b}\bar{c})z^{n+1}}{(-\bar{b}z+\bar{d})^{n+1}}\left((\bar{d}z+\bar{c})^{n-1}f(z)\right)^{(n)}(\sigma(z)) \\ &= f(0)g_0(z) + D_{\sigma,u}^n M_v f(z).\end{aligned}$$

Finally, by analyticity the statement of theorem holds for arbitrary $z \in \mathbb{U}$. ∎

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