EXTRACTA MATHEMATICAE Vol. **31**, Núm. 1, 47 – 67 (2016)

Quasi Grüss Type Inequalities for Complex Functions Defined on Unit Circle with Applications for Unitary Operators in Hilbert Spaces

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Presented by Alfonso Montes Received April 14, 2015

Abstract: Some quasi Grüss type inequalities for the Riemann-Stieltjes integral of continuous complex valued integrands defined on the complex unit circle $\mathcal{C}(0,1)$ and various subclasses of integrators are given. Natural applications for functions of unitary operators in Hilbert spaces are provided.

Key words: Grüss type inequalities, Riemann-Stieltjes integral inequalities, Unitary operators in Hilbert spaces, Spectral theory, Quadrature rules.

AMS *Subject Class.* (2010): 26D15, 47A63.

1. Introduction

The concept of *Riemann-Stieltjes integral* $\int_a^b f(t)\,du(t)$, where f is called *the integrand* and u is called *the integrator*, plays an important role in Mathematics, for instance in the definition of complex integral, the representation of bounded linear functionals on the Banach space of all continuous functions on an interval $[a,b]$, in the spectral representation of selfadjoint operators on complex Hilbert spaces and other classes of operators such as the unitary operators, etc.

One can approximate the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ with the following simpler quantity:

$$\frac{1}{b-a}[u(b)-u(a)]\cdot\int_a^b f(t)\,dt \qquad ([11],\,[12]). \tag{1.1}$$

In order to provide *a priory* sharp bounds for the *approximation error*,

consider the functionals:

$$D(f,u;a,b) := \int_a^b f(t)\,du(t) - \frac{1}{b-a}[u(b)-u(a)]\cdot\int_a^b f(t)\,dt.$$

If the integrand f is *Riemann integrable* on $[a,b]$ and the integrator $u:[a,b]\to\mathbb{R}$ is $L-$*Lipschitzian*, i.e.,

$$|u(t)-u(s)|\le L|t-s| \qquad \text{for each } \ t,s\in[a,b], \tag{1.2}$$

then the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ exists and, as pointed out in [11], the following quasi Grüss type inequality holds

$$|D(f,u;a,b)|\le L\int_a^b\left|f(t)-\int_a^b\frac{1}{b-a}f(s)\,ds\right|dt. \tag{1.3}$$

The inequality (1.3) is sharp in the sense that the multiplicative constant $C=1$ in front of L cannot be replaced by a smaller quantity. Moreover, if there exists the constants $m,M\in\mathbb{R}$ such that $m\le f(t)\le M$ for a.e. $t\in[a,b]$, then [11]

$$|D(f,u;a,b)|\le\frac{1}{2}L(M-m)(b-a). \tag{1.4}$$

The constant $\frac{1}{2}$ is best possible in (1.4).

We call this type of inequalities of *quasi Grüss type* since for integrators of integral form $u(t):=\frac{1}{b-a}\int_a^t g(s)ds$ the left hand side becomes

$$\left|\frac{1}{b-a}\int_a^b f(t)\,du(t)-\frac{1}{b-a}\int_a^b f(t)\,dt\cdot\frac{1}{b-a}\int_a^b g(s)\,ds\right|$$

that is related with the well known Grüss inequality.

A different approach in the case of integrands of bounded variation were considered by the same authors in 2001, [12], where they showed that

$$|D(f,u;a,b)|\le\max_{t\in[a,b]}\left|f(t)-\frac{1}{b-a}\int_a^b f(s)ds\right|\bigvee_a^b(u), \tag{1.5}$$

provided that f is continuous and u is of bounded variation. Here $\bigvee_a^b(u)$ denotes the total variation of u on $[a,b]$. The inequality (1.5) is sharp.

If we assume that f is $K-$Lipschitzian, then [12]

$$|D(f,u;a,b)| \leq \frac{1}{2}K(b-a)\bigvee_a^b(u), \tag{1.6}$$

with $\frac{1}{2}$ the best possible constant in (1.6).

For various bounds on the error functional $D(f,u;a,b)$ where f and u belong to different classes of function for which the Stieltjes integral exists, see [9], [8], [7], and [6] and the references therein.

For other inequalities for the Riemann-Stieltjes integral see [1] - [4], [5] - [10], [14] and the references therein.

For continuous functions $f:\mathcal{C}(0,1)\to\mathbb{C}$, where $\mathcal{C}(0,1)$ is the unit circle from $\mathbb{C}$ centered in 0 and $u:[a,b]\subseteq[0,2\pi]\to\mathbb{C}$ is a function of bounded variation on $[a,b]$, we can define the following *functional of quasi Grüss* type as well:

$$D_{\mathcal{C}}(f;u,a,b) := \int_a^b f(e^{it})du(t) - \frac{1}{b-a}[u(b)-u(a)]\cdot\int_a^b f(e^{it})dt. \tag{1.7}$$

In this paper we establish some bounds for the magnitude of $S_{\mathcal{C}}(f;u,a,b)$ when the integrand $f:\mathcal{C}(0,1)\to\mathbb{C}$ satisfies some *Hölder's type conditions* on the circle $\mathcal{C}(0,1)$ while the integrator u is of bonded variation.

It is shown that this functional can be naturally connected with continuous functions of unitary operators on Hilbert spaces.

We recall here some basic facts on unitary operators and spectral families that will be used in the sequel.

We say that the bounded linear operator $U:H\to H$ on the Hilbert space H is *unitary* iff $U^*=U^{-1}$.

It is well known that (see for instance [13, p. 275-p. 276]), if U is a unitary operator, then there exists a family of *projections* $\{E_\lambda\}_{\lambda\in[0,2\pi]}$, called the *spectral family* of U with the following properties:

a) $E_\lambda \leq E_\mu$ for $0\leq\lambda\leq\mu\leq 2\pi$;

b) $E_0=0$ and $E_{2\pi}=1_H$ (the *identity operator* on H);

c) $E_{\lambda+0}=E_\lambda$ for $0\leq\lambda<2\pi$;

d) $U=\int_0^{2\pi}e^{i\lambda}dE_\lambda$, where the integral is of Riemann-Stieltjes type.

Moreover, if $\{F_\lambda\}_{\lambda\in[0,2\pi]}$ is a family of projections satisfying the requirements a) - d) above for the operator U, then $F_\lambda=E_\lambda$ for all $\lambda\in[0,2\pi]$.

Also, for every continuous complex valued function $f : \mathcal{C}(0,1) \to \mathbb{C}$ on the complex unit circle $\mathcal{C}(0,1)$, we have

$$f(U) = \int_0^{2\pi} f\left(e^{i\lambda}\right) dE_\lambda \tag{1.8}$$

where the integral is taken in the Riemann-Stieltjes sense.

In particular, we have the equalities

$$\langle f(U)x, y\rangle = \int_0^{2\pi} f\left(e^{i\lambda}\right) d\langle E_\lambda x, y\rangle \tag{1.9}$$

and

$$\|f(U)x\|^2 = \int_0^{2\pi} \left|f\left(e^{i\lambda}\right)\right|^2 d\|E_\lambda x\|^2 = \int_0^{2\pi} \left|f\left(e^{i\lambda}\right)\right|^2 d\langle E_\lambda x, x\rangle, \tag{1.10}$$

for any $x, y \in H$.

From the above properties it follows that the function $g_x(\lambda) := \langle E_\lambda x, x\rangle$ is *monotonic nondecreasing* and *right continuous* on $[0, 2\pi]$ for any $x \in H$.

Such functions of unitary operators are

$$\exp(U) = \int_0^{2\pi} \exp\left(e^{i\lambda}\right) dE_\lambda$$

and

$$U^n = \int_0^{2\pi} e^{in\lambda} dE_\lambda$$

for n an integer.

We can also define the *trigonometric functions* for a unitary operator U by

$$\sin(U) = \int_0^{2\pi} \sin\left(e^{i\lambda}\right) dE_\lambda \text{ and } \cos(U) = \int_0^{2\pi} \cos\left(e^{i\lambda}\right) dE_\lambda$$

and the *hyperbolic functions* by

$$\sinh(U) = \int_0^{2\pi} \sinh\left(e^{i\lambda}\right) dE_\lambda \text{ and } \cosh(U) = \int_0^{2\pi} \cosh\left(e^{i\lambda}\right) dE_\lambda$$

where

$$\sinh(z) := \frac{1}{2}[\exp z - \exp(-z)] \text{ and } \cosh(z) := \frac{1}{2}[\exp z + \exp(-z)], \quad z \in \mathbb{C}.$$

2. Inequalities for Riemann-Stieltjes integral

We say that the complex function $f : \mathcal{C}(0,1) \to \mathbb{C}$ satisfies an *H-r-Hölder's type condition* on the circle $\mathcal{C}(0,1)$, where $H > 0$ and $r \in (0,1]$ are given, if

$$|f(z) - f(w)| \leq H|z-w|^r \tag{2.1}$$

for any $w, z \in \mathcal{C}(0,1)$.

If $r = 1$ and $L = H$ then we call it of *L-Lipschitz type*.

Consider the power function $f : \mathbb{C}\backslash\{0\} \to \mathbb{C}$, $f(z) = z^m$ where m is a nonzero integer. Then, obviously, for any z, w belonging to the unit circle $\mathcal{C}(0,1)$ we have the inequality

$$|f(z) - f(w)| \leq |m||z-w|$$

which shows that f is Lipschitzian with the constant $L = |m|$ on the circle $\mathcal{C}(0,1)$.

For $a \neq \pm 1, 0$ real numbers, consider the function $f : \mathcal{C}(0,1) \to \mathbb{C}$, $f_a(z) = \frac{1}{1-az}$. Observe that

$$|f_a(z) - f_a(w)| = \frac{|a||z-w|}{|1-az||1-aw|} \tag{2.2}$$

for any $z, w \in \mathcal{C}(0,1)$.

If $z = e^{it}$ with $t \in [0, 2\pi]$, then we have

$$\begin{aligned} |1-az|^2 &= 1 - 2a\,\mathrm{Re}(\bar{z}) + a^2|z|^2 = 1 - 2a\cos t + a^2 \\ &\geq 1 - 2|a| + a^2 = (1-|a|)^2 \end{aligned}$$

therefore

$$\frac{1}{|1-az|} \leq \frac{1}{|1-|a||} \quad \text{and} \quad \frac{1}{|1-aw|} \leq \frac{1}{|1-|a||} \tag{2.3}$$

for any $z, w \in \mathcal{C}(0,1)$.

Utilising (2.2) and (2.3) we deduce

$$|f_a(z) - f_a(w)| \leq \frac{|a|}{(1-|a|)^2}|z-w| \tag{2.4}$$

for any $z, w \in \mathcal{C}(0,1)$, showing that the function f_a is Lipschitzian with the constant $L_a = \frac{|a|}{(1-|a|)^2}$ on the circle $\mathcal{C}(0,1)$.

THEOREM 1. *Let $f : \mathcal{C}(0,1) \to \mathbb{C}$ satisfies an H-r-Hölder's type condition on the circle $\mathcal{C}(0,1)$, where $H > 0$ and $r \in (0,1]$ are given. If $u : [a,b] \subseteq [0,2\pi] \to \mathbb{C}$ is a function of bounded variation on $[a,b]$, then*

$$\begin{aligned} |D_{\mathcal{C}}(f;u,a,b)| &\leq \frac{2^r H}{b-a} \max_{t\in[a,b]} B_r(a,b;t) \bigvee_a^b (u) \\ &\leq \frac{H}{r+1}(b-a)^r \bigvee_a^b (u) \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} B_r(a,b;t) &:= \int_a^t \sin^r\left(\frac{t-s}{2}\right) ds + \int_t^b \sin^r\left(\frac{s-t}{2}\right) ds \\ &\leq \frac{1}{2^r} \frac{(t-a)^{r+1} + (b-t)^{r+1}}{r+1} \end{aligned} \tag{2.6}$$

for any $t \in [a,b]$.

In particular, if f is Lipschitzian with the constant $L > 0$, and $[a,b] \subset [0,2\pi]$ with $b - a \neq 2\pi$, then we have the simpler inequality

$$|D_{\mathcal{C}}(f;u,a,b)| \leq \frac{8L}{b-a} \sin^2\left(\frac{b-a}{4}\right) \bigvee_a^b (u) \leq \frac{1}{2} L(b-a) \bigvee_a^b (u). \tag{2.7}$$

If $a = 0$ and $b = 2\pi$ and f is Lipschitzian with the constant $L > 0$, then

$$|D_{\mathcal{C}}(f;u,0,2\pi)| \leq \frac{4L}{\pi} \bigvee_0^{2\pi} (u). \tag{2.8}$$

Proof. We have

$$\begin{aligned} D_{\mathcal{C}}(f;u,a,b) &= \int_a^b \left(f\left(e^{it}\right) - \frac{1}{b-a} \int_a^b f\left(e^{is}\right) ds \right) du(t) \\ &= \frac{1}{b-a} \int_a^b \left(\int_a^b \left[f\left(e^{it}\right) - f\left(e^{is}\right) \right] ds \right) du(t). \end{aligned} \tag{2.9}$$

It is known that if $p : [c,d] \to \mathbb{C}$ is a continuous function and $v : [c,d] \to \mathbb{C}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_c^d p(t)dv(t)$ exists and the following inequality holds

$$\left| \int_c^d p(t)dv(t) \right| \leq \max_{t\in[c,d]} |p(t)| \bigvee_c^d (v). \tag{2.10}$$

Utilising this property and (2.9) we have

$$\begin{aligned} |D_{\mathcal{C}}(f;u,a,b)| &= \frac{1}{b-a}\left|\int_a^b \left(\int_a^b \left[f\left(e^{it}\right)-f\left(e^{is}\right)\right] ds\right) du(t)\right| \\ &\leq \frac{1}{b-a}\max_{x\in[a,b]}\left|\int_a^b \left[f\left(e^{it}\right)-f\left(e^{is}\right)\right] ds\right| \bigvee_a^b (u). \end{aligned} \tag{2.11}$$

Utilising the properties of the Riemann integral and the fact that f is of H-r-Hölder's type on the circle $\mathcal{C}(0,1)$ we have

$$\begin{aligned} \left|\int_a^b \left[f\left(e^{it}\right)-f\left(e^{is}\right)\right] ds\right| &\leq \int_a^b \left|f\left(e^{it}\right)-f\left(e^{is}\right)\right| ds \\ &\leq H\int_a^b \left|e^{is}-e^{it}\right|^r ds \end{aligned} \tag{2.12}$$

Since

$$\begin{aligned} \left|e^{is}-e^{it}\right|^2 &= \left|e^{is}\right|^2 - 2\,\mathrm{Re}\left(e^{i(s-t)}\right) + \left|e^{it}\right|^2 \\ &= 2-2\cos(s-t) = 4\sin^2\left(\frac{s-t}{2}\right) \end{aligned}$$

for any $t,s\in\mathbb{R}$, then

$$\left|e^{is}-e^{it}\right|^r = 2^r\left|\sin\left(\frac{s-t}{2}\right)\right|^r \tag{2.13}$$

for any $t,s\in\mathbb{R}$.

Therefore

$$\begin{aligned} \int_a^b \left|e^{is}-e^{it}\right|^r ds &= 2^r\int_a^b \left|\sin\left(\frac{s-t}{2}\right)\right|^r ds \\ &= 2^r\left[\int_a^t \sin^r\left(\frac{t-s}{2}\right) ds + \int_t^b \sin^r\left(\frac{s-t}{2}\right) ds\right] \end{aligned} \tag{2.14}$$

for any $t\in[a,b]$.

On making use of (2.12) and (2.14) we have

$$\max_{x\in[a,b]}\left|\int_a^b \left[f\left(e^{it}\right)-f\left(e^{is}\right)\right] ds\right| \leq 2^r H\max_{t\in[a,b]} B_r(a,b;t)$$

and the first inequality in (2.5) is proved.

Utilising the elementary inequality $|\sin(x)| \leq |x|$, $x \in \mathbb{R}$ we have

$$\begin{aligned} B_r(a,b;t) &\leq \int_a^t \left(\frac{t-s}{2}\right)^r ds + \int_t^b \left(\frac{s-t}{2}\right)^r ds \\ &= \frac{1}{2^r}\frac{(t-a)^{r+1}+(b-t)^{r+1}}{r+1} \end{aligned} \tag{2.15}$$

for any $t \in [a,b]$, and the inequality (2.6) is proved.

If we consider the auxiliary function $\varphi : [a,b] \to \mathbb{R}$,

$$\varphi(t) = (t-a)^{r+1} + (b-t)^{r+1},\ r \in (0,1]$$

then

$$\varphi'(t) = (r+1)\left[(t-a)^r - (b-t)^r\right]$$

and

$$\varphi''(t) = (r+1)r[(t-a)^{r-1} + (b-t)^{r-1}].$$

We have $\varphi'(t) = 0$ iff $t = \frac{a+b}{2}$, $\varphi'(t) < 0$ for $t \in \left(a, \frac{a+b}{2}\right)$ and $\varphi'(t) > 0$ for $t \in (\frac{a+b}{2}, b)$. We also have that $\varphi''(t) > 0$ for any $t \in (a,b)$ showing that φ is strictly decreasing on $\left(a, \frac{a+b}{2}\right)$ and strictly increasing on $(\frac{a+b}{2}, b)$. We also have that

$$\min_{t\in[a,b]} \varphi(t) = \varphi\left(\frac{a+b}{2}\right) = \frac{(b-a)^{r+1}}{2^r}$$

and

$$\max_{t\in[a,b]} \varphi(t) = \varphi(a) = \varphi(b) = (b-a)^{r+1}.$$

Taking the maximum over $t \in [a,b]$ in (2.15) we deduce the second inequality in (2.5).

For $r = 1$ we have

$$\begin{aligned} B(a,b;t) &:= \int_a^t \sin\left(\frac{t-s}{2}\right) ds + \int_t^b \sin\left(\frac{s-t}{2}\right) ds \\ &= 2 - 2\cos\left(\frac{t-a}{2}\right) - 2\cos\left(\frac{b-t}{2}\right) + 2 \\ &= 2\left[1 - \cos\left(\frac{t-a}{2}\right) + 1 - \cos\left(\frac{b-t}{2}\right)\right] \\ &= 2\left[2\sin^2\left(\frac{t-a}{4}\right) + 2\sin^2\left(\frac{b-t}{4}\right)\right] \\ &= 4\left[\sin^2\left(\frac{t-a}{4}\right) + \sin^2\left(\frac{b-t}{4}\right)\right] \end{aligned}$$

for any $t \in [a,b]$.

Now, if we take the derivative in the first equality, we have

$$B'(a,b;t) = \sin\left(\frac{t-a}{2}\right) - \sin\left(\frac{b-t}{2}\right)$$
$$= 2\sin\left(\frac{t-\frac{a+b}{2}}{2}\right)\cos\left(\frac{b-a}{4}\right),$$

for $[a,b] \subset [0,2\pi]$ and $b-a \neq 2\pi$.

We observe that $B'(a,b;t) = 0$ iff $t = \frac{a+b}{2}$, $B'(a,b;t) < 0$ for $t \in \left(a, \frac{a+b}{2}\right)$ and $B'(a,b;t) > 0$ for $t \in (\frac{a+b}{2}, b)$. The second derivative is given by

$$B''(a,b;t) = \cos\left(\frac{t-\frac{a+b}{2}}{2}\right)\cos\left(\frac{b-a}{4}\right)$$

and we observe that $B''(a,b;t) > 0$ for $t \in (a,b)$.

Therefore the function $B(a,b;\cdot)$ is strictly decreasing on $\left(a, \frac{a+b}{2}\right)$ and strictly increasing on $\left(\frac{a+b}{2}, b\right)$. It is also a strictly convex function on (a,b). We have

$$\min_{t\in[a,b]} B(a,b;t) = B\left(a,b;\frac{a+b}{2}\right) = 8\sin^2\left(\frac{b-a}{8}\right)$$

and

$$\max_{t\in[a,b]} B(a,b;t) = B(a,b;a) = B(a,b;b) = 4\sin^2\left(\frac{b-a}{4}\right).$$

This proves the bound (2.7).

If $a = 0$ and $b = 2\pi$, then

$$B(0,2\pi;t) = 4\left[\sin^2\left(\frac{t}{4}\right) + \sin^2\left(\frac{2\pi-t}{4}\right)\right] = 4$$

and by (2.5) we get (2.8).

The proof is complete. ∎

The following result also holds:

THEOREM 2. *Let $f : \mathcal{C}(0,1) \to \mathbb{C}$ satisfies an H-r-Hölder's type condition on the circle $\mathcal{C}(0,1)$, where $H > 0$ and $r \in (0,1]$ are given. If $u : [a,b] \subseteq [0,2\pi] \to \mathbb{C}$ is a function of Lipschitz type with the constant $K > 0$ on $[a,b]$, then*

$$|D_{\mathcal{C}}(f;u,a,b)| \leq \frac{2^r HK}{b-a} C_r(a,b) \leq \frac{2HK(b-a)^{r+1}}{(r+1)(r+2)} \tag{2.16}$$

where

$$\begin{aligned} C_r(a,b) &:= \int_a^b \int_a^t \sin^r\left(\frac{t-s}{2}\right) ds\, dt + \int_a^b \int_t^b \sin^r\left(\frac{s-t}{2}\right) ds\, dt \\ &\leq \frac{(b-a)^{r+2}}{2^{r-1}(r+1)(r+2)}. \end{aligned} \tag{2.17}$$

In particular, if f is Lipschitzian with the constant $L > 0$, then we have the simpler inequality

$$\begin{aligned} |D_{\mathcal{C}}(f;u,a,b)| &\leq \frac{16LK}{b-a}\left[\frac{b-a}{2} - \sin\left(\frac{b-a}{2}\right)\right] \\ &\leq \frac{LK(b-a)^2}{3}. \end{aligned} \tag{2.18}$$

Proof. It is well known that if $p : [c,d] \to \mathbb{C}$ is a Riemann integrable function and $v : [c,d] \to \mathbb{C}$ is Lipschitzian with the constant $M > 0$, then the Riemann-Stieltjes integral $\int_c^d p(t)dv(t)$ exists and the following inequality holds

$$\left|\int_c^d p(t)dv(t)\right| \leq M\int_c^d |p(t)|dt. \tag{2.19}$$

Utilising the equality (2.9) and this property we have

$$\begin{aligned} |D_{\mathcal{C}}(f;u,a,b)| &= \frac{1}{b-a}\left|\int_a^b \left(\int_a^b \left[f\left(e^{it}\right) - f\left(e^{is}\right)\right] ds\right) du(t)\right| \\ &\leq \frac{K}{b-a}\int_a^b \left|\left(\int_a^b \left[f\left(e^{it}\right) - f\left(e^{is}\right)\right] ds\right)\right| dt. \end{aligned} \tag{2.20}$$

From (2.12) and (2.14) we have

$$\begin{aligned} &\left|\int_a^b \left[f\left(e^{it}\right) - f\left(e^{is}\right)\right] ds\right| \\ &\qquad \leq \int_a^b \left|f\left(e^{it}\right) - f\left(e^{is}\right)\right| ds \\ &\qquad \leq H\int_a^b \left|e^{is} - e^{it}\right|^r ds \\ &\qquad = 2^r H\left[\int_a^t \sin^r\left(\frac{t-s}{2}\right) ds + \int_t^b \sin^r\left(\frac{s-t}{2}\right) ds\right] \end{aligned} \tag{2.21}$$

and by (2.20) we deduce the first part of (2.16).

Since, by (2.15), we have

$$\int_a^t \left(\frac{t-s}{2}\right)^r ds + \int_t^b \left(\frac{s-t}{2}\right)^r ds = \frac{1}{2^r}\frac{(t-a)^{r+1}+(b-t)^{r+1}}{r+1},$$

then

$$\begin{aligned} C_r(a,b) &\le \int_a^b \left[\int_a^t \left(\frac{t-s}{2}\right)^r ds + \int_t^b \left(\frac{s-t}{2}\right)^r ds\right] dt \\ &\le \frac{1}{2^r}\int_a^b \frac{(t-a)^{r+1}+(b-t)^{r+1}}{r+1} dt \\ &= \frac{(b-a)^{r+2}}{2^{r-1}(r+1)(r+2)}, \end{aligned}$$

which proves the inequality (2.17).

For $r = 1$, we have

$$\begin{aligned} C_1(a,b) &:= \int_a^b \left[\int_a^t \sin\left(\frac{t-s}{2}\right) ds + \int_t^b \sin\left(\frac{s-t}{2}\right) ds\right] dt \\ &= \int_a^b \left[2 - 2\cos\left(\frac{t-a}{2}\right) - 2\cos\left(\frac{b-t}{2}\right) + 2\right] dt \\ &= 4(b-a) - 4\sin\left(\frac{b-a}{2}\right) - 4\sin\left(\frac{b-a}{2}\right) \\ &= 8\left[\frac{b-a}{2} - \sin\left(\frac{b-a}{2}\right)\right], \end{aligned}$$

which, by (2.16), produces the desired inequality (2.18). ∎

Remark 1. In the case $b = 2\pi$ and $a = 0$ the inequality (2.18) produces the simple inequality

$$|D_{\mathcal{C}}(f;u,0,2\pi)| \le 8LK. \tag{2.22}$$

The following result for monotonic integrators also holds.

THEOREM 3. *Let $f : \mathcal{C}(0,1) \to \mathbb{C}$ satisfies an H-r-Hölder's type condition on the circle $\mathcal{C}(0,1)$, where $H > 0$ and $r \in (0,1]$ are given. If $u : [a,b] \subseteq [0,2\pi] \to \mathbb{R}$ is a monotonic nondecreasing function on $[a,b]$, then*

$$\begin{aligned} |D_{\mathcal{C}}(f;u,a,b)| &\leq \frac{2^r H}{b-a} D_r(a,b) \\ &\leq \frac{H}{(r+1)(b-a)} \int_a^b \left[(t-a)^{r+1} + (b-t)^{r+1}\right] du(t) \\ &\leq \frac{H}{(r+1)} (b-a)^r [u(b) - u(a)] \end{aligned} \tag{2.23}$$

where

$$D_r(a,b) := \int_a^b B_r(a,b;t) du(t) \tag{2.24}$$

and $B_r(a,b;t)$ is given by (2.6).

In particular, if f is Lipschitzian with the constant $L > 0$, then we have the simpler inequality

$$\begin{aligned} |D_{\mathcal{C}}(f;u,a,b)| &\leq \frac{8L}{b-a} \int_a^b \left[\sin^2\left(\frac{t-a}{4}\right) + \sin^2\left(\frac{b-t}{4}\right)\right] du(t) \\ &\leq \frac{L}{2}(b-a)[u(b) - u(a)]. \end{aligned} \tag{2.25}$$

Proof. It is well known that if $p : [c,d] \to \mathbb{C}$ is a continuous function and $v : [c,d] \to \mathbb{R}$ is monotonic nondecreasing on $[c,d]$, then the Riemann-Stieltjes integral $\int_c^d p(t)dv(t)$ exists and the following inequality holds

$$\left| \int_c^d p(t) dv(t) \right| \leq \int_c^d |p(t)| dv(t). \tag{2.26}$$

Utilising this property and the identity (2.9) we have

$$
\begin{aligned}
&|D_{\mathcal{C}}(f;u,a,b)| \\
&\quad = \frac{1}{b-a}\left|\int_a^b\left(\int_a^b\left[f\left(e^{it}\right)-f\left(e^{is}\right)\right]ds\right)du(t)\right| \\
&\quad \le \frac{1}{b-a}\int_a^b\left|\left(\int_a^b\left[f\left(e^{it}\right)-f\left(e^{is}\right)\right]ds\right)\right|du(t) \\
&\quad \le \frac{1}{b-a}\int_a^b\left(\int_a^b\left|\left(f\left(e^{it}\right)-f\left(e^{is}\right)\right)\right|ds\right)du(t) \\
&\quad \le \frac{H}{b-a}\int_a^b\left(\int_a^b\left|e^{is}-e^{it}\right|^r ds\right)du(t) \\
&\quad = \frac{2^r H}{b-a}\int_a^b\left[\int_a^t \sin^r\left(\frac{t-s}{2}\right)ds+\int_t^b \sin^r\left(\frac{s-t}{2}\right)ds\right]du(t).
\end{aligned}
\tag{2.27}
$$

We also have that

$$
\begin{aligned}
&\int_a^b\left[\int_a^t \sin^r\left(\frac{t-s}{2}\right)ds+\int_t^b \sin^r\left(\frac{s-t}{2}\right)ds\right]du(t) \\
&\qquad \le \int_a^b\left[\int_a^t\left(\frac{t-s}{2}\right)^r ds+\int_t^b\left(\frac{s-t}{2}\right)^r ds\right]du(t) \\
&\qquad = \frac{1}{2^r}\int_a^b \frac{(t-a)^{r+1}+(b-t)^{r+1}}{r+1}du(t) \\
&\qquad = \frac{1}{2^r(r+1)}\int_a^b[(t-a)^{r+1}+(b-t)^{r+1}]du(t)
\end{aligned}
$$

and the first part of the inequality (2.23) is proved.

Since

$$\max_{t\in[a,b]}\left[(t-a)^{r+1}+(b-t)^{r+1}\right]=(b-a)^{r+1}$$

then the last part of (2.23) is also proved

For $r=1$ we have

$$
\begin{aligned}
D_1(a,b) &:= \int_a^b B_1(a,b;t)du(t) \\
&= 4\int_a^b\left[\sin^2\left(\frac{t-a}{4}\right)+\sin^2\left(\frac{b-t}{4}\right)\right]du(t)
\end{aligned}
$$

and the inequality (2.25) is obtained. ∎

Remark 2. The case $a = 0, b = 2\pi$ can be stated as

$$|D_{\mathcal{C}}(f; u, 0, 2\pi)| \leq \frac{4L}{\pi}[u(2\pi) - u(0)]. \tag{2.28}$$

Indeed, by (2.25) we have

$$\begin{aligned}
|D_{\mathcal{C}}(f; u, 0, 2\pi)| &\leq \frac{8L}{2\pi}\int_0^{2\pi}\left[\sin^2\left(\frac{t}{4}\right) + \sin^2\left(\frac{2\pi - t}{4}\right)\right] du(t) \\
&= \frac{4L}{\pi}\int_0^{2\pi}\left[\sin^2\left(\frac{t}{4}\right) + \sin^2\left(\frac{\pi}{2} - \frac{t}{4}\right)\right] du(t) \\
&= \frac{4L}{\pi}\int_0^{2\pi}\left[\sin^2\left(\frac{t}{4}\right) + \cos^2\left(\frac{t}{4}\right)\right] du(t) \\
&= \frac{4L}{\pi}[u(2\pi) - u(0)].
\end{aligned}$$

3. Applications for functions of unitary operators

We have the following vector inequality for functions of unitary operators.

Theorem 4. *Assume that $f : \mathcal{C}(0,1) \to \mathbb{C}$ satisfies an L-Lipschitz type condition on the circle $\mathcal{C}(0,1)$, where $L > 0$ is given. If the operator $U : H \to H$ on the Hilbert space H is unitary and $\{E_\lambda\}_{\lambda\in[0,2\pi]}$ is its spectral family, then*

$$\begin{aligned}
&\left|\langle f(U)x, y\rangle - \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})dt \cdot \langle x, y\rangle\right| \\
&\qquad\qquad \leq \frac{4L}{\pi}\bigvee_0^{2\pi}(\langle E_{(\cdot)}x, y\rangle) \leq \frac{4L}{\pi}\|x\|\|y\|
\end{aligned} \tag{3.1}$$

for any $x, y \in H$.

Proof. For given $x, y \in H$, define the function $u(\lambda) := \langle E_\lambda x, y\rangle, \lambda \in [0, 2\pi]$. We will show that u is of bounded variation and

$$\bigvee_0^{2\pi}(u) =: \bigvee_0^{2\pi}\left(\langle E_{(\cdot)}x, y\rangle\right) \leq \|x\|\|y\|. \tag{3.2}$$

It is well known that, if P is a nonnegative selfadjoint operator on H, i.e., $\langle Px, x\rangle \geq 0$ for any $x \in H$, then the following inequality is a *generalization of*

the Schwarz inequality in H

$$|\langle Px, y\rangle|^2 \leq \langle Px, x\rangle\langle Py, y\rangle, \tag{3.3}$$

for any $x, y \in H$.

Now, if $d : 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 2\pi$ is an arbitrary partition of the interval $[0, 2\pi]$, then we have by Schwarz's inequality for nonnegative operators (3.3) that

$$\begin{aligned}
&\bigvee_0^{2\pi} \left(\langle E_{(\cdot)}x, y\rangle\right) \\
&\quad = \sup_d \left\{\sum_{i=0}^{n-1} \left|\left\langle \left(E_{t_{i+1}} - E_{t_i}\right) x, y\right\rangle\right|\right\} \\
&\quad \leq \sup_d \left\{\sum_{i=0}^{n-1} \left[\left\langle \left(E_{t_{i+1}} - E_{t_i}\right) x, x\right\rangle^{1/2} \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) y, y\right\rangle^{1/2}\right]\right\} := I.
\end{aligned} \tag{3.4}$$

By the Cauchy-Buniakovski-Schwarz inequality for sequences of real numbers we also have that

$$\begin{aligned}
I &\leq \sup_d \left\{\left[\sum_{i=0}^{n-1} \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) x, x\right\rangle\right]^{1/2} \left[\sum_{i=0}^{n-1} \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) y, y\right\rangle\right]^{1/2}\right\} \\
&\leq \sup_d \left[\sum_{i=0}^{n-1} \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) x, x\right\rangle\right]^{1/2} \sup_d \left[\sum_{i=0}^{n-1} \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) y, y\right\rangle\right]^{1/2} \\
&= \left[\bigvee_0^{2\pi} \left(\langle E_{(\cdot)}x, x\rangle\right)\right]^{1/2} \left[\bigvee_0^{2\pi} \left(\langle E_{(\cdot)}y, y\rangle\right)\right]^{1/2} = \|x\|\|y\|
\end{aligned} \tag{3.5}$$

for any $x, y \in H$.

Utilising the inequality (2.8) we can write that

$$\begin{aligned}
&\left|\int_0^{2\pi} f(e^{it})\, d\langle E_t x, y\rangle - \frac{1}{2\pi}[\langle E_{2\pi}x, y\rangle - \langle E_0 x, y\rangle] \cdot \int_0^{2\pi} f(e^{it})dt\right| \\
&\qquad \leq \frac{4L}{\pi} \bigvee_0^{2\pi} \left(\langle E_{(\cdot)}x, y\rangle\right),
\end{aligned} \tag{3.6}$$

for any $x, y \in H$.

On making use of the representation theorem (1.9) and the inequality (3.2) we deduce the desired result (3.1). ∎

Remark 3. Consider the function $f:\mathcal{C}(0,1)\to\mathbb{C}$, $f_a(z)=\frac{1}{1-az}$ with a real and $0<|a|<1$. We know that this function is Lipschitzian with the constant $L=\frac{|a|}{(1-|a|)^2}$. Since $|ae^{it}|=|a|<1$, then

$$\begin{aligned}\int_0^{2\pi} f\left(e^{it}\right)dt &= \int_0^{2\pi}\frac{1}{1-ae^{it}}dt=\int_0^{2\pi}\sum_{n=0}^{\infty}\left(ae^{it}\right)^n dt\\ &= \sum_{n=0}^{\infty}a^n\int_0^{2\pi}\left(e^{it}\right)^n dt=\int_0^{2\pi}dt=2\pi,\end{aligned}$$

since for any natural number $n\geq 1$ we have $\int_0^{2\pi}(e^{it})^n dt=0$.

Applying the inequality (3.1) we have

$$\begin{aligned}&\left|\left\langle\left(1_H-aU\right)^{-1}x,y\right\rangle-\langle x,y\rangle\right|\\ &\qquad\leq\frac{4|a|}{\pi(1-|a|)^2}\bigvee_0^{2\pi}\left(\langle E_{(\cdot)}x,y\rangle\right)\leq\frac{4|a|}{\pi(1-|a|)^2}\|x\|\|y\|\end{aligned}\tag{3.7}$$

for any $x,y\in H$.

4. A quadrature rule

We consider the following partition of the interval $[a,b]$

$$\Delta_n: a=x_0<x_1<\cdots<x_{n-1}<x_n=b.$$

Define $h_k:=x_{k+1}-x_k$, $0\leq k\leq n-1$ and $\nu(\Delta_n)=\max\{h_k:0\leq k\leq n-1\}$ the norm of the partition Δ_n.

For the continuous function $f:\mathcal{C}(0,1)\to\mathbb{C}$ and the function $u:[a,b]\subseteq[0,2\pi]\to\mathbb{C}$ of bounded variation on $[a,b]$, define the quadrature rule

$$D_n(f,u,\Delta_n):=\sum_{k=0}^{n-1}\frac{u(x_{k+1})-u(x_k)}{x_{k+1}-x_k}\int_{x_k}^{x_{k+1}}f\left(e^{it}\right)dt\tag{4.1}$$

and the remainder $R_n(f,u,\Delta_n)$ in approximating the Riemann-Stieltjes integral $\int_a^b f(e^{it})du(t)$ by $D_n(f,u,\Delta_n)$. Then we have

$$\int_a^b f\left(e^{it}\right)du(t)=D_n(f,u,\Delta_n)+R_n(f,u,\Delta_n).\tag{4.2}$$

The following result provides a *priory* bounds for $R_n(f,u,\Delta_n)$ in several instances of f and u as above.

PROPOSITION 1. *Assume that $f : \mathcal{C}(0,1) \to \mathbb{C}$ satisfies the following Lipschitz type condition*

$$|f(z) - f(w)| \leq L|z - w|$$

for any $w, z \in \mathcal{C}(0,1)$, where $L > 0$ is given given.

If $[a,b] \subseteq [0,2\pi]$ and the function $u : [a,b] \to \mathbb{C}$ is of bounded variation on $[a,b]$, then for any partition $\Delta_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ with the norm $\nu(\Delta_n) < 2\pi$ we have the error bound

$$\begin{aligned} |R_n(f,u,\Delta_n)| &\leq 8L \sum_{k=0}^{n-1} \frac{1}{x_{k+1} - x_k} \sin^2\left(\frac{x_{k+1} - x_k}{4}\right) \bigvee_{x_k}^{x_{k+1}} (u) \\ &\leq \frac{1}{2} L \nu(\Delta_n) \bigvee_{a}^{b} (u). \end{aligned} \tag{4.3}$$

Proof. Since $\nu(\Delta_n) < 2\pi$, then on writing inequality (2.7) on each interval $[x_k, x_{k+1}]$, where $0 \leq k \leq n-1$, we have

$$\begin{aligned} &\left| \int_{x_k}^{x_{k+1}} f\left(e^{it}\right) du(t) - \frac{u(x_{k+1}) - u(x_k)}{x_{k+1} - x_k} \int_{x_k}^{x_{k+1}} f\left(e^{it}\right) dt \right| \\ &\qquad \leq \frac{8L}{x_{k+1} - x_k} \sin^2\left(\frac{x_{k+1} - x_k}{4}\right) \bigvee_{x_k}^{x_{k+1}} (u). \end{aligned}$$

Utilising the generalized triangle inequality we then have

$$\begin{aligned} &|R_n(f,u,\Delta_n)| \\ &\quad = \left| \sum_{k=0}^{n-1} \left[\int_{x_k}^{x_{k+1}} f\left(e^{it}\right) du(t) - \frac{u(x_{k+1}) - u(x_k)}{x_{k+1} - x_k} \int_{x_k}^{x_{k+1}} f\left(e^{it}\right) dt \right] \right| \\ &\quad \leq \sum_{k=0}^{n-1} \left| \left[\int_{x_k}^{x_{k+1}} f\left(e^{it}\right) du(t) - \frac{u(x_{k+1}) - u(x_k)}{x_{k+1} - x_k} \int_{x_k}^{x_{k+1}} f\left(e^{it}\right) dt \right] \right| \\ &\quad \leq \sum_{k=0}^{n-1} \frac{8L}{x_{k+1} - x_k} \sin^2\left(\frac{x_{k+1} - x_k}{4}\right) \bigvee_{x_k}^{x_{k+1}} (u) \\ &\quad \leq 8L \max_{0 \leq k \leq n-1} \left\{ \frac{1}{x_{k+1} - x_k} \sin^2\left(\frac{x_{k+1} - x_k}{4}\right) \right\} \sum_{k=0}^{n-1} \bigvee_{x_k}^{x_{k+1}} (u) \\ &\quad = 8L \max_{0 \leq k \leq n-1} \left\{ \frac{1}{x_{k+1} - x_k} \sin^2\left(\frac{x_{k+1} - x_k}{4}\right) \right\} \bigvee_{a}^{b} (u). \end{aligned}$$

Since

$$\frac{1}{x_{k+1}-x_k}\sin^2\left(\frac{x_{k+1}-x_k}{4}\right)\leq\frac{1}{16}(x_{k+1}-x_k)$$

then

$$\max_{0\leq k\leq n-1}\left\{\frac{1}{x_{k+1}-x_k}\sin^2\left(\frac{x_{k+1}-x_k}{4}\right)\right\}\leq\frac{1}{16}\nu(\Delta_n)$$

and the last part of (4.3) also holds. ∎

Remark 4. The above proposition has some particular cases of interest. If we take for instance $a=0$, $x_1=\pi$ and $b=2\pi$, then we have from (4.3) that

$$\left|\int_0^{2\pi}f\left(e^{it}\right)du(t)-\frac{u(\pi)-u(0)}{\pi}\int_0^{\pi}f\left(e^{it}\right)dt-\frac{u(2\pi)-u(\pi)}{\pi}\int_{\pi}^{2\pi}f(e^{it})dt\right|\leq\frac{8L}{\pi}\bigvee_0^{2\pi}(u).$$

Remark 5. We observe that the last bound in (4.3) provides a simple way to choose a division such that the accuracy in approximation is better that a given small $\varepsilon>0$. Indeed, if we want

$$\frac{1}{2}L\nu(\Delta_n)\bigvee_a^b(u)\leq\varepsilon$$

then we need to take Δ_n such that

$$\nu(\Delta_n)\leq\frac{2\varepsilon}{\bigvee_a^b(u)L}.$$

The above proposition can be also utilized to approximate functions of unitary operators as follows.

We consider the following partition of the interval $[0,2\pi]$

$$\Gamma_n:0=\lambda_0<\lambda_1<\cdots<\lambda_{n-1}<\lambda_n=2\pi$$

where $0\leq k\leq n-1$.

If U is a unitary operator on the Hilbert space H and $\{E_\lambda\}_{\lambda\in[0,2\pi]}$, the spectral family of U, then we can introduce the following sums:

$$D_n(f,U,\Gamma_n;x,y):=\sum_{k=0}^{n-1}\frac{1}{\lambda_{k+1}-\lambda_k}\int_{\lambda_k}^{\lambda_{k+1}}f\left(e^{it}\right)dt\cdot\left\langle\left(E_{\lambda_{k+1}}-E_{\lambda_k}\right)x,y\right\rangle. \tag{4.4}$$

COROLLARY 1. *Assume that $f : \mathcal{C}(0,1) \to \mathbb{C}$ satisfies the following Lipschitz type condition*

$$|f(z) - f(w)| \leq L|z - w|$$

for any $w, z \in \mathcal{C}(0,1)$, where $L > 0$ is given. Assume also that U is a unitary operator on the Hilbert space H and $\{E_\lambda\}_{\lambda\in[0,2\pi]}$ is the spectral family of U.

If Γ_n is a partition of the interval $[0, 2\pi]$ with $\nu(\Gamma_n) < 2\pi$ then we have the representation

$$\langle f(U)x, y\rangle = D_n(f, U, \Gamma_n; x, y) + R_n(f, U, \Gamma_n; x, y) \tag{4.5}$$

with the error $R_n(f, U, \Delta_n; x, y)$ satisfying the bounds

$$\begin{aligned} &|R_n(f, U, \Gamma_n; x, y)| \\ &\leq 8L \sum_{k=0}^{n-1} \frac{1}{\lambda_{k+1} - \lambda_k} \sin^2\left(\frac{\lambda_{k+1} - \lambda_k}{4}\right) \bigvee_{\lambda_k}^{\lambda_{k+1}} \left(\langle E_{(\cdot)}x, y\rangle\right) \\ &\leq \frac{1}{2} L\nu(\Gamma_n) \bigvee_{0}^{2\pi} \left(\langle E_{(\cdot)}x, y\rangle\right) \leq \frac{1}{2} L\nu(\Gamma_n)\|x\|\|y\| \end{aligned} \tag{4.6}$$

for any $x, y \in H$.

Remark 6. Consider the exponential mean

$$E_z(p, q) := \frac{\exp(pz) - \exp(qz)}{p - q}$$

defined for complex numbers z and the real numbers p, q with $p \neq q$.

For the function $f(z) = z^m$ with m an integer we have

$$\int_q^p f\left(e^{it}\right) dt = \int_q^p e^{imt} dt = \frac{1}{im}\left(e^{imp} - e^{imq}\right) = \frac{1}{im}(p - q)E_{e^{im}}(p, q).$$

For a partition Γ_n as above, define the sum

$$P_n(U, \Gamma_n; x, y) := \frac{1}{im} \sum_{k=0}^{n-1} E_{e^{im}}(\lambda_{k+1}, \lambda_k) \left\langle \left(E_{\lambda_{k+1}} - E_{\lambda_k}\right) x, y\right\rangle. \tag{4.7}$$

We can approximate the power m of an unitary operator as follows:

$$\langle U^m x, y\rangle = P_n(U, \Gamma_n; x, y) + T_n(U, \Gamma_n; x, y) \tag{4.8}$$

where the error $T_n(U,\Gamma_n;x,y)$ satisfies the bounds

$$\begin{aligned}
&|T_n(U,\Gamma_n;x,y)| \\
&\qquad \leq 8|m|\sum_{k=0}^{n-1}\frac{1}{\lambda_{k+1}-\lambda_k}\sin^2\left(\frac{\lambda_{k+1}-\lambda_k}{4}\right)\bigvee_{\lambda_k}^{\lambda_{k+1}}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\
&\qquad \leq \frac{1}{2}|m|\nu(\Gamma_n)\bigvee_{0}^{2\pi}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \leq \frac{1}{2}|m|\nu(\Gamma_n)\|x\|\|y\|
\end{aligned} \tag{4.9}$$

for any vectors $x,y\in H$.

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