Bounded normal generation is not equivalent to topological bounded normal generation

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Abstract: We show that some derived L^1 full groups provide examples of non simple Polish groups with the topological bounded normal generation property. In particular, it follows that there are Polish groups with the topological bounded normal generation property but not the bounded normal generation property.

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1. Introduction

The study of how quickly the conjugacy class of a non-central element generates the whole group has often turned out to be useful in understanding both the algebraic and topological structure of the given group. For example, such results on involutions by Broise [2] were crucial in de la Harpe's proof of simplicity of projective unitary groups of some von Neumann factors [3], while analogous results by Ryzhikov [12] are used to prove automatic continuity of homomorphisms for full groups of measure-preserving equivalence relations by Kittrell and Tsankov [9].

A group G has *bounded normal generation* (BNG) if the union of the conjugacy classes of any nontrivial element and of its inverse generate G in finitely many steps. Observe that any group with bounded normal generation is simple. Bounded normal generation already appears in Fathi's result that

the group of measure-preserving transformations of a standard probability space is simple [7], but the name was only given recently in [4, 5].

Obtaining bounded normal generation with a sharp number of steps is a crucial point of the seminal article [11] by Liebeck and Shalev concerning bounds on the diameter of the Cayley graph of a finite simple group. Other applications of bounded normal generation can be found in the first named author's work together with Thom [4, 5] concerning invariant automatic continuity and the uniqueness of Polish group topologies for projective unitary groups of certain operator algebras, in particular for separable type II_1 von Neumann factors.

The notion of bounded normal generation can be topologized: a topological group G has *topological bounded normal generation* if the conjugacy classes of every nontrivial element and of its inverse generate a dense subset of G in finitely many steps. This definition was introduced in [4] to give a clean structure to the proof of bounded normal generation for projective unitary groups of II_1 factors: prove topological bounded normal generation via some finite-dimensional approximation methods in the strong operator topology, then use some commutator tricks and results on symmetries to obtain bounded normal generation.

However, it is a priori not clear whether bounded normal generation and topological bounded normal generation are the same properties. Basic examples of non simple topologically simple groups such as the symmetric group over an infinite set or the projective unitary group of an infinite dimensional Hilbert space actually fail topological bounded normal generation (see Corollary 2.6). In fact, this problem was left open in [4] and is the main point of this article: we provide examples that separate bounded normal generation from its topological counterpart. Actually, we show that there are Polish groups which have topological bounded normal generation but which are not even simple.

The groups that serve our purposes are called derived L^1 full groups, introduced by the second named author in [10] as a measurable analogue the derived groups of small topological full groups. Let us now state our main result, see Section 2 for definitions.

THEOREM A. (See Theorem 4.4) *The derived L^1 full group of any hyperfinite ergodic graphing has topological bounded normal generation.*

In the above theorem we can moreover estimate how many products of conjugacy classes of a non-trivial element and its inverse in the derived L^1 full

group we need to generate a dense subset, see Theorem 4.4.

We do not know wether one can remove the hyperfiniteness asumption in Theorem A. As a concrete example of this result, one can consider the derived L^1 full group of a measure-preserving ergodic transformation, which was already known to be non-simple but topologically simple [10]. Using reconstruction-type results, we can show that this provides many examples of non simple groups with topological bounded normal generation.

THEOREM B. (See Corollary 5.4) *There are uncountably many pairwise non-isomorphic Polish groups which have topological bounded normal generation but fail to be simple (in particular they fail bounded normal generation).*

The article is structured as follows: in Section 2 we provide preliminaries around bounded normal generation and define derived L^1 full groups of graphings, in Section 3 we show that for hyperfinite graphings these groups contain a dense subgroup in which every element is a product of two involutions, in Section 4 we use this to prove Theorem A (see Theorem 4.4) and finally in Section 5 we prove a reconstruction theorem for derived L^1 full groups and use it to prove Theorem B (see Corollary 5.4).

2. PRELIMINARIES

2.1. TOPOLOGICAL BOUNDED NORMAL GENERATION.
If G is a group and $g \in G$, we write

$$g^{\pm G} = \left\{ hgh^{-1} \,:\, h \in G \right\} \cup \left\{ hg^{-1}h^{-1} \,:\, h \in G \right\}.$$

Given a subset A of G, we moreover write $A^{\cdot n}$ for the set of products of n elements of A, that is

$$A^{\cdot n} = \{a_1 \cdots a_n \,:\, a_1, \ldots, a_n \in A\}.$$

The following definitions are taken from [4, 5].

DEFINITION 2.1. A group G has *bounded normal generation* (BNG) if for every non-trivial $g \in G$, there is $n(g) \in \mathbb{N}$ such that

$$\left(g^{\pm G}\right)^{\cdot n(g)} = G.$$

A function $n : G \setminus \{1\} \to \mathbb{N}$ satisfying the above assumption is then called a *normal generation function*.

Observe that BNG implies simplicity. An example of a simple group without BNG is given by finitely supported permutations of signature 1 on an infinite set. Let us now define the topological version of BNG.

DEFINITION 2.2. A topological group G has *topological bounded normal generation* (TBNG) if for every non-trivial $g \in G$, there is $n(g) \in \mathbb{N}$ such that

$$\overline{(g^{\pm G})^{\cdot n(g)}} = G.$$

We still call a function $n : G \setminus \{1\} \to \mathbb{N}$ satisfying the above assumption a normal generation function.

The purpose of this paper is to exhibit examples of non-simple topological groups with TBNG. Let us first explain why the first examples of non simple topologically simple groups which come to mind actually fail TBNG.

DEFINITION 2.3. Let G be a topological group. A *lower semi-continuous invariant length function* on G is a map $l : G \to [0, +\infty]$ such that

(i) $l(1_G) = 0$;

(ii) $l(g^{-1}) = l(g)$ for all $g \in G$;

(iii) $l(gh) \leq l(g) + l(h)$ for all $g, h \in G$;

(iv) $l(ghg^{-1}) = l(h)$ for all $g, h \in G$;

(v) for all $r \geq 0$, the set $l^{-1}([0, r])$ is closed.

Such a function is *unbounded* if for all $K \in \mathbb{N}$ there is $g \in G$ such that $l(g) \in]K, +\infty[$.

EXAMPLE 2.4. On the symmetric group over a discrete set X endowed with the pointwise convergence topology, one can let $l(g)$ be the cardinality of the support of g. If X is infinite, then such a length function is unbounded.

On the unitary group of a Hilbert space endowed with the strong topology, one can let $l(g)$ be the minimum over $\lambda \in \mathbb{S}^1$ of the rank of $(g - \lambda \mathrm{id}_{\mathcal{H}})$ or of the trace class norm of $(g - \lambda \mathrm{id}_{\mathcal{H}})$. If the Hilbert space is infinite dimensional, then such a length function is unbounded and quotients down to an unbounded lower semi-continuous length function on the projective unitary group.

Observe that if we have an invariant length function on G then the group of elements of finite length is a F_σ normal subgroup of G. Moreover, we have the following straightforward proposition.

PROPOSITION 2.5. *If a topological group G admits an unbounded lower semi-continuous invariant length function, then it fails TBNG.*

Proof. Let l be an unbounded lower semi-continuous invariant length function on G. By unboundness we find $g \in G$ with $l(g) \in]0, +\infty[$ (in particular g is non-trivial). Fix $n \in \mathbb{N}$ and let $N = n \cdot l(g)$. Observe that $(g^{\pm G})^{\cdot n} \subseteq l^{-1}([0, N])$, and since the latter is closed we also have $\overline{(g^{\pm G})^{\cdot n}} \subseteq l^{-1}([0, N])$. Since l is unbounded, $l^{-1}([0, N])$ is not equal to G and we conclude that for every $n \in \mathbb{N}$ we have $\overline{(g^{\pm G})^{\cdot n}} \neq G$. So G fails TBNG. ∎

Applying this proposition to our two examples, we get the following.

COROLLARY 2.6. *The symmetric group over an infinite discrete set fails TBNG for the topology of pointwise convergence. The projective unitary group of an infinite-dimensional Hilbert space fails TBNG for the strong operator topology.*

Remark. The fact that the projective unitary group of an infinite-dimensional Hilbert space fails TBNG can also be seen via generalized projective s-numbers as in [4, pp. 80-81].

Let us conclude this section by introducing another natural topological version of BNG. Define *strong topological bounded normal generation* (STBNG) for a topological group G by asking that for every non-trivial $g \in G$, there is $n(g) \in \mathbb{N}$ such that

$$\left(\overline{g^{\pm G}} \right)^{\cdot n(g)} = G.$$

Since we always have $\left(\overline{g^{\pm G}} \right)^{\cdot n} \subseteq \overline{(g^{\pm G})^{\cdot n}}$, STBNG implies TBNG. In a compact group finite pointwise products of closed subsets are compact hence closed, so we actually always have $\left(\overline{g^{\pm G}} \right)^{\cdot n} = \overline{(g^{\pm G})^{\cdot n}}$. Hence STBNG is equivalent to TBNG for compact groups. However it is not true in general that products of closed sets are closed (for instance in $G = \mathbb{R}$ consider $A = B = \mathbb{N} \cup -\sqrt{2}\mathbb{N}$), so the following question is natural.

QUESTION 2.7. *Is it true in general that TBNG is equivalent to STBNG?*

2.2. DERIVED L^1 FULL GROUPS.

Let (X, μ) be a standard probability space. We will ignore null sets, in particular we identify two Borel functions on (X, μ) if they are the same up

to a null set. Given A, B Borel subsets of X, a *partial isomorphism* of (X, μ) of *domain* A and *range* B is a Borel bijection $f : A \to B$ which is measure preserving for the measures induced by μ on A and B respectively. We denote by $\operatorname{dom} f = A$ its domain, and by $\operatorname{rng} f = B$ its range. Note that in particular, $\mu(\operatorname{dom} f) = \mu(\operatorname{rng} f)$. When $\operatorname{dom} f = \operatorname{rng} f = X$ we say that f is a *measure-preserving transformation* and we denote by $\operatorname{Aut}(X, \mu)$ the group of such transformations. A measure-preserving transformation is *periodic* when all its orbits are finite.

Given a measure-preserving transformation T and a Borel T-invariant subset A of X, we denote by T_A the measure-preserving transformation defined by

$$T_A(x) = \begin{cases} T(x) & \text{if } x \in A\,, \\ x & \text{otherwise}\,. \end{cases}$$

A *graphing* is a countable set Φ of partial isomorphisms of (X, μ). It *generates* a *measure preserving equivalence relation* $\mathcal{R}_\Phi$, defined to be the smallest equivalence relation such that $\varphi(x) \, \mathcal{R}_\Phi \, x$ for every $\varphi \in \Phi$.

Every graphing Φ induces a natural graph structure on X where we connect x to y whenever there is $\varphi \in \Phi$ such that $y = \varphi(x)$. The connected components of this graph are precisely the $\mathcal{R}_\Phi$ classes, and we thus have a *graph metric* $d_\Phi : \mathcal{R}_\Phi \to \mathbb{N}$ given by

$$d_\Phi(x, y) = \min \big\{ n \in \mathbb{N} : \exists\, \varphi_1, \ldots, \varphi_n \in \Phi,\ \epsilon_1, \ldots, \epsilon_n \in \{-1, +1\},$$
$$y = \varphi_n^{\epsilon_n} \circ \cdots \circ \varphi_1^{\epsilon_1}(x) \big\}.$$

The *full group* of the measure-preserving equivalence relation $\mathcal{R}_\Phi$ is the group of all measure-preserving transformations T such that for all $x \in X$ we have $(x, T(x)) \in \mathcal{R}_\Phi$. We denote it by $[\mathcal{R}_\Phi]$.

The L^1 *full group* of the graphing Φ is the group of all $T \in [\mathcal{R}_\Phi]$ such that

$$\int_X d_\Phi(x, T(x)) d\mu(x) < +\infty\,.$$

It is denoted by $[\Phi]_1$. It has a natural complete separable right-invariant metric $\tilde{d}_\Phi$ given by

$$\tilde{d}_\Phi(T, U) = \int_X d_\Phi(T(x), U(x)) d\mu(x)\,.$$

We refer the reader to [10] for proofs and more background.

For the next definition, recall that the *derived group* of a group G is the subgroup generated by elements of the form $ghg^{-1}h^{-1}$ where $g, h \in G$. It is the smallest normal subgroup N of G such that G/N is abelian.

DEFINITION 2.8. The *derived* L^1 *full group* of a graphing Φ is the closure of the derived group of the L^1 full group of Φ. It is denoted by $[\Phi]'_1$.

The derived L^1 full group of a graphing Φ is by construction the smallest *closed* normal subgroup N of $[\Phi]_1$ such that $[\Phi]_1/N$ is abelian.

A graphing is *aperiodic* if all its connected components are infinite. An important fact is that as soon as Φ is an aperiodic graphing, every periodic element of the L^1 full group of Φ actually belongs to the derived L^1 full group of Φ (see [10, Lemma 3.11]). Moreover, we have the following result which will be strengthened in the next section for hyperfinite graphings.

THEOREM 2.9. ([10, THEOREM 3.13]) *Let Φ be an aperiodic graphing. Then its derived L^1 full group $[\Phi]'_1$ is topologically generated by involutions.*

3. PRODUCTS OF TWO INVOLUTIONS IN THE DERIVED L^1 FULL GROUP OF HYPERFINITE GRAPHINGS

DEFINITION 3.1. A measure-preserving equivalence relation is called *finite* if all its equivalence classes are finite.

It is *hyperfinite* if it can be written as an increasing union of measure-preserving equivalence relations $\mathcal{R}_n$ whose equivalence classes are finite.

A graphing Φ is hyperfinite if the equivalence relation it generates is hyperfinite.

Let Φ be a hyperfinite graphing, our aim is to show that its derived L^1 full group contains a dense subgroup all of whose elements are products of two involutions.

LEMMA 3.2. *Let Φ be an aperiodic graphing and suppose that $\mathcal{R}_\Phi$ is written as an increasing union of equivalence relations $\mathcal{R}_n$. Then $\bigcup_{n \in \mathbb{N}} [\mathcal{R}_n] \cap [\Phi]'_1$ is dense in $[\Phi]'_1$.*

Proof. By Theorem 2.9 we only need to be able to approximate involutions from $[\Phi]'_1$ by elements from $\bigcup_{n \in \mathbb{N}} [\mathcal{R}_n] \cap [\Phi]'_1$. Let $U \in [\Phi]'_1$ be an involution, then the U-invariant sets $A_n = \{x \in X : x\mathcal{R}_n U(x)\}$ satisfy $\bigcup_{n \in \mathbb{N}} A_n = X$ since $\bigcup_n \mathcal{R}_n = \mathcal{R}_\Phi$. By the dominated convergence theorem, we then have

$U_{A_n} \to U$. Moreover U_{A_n} is an involution so $U_{A_n} \in [\Phi]'_1$ and by construction $U_{A_n} \in [\mathcal{R}_n]$ so $U_{A_n} \in [\mathcal{R}_n] \cap [\Phi]'_1$ and the lemma is proved. ∎

DEFINITION 3.3. Given a graphing Φ, we say that the $\mathcal{R}_\Phi$-classes have a *uniformly bounded Φ-diameter* if there is $N \in \mathbb{N}$ such that for any $(x, y) \in \mathcal{R}_\Phi$ we have $d_\Phi(x, y) \leq N$.

LEMMA 3.4. *Let Φ be an aperiodic graphing, let $\mathcal{R}$ be a finite subequivalence relation of $\mathcal{R}_\Phi$ whose equivalence classes have a uniformly bounded Φ-diameter. Then $[\mathcal{R}]$ is a subgroup of $[\Phi]'_1$.*

Proof. The fact that there is a uniform bound M on the diameter of the $\mathcal{R}$-classes ensures that $[\mathcal{R}]$ is a subgroup of $[\Phi]_1$. Moreover all the elements of $[\mathcal{R}]$ are periodic and hence belong to $[\Phi]'_1$. ∎

LEMMA 3.5. *Let $\mathcal{R}$ be an equivalence relation. Suppose that there is $n \in \mathbb{N}$ such that all the $\mathcal{R}$-classes have cardinality at most n. Then $[\mathcal{R}]$ is locally finite.*

Proof. Let $U_1, \dots, U_k \in [\mathcal{R}]$ for some arbitrary $k \in \mathbb{N}$. By using a Borel linear order on X, we can identify in an $\mathcal{R}$-invariant manner the $\mathcal{R}$ orbit of each $x \in X$ with a set of the form $\{1, \dots, m_x\}$ with $m_x \leq n$. For each $x \in X$ there are only finitely many ways that the marked group generated by $\langle U_1, \dots, U_k \rangle$ can act on the finite set $\{1, \dots, m_x\}$. So we get a finite partition of X into $\mathcal{R}$-invariant sets such that on each atom of this partition, the marked group $\langle U_1, \dots, U_k \rangle$ acts on each orbit the same way, and we thus embed $\langle U_1, \dots, U_k \rangle$ in a finite product of finite permutation groups. ∎

THEOREM 3.6. *Let Φ be a hyperfinite graphing. Then $[\Phi]'_1$ contains a dense locally finite subgroup, all whose elements are products of two involutions.*

Proof. By definition we may write $\mathcal{R}_\Phi$ as an increasing union of finite equivalence relations $\mathcal{R}_n$. It is well known that we can moreover assume that each $\mathcal{R}_n$ has equivalence classes of size at most n. Let us show that we can also assume each $\mathcal{R}_n$ has equivalence classes of uniformly bounded Φ-diameter and cardinality.

We first find an increasing sequence of integers $(\varphi(n))_{n \in \mathbb{N}}$ such that for all $n \in \mathbb{N}$,

$$\mu\Big(\big\{x \in X \,:\, \big|[x]_{\mathcal{R}_n}\big| > \varphi(n) \text{ or } \operatorname{diam}_\Phi\big([x]_{\mathcal{R}_n}\big) > \varphi(n)\big\}\Big) < \frac{1}{2^n}.$$

Then by the Borel-Cantelli lemma, for almost all $x \in X$ there are only finitely many $n \in \mathbb{N}$ such that $|[x]_{\mathcal{R}_n}| > \varphi(n)$ or $\mathrm{diam}_\Phi([x]_{\mathcal{R}_n}) > \varphi(n)$. So if we define new equivalence relations $\mathcal{S}_n$ by $(x, y) \in \mathcal{S}_n$ if $(x, y) \in \mathcal{R}_n$ and $\mathrm{diam}_\Phi([x]_{\mathcal{R}_n}) \leq \varphi(n)$ and $|[x]_{\mathcal{R}_n}| \leq \varphi(n)$, we still have $\bigcup_{n \in \mathbb{N}} \mathcal{S}_n = \mathcal{R}$ and the $\mathcal{S}_n$-classes have a uniformly bounded diameter and cardinality as wanted.

By Lemma 3.2, $\bigcup_{n \in \mathbb{N}} [\mathcal{S}_n] \cap [\Phi]_1'$ is dense in $[\Phi]_1'$. But Lemma 3.4 yields that $[\mathcal{S}_n] \cap [\Phi]_1' = [\mathcal{S}_n]$. Since $\mathcal{S}_n$ is finite and all its classes have cardinality at most $\varphi(n)$, $[\mathcal{S}_n]$ is locally finite by Lemma 3.5. Finally each element of $[\mathcal{S}_n]$ is a product of two involutions by [8, Sublemma 4.3]. So the subgroup $\bigcup_{n \in \mathbb{N}} [\mathcal{S}_n]$ is as wanted. ∎

4. Topological bounded normal generation for derived L^1 full groups

In this section we will prove that derived L^1 full groups of ergodic amenable graphings have TBNG. A key tool will be Theorem 3.6, but we also need a better understanding of products of conjugates of involutions.

LEMMA 4.1. ([10, LEMMA 3.22]) *Let Φ be an ergodic graphing, and let $U \in [\Phi]_1$ be an involution whose support has measure $\alpha \leq 1/2$. Then the closure of the $[\Phi]_1'$ conjugacy class of U contains all involutions in $[\Phi]_1$ whose support has measure α and is disjoint from the support of U.*

LEMMA 4.2. *Let Φ be an ergodic graphing, let $U \in [\Phi]_1$ be an involution whose support has measure $\alpha < 1/3$. Then the closure of the $[\Phi]_1'$ conjugacy class of U contains all involutions in $[\Phi]_1$ whose support has measure α.*

Proof. Let $V \in [\Phi]_1$ have a support of measure α. Observe that $\mu(\mathrm{supp}\, U \cup \mathrm{supp}\, V) < 2/3$. By ergodicity, we find an involution $W \in [\mathcal{R}_\Phi]$ whose support has measure $1/3$ and is disjoint from $\mathrm{supp}\, U \cup \mathrm{supp}\, V$. By shrinking down W, we may actually assume that $W \in [\Phi]_1$ and that $\mu(\mathrm{supp}\, W) = \alpha$.

By the previous lemma W belongs to the closure of the conjugacy class of U and V belongs to the closure of the conjugacy class of W so V belongs to the closure of the conjugacy class of U. ∎

LEMMA 4.3. *Let Φ be an ergodic graphing and let $U \in [\Phi]_1$ be an involution whose support has measure $\alpha < 1/3$. Then every involution whose support has measure $\leq 2\alpha$ is the product of 2 elements from the closure of the $[\Phi]_1'$ conjugacy class of U.*

Proof. Let $V \in [\Phi]_1$ be an involution whose support has measure $\beta \leq 2\alpha$. Cut the support of V into two disjoint V-invariant sets A_1 and A_2 of measure $\beta/2$, and let $W \in [\Phi]_1$ be an involution with support of measure $\alpha - \beta/2$ disjoint from the support of V (in particular, W commutes with V).

Then $V = V_{A_1}V_{A_2} = (V_{A_1}W)(V_{A_2}W)$ and by the previous lemma both $V_{A_1}W$ and $V_{A_2}W$ belong to the closure of the conjugacy class of U. ∎

THEOREM 4.4. *Let Φ be a hyperfinite ergodic graphing. Then $[\Phi]_1'$ has TBNG. A normal generation function is given by $n(T) = 2 + 2\lceil \frac{7}{2\mu(\operatorname{supp} T)}\rceil$.*

Proof. Let $T \in [\Phi]_1'$. Let $A \subseteq \operatorname{supp} T$ be a maximal subset such that A and $T(A)$ are disjoint. Then by maximality $\operatorname{supp} T \subseteq T^{-1}(A) \cup A \cup T(A)$, and thus $\mu(A) \geq \mu(\operatorname{supp} T)/3$.

We may now find an involution $U \in [\Phi]_1$ whose support is contained in A and has measure $\mu(\operatorname{supp} T)/7$. Now $[T, U]$ is an involution whose support has measure $\mu(\operatorname{supp}[T, U]) = 2\mu(\operatorname{supp} T)/7 < 1/3$.

Let V be an arbitrary involution, we cut down its support into V-invariant pieces of measure $\mu(\operatorname{supp}[T, U])$ plus a remaining piece of measure less than $\mu(\operatorname{supp}[T, U])$. Observe that V is equal to the product of the transformations it induces on these pieces. There are at most $\lceil \frac{1}{\mu(\operatorname{supp}[T,U])}\rceil = \lceil \frac{7}{2\mu(\operatorname{supp} T)}\rceil$ such pieces, and by Lemma 4.2 and Lemma 4.3 applied to the involutions induced by V on the pieces, we can then write V as a product of at most $1 + \lceil \frac{7}{2\mu(\operatorname{supp} T)}\rceil$ elements of the closure of the conjugacy class of $[T, U]$.

Since there is a dense subgroup of $[\Phi]_1'$ whose elements are products of 2 involutions, we have a dense subgroup of $[\Phi]_1'$ all whose elements are in $\overline{(T^{\pm[\Phi]_1'})^k}$ where

$$k = 2\left(1 + \left\lceil \frac{7}{2\mu(\operatorname{supp} T)}\right\rceil\right).$$

So $[\Phi]_1'$ has TBNG. ∎

Remark. To get STBNG via the same approach, we would need to know that each element of $[\Phi]_1'$ is a product of at most k involutions for some fixed k. We do not know wether this is true, and we also cannot exclude that every element of the derived L^1 full group is a product of two involutions.

COROLLARY 4.5. *Let T be an ergodic measure-preserving transformation. Then $[T]_1'$ has TBNG but is not simple.*

Proof. By [6, Theorem 1] the graphing $\{T\}$ is hyperfinite, so by Theorem 4.4 the derived L^1 full group $[T_\alpha]'_1$ has TBNG. But by [10, Theorem 4.26] we also have that $[T]'_1$ is not simple. ∎

5. Reconstruction for the derived L^1 full group

Recall that two measure-preserving transformations T and T' are *flip conjugate* if there is a measure-preserving transformation S such that $T = ST'S^{-1}$ or $T^{-1} = ST'S^{-1}$. In [10] it was shown that when the L^1 full groups of two ergodic measure-preserving transformations T and T' are isomorphic, then T and T' are flip conjugate. The purpose of this section is to prove the same result for the *derived* L^1 full group.

Let us first note that by [10, Propoposition 3.17] and a reconstruction theorem of Fremlin as stated in [10, Theorem 3.18], we have the following result:

LEMMA 5.1. *Let Φ and Ψ be a aperiodic graphings, and let $\rho : [\Phi]'_1 \to [\Psi]'_1$ be a group isomorphism. Then there is a non-singular transformation S such that for all $U \in [\Phi]'_1$ we have $\rho(U) = SUS^{-1}$.*

In the ergodic case, we can upgrade this as in [10, Corollary 3.19].

COROLLARY 5.2. *Let Φ and Ψ be a ergodic graphings, and let $\rho : [\Phi]'_1 \to [\Psi]'_1$ be a group isomorphism. Then there is a measure-preserving transformation S such that for all $U \in [\Phi]'_1$ we have $\rho(U) = SUS^{-1}$.*

Proof. Same as the proof of [10, Corollary 3.18]. ∎

We then have the following reconstruction theorem, whose proof is inspired from a similar result of Bezuglyi and Medynets for topological full groups [1, Lemma 5.12].

THEOREM 5.3. *Let T_a and T_b be two ergodic measure-preserving transformations. Then the following are equivalent:*

  (i) *T_a and T_b are flip conjugate;*
 (ii) *$[T_a]'_1$ and $[T_b]'_1$ are topologically isomorphic;*
(iii) *$[T_a]'_1$ and $[T_b]'_1$ are abstractly isomorphic.*

Proof. The only non-trivial implication is (iii)⇒(i). Let $\rho : [T_a]_1' \to [T_b]_1'$ be a group isomorphism. By the previous corollary we find a measure-preserving transformation S such that for all $U \in [T_a]_1'$ one has

$$\rho(U) = SUS^{-1}.$$

We then find $A, B \subseteq X$ such that

$$X = A \sqcup T_a(A) \sqcup B \sqcup T_a(B) \sqcup T_a^2(B)$$

(see e.g. [10, Proposition 2.7]) and let

$$A_1 = A, \quad A_2 = T_a(A), \quad A_3 = B, \quad A_4 = T_a(B), \quad A_5 = T_a^2(B).$$

Consider then the involutions $U_1, \cdots, U_5$ defined by

$$U_i(x) = \begin{cases} T_a(x) & \text{if } x \in A_i, \\ T_a^{-1}(x) & \text{if } x \in T_a(A_i), \\ x & \text{else}. \end{cases}$$

These involutions all belong to the L^1 full group of T_a, so they actually belong to $[T_a]_1'$ by [10, Lemma 3.10]. Now for every $x \in A_i$ we have $T_a(x) = U_i(x)$ so for every $x \in S(A_i)$ we have $ST_aS^{-1}(x) = SU_iS^{-1}(x)$. Since for each $i \in \{1, \ldots, 5\}$ we have $SU_iS^{-1} \in [T_b]_1' \le [T_b]_1$, we conclude that ST_aS^{-1} can be obtained by cutting and pasting finitely many elements of $[T_b]_1$, hence $ST_aS^{-1} \in [T_b]_1$.

By the same argument we also have $T_b \in [ST_aS^{-1}]_1$ so T_b and ST_aS^{-1} share the same orbits. Belinskaya's theorem (as stated in [10, Theorem 4.1]) now implies that T_a and T_b are flip conjugate. ∎

COROLLARY 5.4. *There is a continuum of pairwise non-isomorphic non-simple Polish groups having TBNG. In particular, TBNG does not imply BNG.*

Proof. Consider, for an irrational $\alpha \in]0, 1/2[$, the rotation $T_\alpha(x) = x$ mod 1. By Corollary 4.5 the derived L^1 full groups $[T_\alpha]_1'$ have TBNG and are not simple. Moreover all those groups are non-isomorphic by the previous theorem and the fact that two irrational rotations T_α and T_β are conjugate iff $\alpha = \pm\beta$ mod 1. ∎

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