



On isolated points of the approximate point spectrum of a closed linear relation

M. LAJNEF, M. MNIF

*Department of Mathematics, Faculty of Sciences of Sfax, University of Sfax
Route de Soukra Km 3.5, BP 1171, 3000 Sfax, Tunisia*

maliklajnaf@gmail.com, maher.mnif@gmail.com

Received June 28, 2021
Accepted September 30, 2021

Presented by M. Mbekhta

Abstract: We investigate in this paper the isolated points of the approximate point spectrum of a closed linear relation acting on a complex Banach space by using the concepts of quasinilpotent part and the analytic core of a linear relation.

Key words: Linear relation, isolated point of the approximate point spectrum, analytic core, quasinilpotent part.

MSC (2020): 47A06, 47A10.

1. INTRODUCTION AND PRELIMINARIES

Throughout this paper, $(X, \|\cdot\|)$ will denote a complex Banach space. In 2008, González et al. [8] have shown that if 0 is isolated in the approximate point spectrum of a bounded operator T , then the quasinilpotent part $H_0(T)$ and the analytic core $K(T)$ of T are closed, $H_0(T) \cap K(T) = \{0\}$, $H_0(T) \oplus K(T)$ is closed and there exists $\lambda_0 \neq 0$ such that

$$H_0(T) \oplus K(T) = K(T - \lambda_0 I) = \bigcap_{n=0}^{\infty} \text{Im}(T - \lambda_0 I)^n.$$

In recent years, the study of isolated spectral points of a multivalued linear operator (linear relation) has generated a great deal of research attention. It was proved in [9] that for a closed and bounded linear relation T such that 0 is a point of its spectrum, we have the equivalence:

$$0 \text{ is isolated in the spectrum of } T \iff \begin{cases} H_0(T) \text{ and } K(T) \text{ are closed} \\ \text{and } X = H_0(T) \oplus K(T). \end{cases}$$

The previous studies on isolated spectral points in the two cases of linear operators and relations and their extensions motivate us to focus on establishing



some necessary conditions for which a point of the approximate point spectrum of a closed linear relation be isolated. By the way, this work could be considered as an extension of the study carried out for the case of operators since it covers the case of closed operators which are not necessary bounded. The importance of the investigation of linear relations is shown by the examples of issues in the study of some Cauchy problems associated with parabolic type equations in Banach spaces [6]. Thus, the generalization of the existing results for bounded operators to the general setting of closed linear relations seems to appear quite naturally. We recall now some basic definitions and properties which are needed in the sequel. A linear relation (or a multivalued linear operator) in a Banach space X , $T : X \rightarrow X$, is a mapping from a subspace $D(T)$, called the domain of T into the set of nonempty subsets of X verifying $T(\alpha_1x + \alpha_2y) = \alpha_1T(x) + \alpha_2T(y)$ for all non zero scalars α_1, α_2 and vectors x and $y \in D(T)$. We denote by $\mathcal{LR}(X)$ the class of all linear relations in X . A linear relation $T \in \mathcal{LR}(X)$ is completely determined by its graph defined by $G(T) := \{(x, y) \in X \times X : x \in D(T), y \in Tx\}$. Let $T \in \mathcal{LR}(X)$. The inverse of T is the relation T^{-1} given by $G(T^{-1}) := \{(u, v) \in X \times X : (v, u) \in G(T)\}$. We say that T is closed if its graph is a closed subspace of $X \times X$, open if $\gamma(T) > 0$, where

$$\gamma(T) = \begin{cases} \infty & \text{if } D(T) \subseteq \overline{\text{Ker}(T)}, \\ \inf \left\{ \frac{\|Tx\|}{d(x, \text{Ker}(T))} : x \in D(T) \setminus \overline{\text{Ker}(T)} \right\} & \text{otherwise.} \end{cases}$$

The set of all closed linear relations is denoted by $\mathcal{CR}(X)$. We say that T is continuous if the operator $Q_T T$ is continuous when Q_T denoted the quotient map from X onto $\frac{X}{T(0)}$. In such a case the norm of T is defined by $\|T\| := \|Q_T T\|$. We say that T is bounded if it is continuous and everywhere defined. The set of all bounded and closed linear relations acting between two Banach spaces X and Y is denoted by $\mathcal{BCR}(X, Y)$. If $X = Y$, we write $\mathcal{BCR}(X, X) := \mathcal{BCR}(X)$. The subspaces $\text{Ker}(T) := T^{-1}(0)$ and $\text{Im}(T) := T(D(T))$ are called respectively the null space and the range space of T . We say that T is surjective if $T(D(T)) = X$ and injective if $\text{Ker}(T) = \{0\}$. Note that T is an operator if and only if $T(0) = \{0\}$. In addition, the generalized range of T is defined by

$$R^\infty(T) := \bigcap_{n \in \mathbb{N}} \text{Im}(T^n).$$

For linear relations $S, T \in \mathcal{LR}(X)$ the relations $S + T$, ST and $S\hat{+}T$ are defined respectively by

$$S + T := \{(x, y + z) : (x, y) \in G(S) \text{ and } (x, z) \in G(T)\},$$

$$ST := \{(x, z) : (x, y) \in G(T) \text{ and } (y, z) \in G(S) \text{ for some } y \in X\},$$

$$S\hat{+}T := \{(x + u, y + v) : (x, y) \in G(S) \text{ and } (u, v) \in G(T)\}.$$

This last sum is direct when $G(S) \cap G(T) = \{(0, 0)\}$. In such case, we write $S \oplus T$. We denote by $\mathcal{B}(X, Y)$ the Banach algebra of all bounded operators on X and Y . If $X = Y$, we write $\mathcal{B}(X, X) := \mathcal{B}(X)$. Recall that a linear relation T is regular if $\text{Im}(T)$ is closed and $\text{Ker}(T^n) \subseteq \text{Im}(T^m)$ for all $n, m \in \mathbb{N}$. The class of all regular linear relations in X will be denoted by $\mathcal{R}(X)$. In what follows we write

$$\text{reg}(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is regular}\}.$$

For $r > 0$ we denote $D(0, r) := \{\lambda \in \mathbb{C} : 0 \leq |\lambda| < r\}$ and $D^*(0, r) := D(0, r) \setminus \{0\}$. Now, we essentially aim to define and study some basic tools of the spectral theory. Given a closed linear relation T . For $\lambda \in \mathbb{C}$, we denote by $R_\lambda(T) = (\lambda I - T)^{-1}$ the resolvent of T at λ . The resolvent set of T is the set defined by

$$\rho(T) = \{\lambda \in \mathbb{C} : (\lambda I - T)^{-1} \text{ is everywhere defined and single valued}\}.$$

We say that T is invertible if $0 \in \rho(T)$. The spectrum of T is the set $\sigma(T) = \mathbb{C} \setminus \rho(T)$.

Furthermore, we say that T is bounded below if there exists some $\delta > 0$ such that $\delta\|x\| \leq \|Tx\|$ for every $x \in D(T)$. The approximate point spectrum of T is defined by

$$\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not bounded below}\}.$$

Let us state now a useful lemma that we use below.

LEMMA 1.1. *Let $T \in \mathcal{CR}(X)$. Then we have the following assertions.*

- (i) *If T is surjective then there exists $\epsilon > 0$ such that $T - \lambda I$ is surjective for every $|\lambda| < \epsilon$.*
- (ii) *If T is bounded below then there exists $\epsilon > 0$ such that $T - \lambda I$ is bounded below for every $|\lambda| < \epsilon$.*

Proof. The proof is similar to the proof of [10, Lemma 15]. ■

The structure of this paper is as follows. In Section 2, we are mainly interested in studying the quasinilpotent part and the analytic core of a closed linear relation. Most properties of these latter subspaces are also gathered. The stated results generalize the concepts of quasinilpotent part and the analytic core recently introduced in [12] to the setting of closed not necessary bounded linear relations. Section 3 begins by a generalization to the case of closed linear relations of [9, Theorem 3.1] stated above. After that, we develop a significant quantity of interesting technical lemmas. In particular, we set out the concepts of regular linear relations and gap of two subspaces. This leads us to find some necessary conditions for which a point of the approximate point spectrum be isolated.

2. QUASINILPOTENT PART AND ANALYTIC CORE OF A CLOSED LINEAR RELATION

Let $T \in \mathcal{CR}(X)$. Consider the graph norm $\|\cdot\|_T$ on $D(T)$ defined by

$$\|x\|_T := \|x\| + \|Tx\|.$$

In what follows X_T denotes the space $D(T)$ endowed with the graph norm. Observe that X_T is a Banach space (since $Q_T T$ is a closed operator). Consider the relation \tilde{T} defined by

$$\begin{aligned} \tilde{T} : X_T &\longrightarrow X \\ x &\longmapsto Tx. \end{aligned}$$

Evidently, \tilde{T} is closed and $D(\tilde{T}) = D(T)$. Then, by virtue of [5, II.5.1] we get that $\tilde{T} \in \mathcal{BCR}(X_T, X)$.

Remark 2.1. Note that $Q_T \tilde{T} : X_T \rightarrow \frac{X}{\tilde{T}(0)}$ is bounded. Moreover, for all $x \in D(T)$, $\|Q_T i_T x\| = d(x, T(0)) \leq \|x\| \leq \|x\|_T$, then, $Q_T i_T$ is bounded, where

$$\begin{aligned} i_T : X_T &\longrightarrow X \\ x &\longmapsto x. \end{aligned}$$

Now, let's further extend the concepts of quasinilpotent part and the analytic core developed in [11, 12] to the case of closed not necessary bounded linear relations.

DEFINITION 2.1. Let $T \in \mathcal{CR}(X)$.

- (i) The quasinilpotent part of T , denoted by $H_0(T)$, is the set of all $x \in D(T)$ for which there exists a sequence $(x_n)_n \subseteq D(T)$ satisfying

$$x_0 = x, x_{n+1} \in Tx_n \text{ for all } n \in \mathbb{N} \text{ and } \|x_n\|_T^{\frac{1}{n}} \rightarrow 0.$$

- (ii) The analytic core of T , denoted by $K(T)$, is defined as the set of all $x \in X$ for which there exist $c > 0$ and a sequence $(x_n)_{n \in \mathbb{N}}$ satisfying $x_0 = x$ and for all $n \geq 0$, $x_{n+1} \in D(T)$, $x_n \in Tx_{n+1}$ and

$$d(x_n, \text{Ker}(T) \cap T(0)) \leq c^n d(x, \text{Ker}(T) \cap T(0)).$$

In the next lemma, we collect some elementary properties of $H_0(T)$ and $K(T)$.

LEMMA 2.1. Let $T \in \mathcal{CR}(X)$. Then the following statements hold.

- (i) If F is a closed subspace of X such that $T(F \cap D(T)) \subseteq F$, then $H_0(T) \cap F = H_0(T|F)$.
- (ii) For $\lambda \neq 0$, $H_0(T) \subseteq \overline{(\lambda I - T)H_0(T)}$ (the closure of $(\lambda I - T)H_0(T)$ in X).
- (iii) $T(D(T) \cap K(T)) = K(T)$.
- (iv) If F is a closed subspace of X such that $T(D(T) \cap F) = F$, then $F \subseteq K(T)$.
- (v) If $T \in \mathcal{R}(X)$, then $K(T) = R^\infty(T)$ and it is closed.

Proof. (i) Let $x \in H_0(T|F)$. Then, by Definition 2.1, there exists $(x_n)_n \subseteq F \cap D(T)$ such that

- $x_0 = x$,
- $x_{n+1} \in T|F x_n$,
- $\|x_n\|_{T|F}^{\frac{1}{n}} \rightarrow 0$.

But $T|F x_n = Tx_n$ and $\|x_n\|_{T|F} = \|x_n\|_T$, then $x \in H_0(T) \cap F$.

Conversely, assume that $x \in H_0(T) \cap F$. Then there exists $(x_n)_n \subseteq D(T)$ such that

- $x_0 = x \in F \cap D(T)$,
- $x_{n+1} \in Tx_n$,
- $\|x_n\|_T^{\frac{1}{n}} \rightarrow 0$.

Since $T(F \cap D(T)) \subseteq F$, then $(x_n) \subseteq F$ and so, $x_{n+1} \in T|Fx_n$ and $\|x_n\|_{T|F} = \|x_n\|_T$. Consequently, $x \in H_0(T|F)$.

- (ii) Let $x \in H_0(T)$. Then, there exists $(x_n)_n$ such that $x_0 = x$, $x_{n+1} \in Tx_n$ and $\|x_n\|_T^{\frac{1}{n}} \xrightarrow{n \rightarrow \infty} 0$. Let $y_n = \sum_{k=0}^n \frac{x_k}{\lambda^{k+1}}$. As, for all $n \in \mathbb{N}$, $x_n \in H_0(T)$, then $y_n \in H_0(T)$ and we have

$$(\lambda I - T)y_n = x - \frac{x_{n+1}}{\lambda^{n+1}} + T(0).$$

Therefore,

$$x - \frac{x_{n+1}}{\lambda^{n+1}} \in (\lambda I - T)y_n.$$

Whence,

$$x - \frac{x_{n+1}}{\lambda^{n+1}} \in (\lambda I - T)H_0(T).$$

Using the fact that $\|x_n\|_T^{\frac{1}{n}} \leq \|x_n\|_T^{\frac{1}{n}} \rightarrow 0$, one can deduce that

$$\frac{x_{n+1}}{\lambda^{n+1}} \xrightarrow{n \rightarrow \infty} 0,$$

which implies that $x \in \overline{(\lambda - T)H_0(T)}$. Thus, $H_0(T) \subseteq \overline{(\lambda I - T)H_0(T)}$.

- (iii) Let prove the first inclusion $T(D(T) \cap K(T)) \subseteq K(T)$. Let $y \in T(D(T) \cap K(T))$. Then, there exists $x \in D(T) \cap K(T)$ such that $y \in Tx$. Since $x \in K(T)$ then there exist $\delta > 0$ and a sequence $(x_n)_n$ such that

- $x_0 = x$,
- for all $n \geq 0$, $x_{n+1} \in D(T)$ and $x_n \in Tx_{n+1}$,
- $d(x_n, \text{Ker}(T) \cap T(0)) \leq \delta^n d(x, \text{Ker}(T) \cap T(0))$ for all $n \in \mathbb{N}$.

Let $(y_n)_n$ be the sequence defined by $y_{n+1} = x_n$ for all $n \in \mathbb{N}$ and $y_0 = y$. Since $x_{n-1} \in Tx_n$, then $y_n \in Ty_{n+1}$ for all $n \geq 1$. On the other hand, since $y_1 = x_0 = x$ and $y \in Tx$, then $y_0 \in Ty_1$. We need only to prove that $d(y_n, T(0) \cap \text{Ker}(T)) \leq \delta'^n d(y, T(0) \cap \text{Ker}(T))$ for some $\delta' > 0$. Trivially, if $y \in T(0) \cap \text{Ker}(T)$ there is nothing to prove. If not we get

that $d(y_n, T(0) \cap \text{Ker}(T)) \leq \delta^m d(y, T(0) \cap \text{Ker}(T))$ with $\delta' > 0$ be such that

$$\delta' = \max \left\{ \delta, \frac{d(x, T(0) \cap \text{Ker}(T))}{d(y, T(0) \cap \text{Ker}(T))} \right\}.$$

Thus, $y \in K(T)$.

For the reverse inclusion, let $x \in K(T)$. Then there exists $\delta > 0$ and a sequence $(u_n)_n$ such that

- $u_0 = x$,
- for all $n \geq 0$, $u_{n+1} \in D(T)$ and $u_n \in Tu_{n+1}$,
- $d(u_n, \text{Ker}(T) \cap T(0)) \leq \delta^n d(x, \text{Ker}(T) \cap T(0))$ for all $n \in \mathbb{N}$.

Since $x \in Tu_1$ then, in order to show that $K(T) \subseteq T(D(T) \cap K(T))$, it is sufficient to prove that $u_1 \in K(T)$. If $u_1 \in T(0) \cap \text{Ker}(T)$, then there is nothing to prove. If not, let $(w_n)_n$ be the sequence such that $w_n = u_{n+1}$. We have $w_n = u_{n+1} \in Tu_{n+2} = Tw_{n+1}$. Furthermore,

$$d(w_n, \text{Ker}(T) \cap T(0)) = d(u_{n+1}, \text{Ker}(T) \cap T(0)) \leq \delta^n d(u_1, \text{Ker}(T) \cap T(0)),$$

where $\delta' = \delta^2 \frac{d(x, \text{Ker}(T) \cap T(0))}{d(u_1, \text{Ker}(T) \cap T(0))}$. Hence, $u_1 \in K(T)$.

- (iv) First, we claim that $F \cap D(T)$ is closed in X_T . Indeed, let $(x_n)_n \subseteq F \cap D(T)$ be such that $x_n \xrightarrow[n \rightarrow \infty]{X_T} x$. Trivially, $x \in D(T)$. On the other hand, we have $\|x_n - x\|_T \xrightarrow[n \rightarrow \infty]{} 0$. As F is closed in X , then $x \in F$. Hence, $F \cap D(T)$ is closed in X_T , as claimed. Recall that the relation \tilde{T} is closed. Let us consider $T_0 : D(T) \cap F \rightarrow F$, the restriction of \tilde{T} . We have $G(T_0) = G(\tilde{T}) \cap ((D(\tilde{T}) \cap F) \times F)$ is closed in $X_T \times X$. Then, T_0 is closed. We have, by hypothesis, $\text{Im}(T_0) = F$ then, by the open mapping theorem [5, Theorem III.4.2], we deduce that T_0 is open. Thus, there exists a constant $\gamma > 0$ such that for all $x \in D(T_0) = D(T) \cap F$,

$$d_T(x, \text{Ker}(T_0)) \leq \gamma \|T_0 x\|,$$

where $d_T(x, G) := \inf_{\alpha \in G} \|x - \alpha\|_T$. As, for all $x \in D(T_0)$ and $\alpha \in \text{Ker}(T_0)$, $\|x - \alpha\| \leq \|x - \alpha\|_T$, then $d(x, \text{Ker}(T_0)) \leq d_T(x, \text{Ker}(T_0))$. Hence,

$$d(x, \text{Ker}(T_0)) \leq \gamma \|T_0 x\|. \quad (2.1)$$

Now, consider $\epsilon > 0$ and let $u \in F$. Then, there exists $x \in D(T) \cap F$ such that $u \in Tx$. By (2.1) there exists $y \in \text{Ker}(T_0) \subseteq \text{Ker}(T)$ such that

$\|x - y\| \leq (\gamma + \epsilon)d(u, T(0))$. Take $u_1 = x - y \in D(T) \cap F$. We have $u \in T(u_1)$ and

$$d(u_1, T(0) \cap \text{Ker}(T)) \leq (\gamma + \epsilon)d(u, T(0) \cap \text{Ker}(T)).$$

Continuing in the same manner, we build a sequence $(u_n)_n$ such that $u_0 = u$, for all $n \geq 0$, $u_{n+1} \in D(T) \cap F$, $u_n \in Tu_{n+1}$ and

$$d(u_n, T(0) \cap \text{Ker}(T)) \leq (\gamma + \epsilon)^n d(u, T(0) \cap \text{Ker}(T)).$$

Hence, $u \in K(T)$. Thus, $F \subseteq K(T)$.

- (v) As T is regular then, by [2, Proposition 2.5] and [1, Lemma 20], we get that $R^\infty(T)$ is closed and $T(R^\infty(T) \cap D(T)) = R^\infty(T)$. It follows from (iv) that $R^\infty(T) \subseteq K(T)$. On the other hand, it is clear that $K(T) \subseteq R^\infty(T)$. So, $K(T) = R^\infty(T)$ which is closed, as desired. ■

3. ISOLATED POINT OF THE APPROXIMATE POINT SPECTRUM

The main objective of this section is to give necessary conditions to ensure that the approximate point spectrum of a closed linear relation T does not cluster at a point λ . To do this, we begin with a generalization to the case of closed linear relations of a theorem stated in [9] dealing with the characterization of isolated points of the spectrum of a bounded closed linear relation. For this, we need the following technical lemma.

LEMMA 3.1. *Let $T \in \mathcal{CR}(X)$ and $x \in X$. Then,*

$$\begin{aligned} \tilde{R}_\cdot(T)x : \rho(T) &\longrightarrow X_T \\ \mu &\longmapsto \tilde{R}_\mu(T)x = R_\mu(T)x := (\mu I - T)^{-1}x \end{aligned}$$

is analytic.

Proof. Let $\lambda \in \rho(T)$. By virtue of [5, Corollary VI.1.9], we get that if $|\lambda - \mu| < \|R_\lambda(T)\|^{-1}$, then

$$R_\mu(T) = \sum_{n=0}^{\infty} R_\lambda(T)^{n+1}(\mu - \lambda)^n,$$

which implies that $\tilde{R}_\mu(T)x = \sum_{n=0}^{\infty} \tilde{R}_\lambda(T)^{n+1}x(\mu - \lambda)^n$. It was like proving that $\sum_{n=0}^{\infty} \tilde{R}_\lambda(T)^{n+1}x(\mu - \lambda)^n$ is convergent on X_T . Observe that

$\|\tilde{R}_\lambda(T)^{n+1}x\|_T = \|\tilde{R}_\lambda(T)^{n+1}x\| + \|T\tilde{R}_\lambda(T)^{n+1}x\|$. Moreover, we have

$$\begin{aligned} \|T\tilde{R}_\lambda(T)^{n+1}x\| &= \|Q_T T \tilde{R}_\lambda(T)^{n+1}x\| \\ &= \|Q_T(T - \lambda I + \lambda I)\tilde{R}_\lambda(T)^{n+1}x\| \\ &= \|Q_T[(T - \lambda I)\tilde{R}_\lambda(T)^{n+1}x + \lambda\tilde{R}_\lambda(T)^{n+1}x]\| \\ &= \|Q_T R_\lambda(T)^n x + \lambda Q_T R_\lambda(T)^{n+1}x\| \\ &\leq \|Q_T R_\lambda(T)^n x\| + |\lambda| \|Q_T R_\lambda(T)^{n+1}x\| \\ &\leq \|R_\lambda(T)^n x\| + |\lambda| \|R_\lambda(T)^{n+1}x\|. \end{aligned}$$

Then, $\|\tilde{R}_\lambda(T)^{n+1}x\|_T \leq (1 + |\lambda|)\|R_\lambda(T)^{n+1}x\| + \|\tilde{R}_\lambda(T)^n x\|$. Since

$$\sum_{n \geq 0} R_\lambda(T)^{n+1}x(\mu - \lambda)^n, \quad \sum_{n \geq 0} R_\lambda(T)^n x(\mu - \lambda)^n$$

are absolutely convergent in X , then $\sum_{n \geq 0} \|\tilde{R}_\lambda(T)^n x\|_T |\mu - \lambda|^n$ is convergent. Therefore, $\sum_{n \geq 0} \tilde{R}_\lambda(T)^{n+1}x(\mu - \lambda)^n$ is convergent on X_T , as required. ■

THEOREM 3.1. *Let $T \in \mathcal{CR}(X)$ and let $\lambda_0 \in \sigma(T)$. Then $K(\lambda_0 I - T)$ and $H_0(\lambda_0 I - T)$ are closed and $X = H_0(\lambda_0 I - T) \oplus K(\lambda_0 I - T)$ if and only if λ_0 is an isolated point of $\sigma(T)$.*

Proof. Recall that the quasinilpotent part of a closed linear relation is a subspace of X_T . Then, proceeding as in the proof of [9, Theorem 3.1], and taking into account Lemma 3.1, we obtain that $H_0(T) = \text{Im}(B_1)$ and $K(T) = \text{Ker}(B_1)$ where B_1 is the bounded projection defined as follows:

$$B_1 := \frac{1}{2\pi i} \oint_{\Gamma_{\lambda_0}} R_\lambda(T) d\lambda,$$

where Γ_{λ_0} is a simple closed curve around λ_0 such that the closure of the region bounded by Γ_{λ_0} and containing λ_0 intersects $\sigma(T)$ only at λ_0 . ■

Now, we develop some supplementary technical lemmas that enable us to provide some necessary conditions for which a point of the approximate point spectrum be isolated. The stated notations and terminology are essentially adhered to [8]. Using the proof of [1, Theorem 23] and Lemma 2.1 (v), we get the following lemma.

LEMMA 3.2. *Let $T \in \mathcal{R}(X)$ and $\lambda \in \mathbb{K}$. Then:*

- (i) $\gamma(\lambda I - T) \geq \gamma(T) - 3|\lambda|$.
- (ii) $K(T) \subseteq K(\lambda I - T)$ for all $0 < |\lambda| < \frac{\gamma(T)}{3}$.

The following lemma gives an analytic core stability result for a regular linear relation.

LEMMA 3.3. *Let $T \in \mathcal{R}(X)$ and $\lambda \in \mathbb{K}$. Then there exists $\nu > 0$ such that $K(T) = K(T - \lambda I)$ for all $|\lambda| < \nu$.*

Proof. Let $|\mu_0| < \frac{1}{7}\gamma(T)$. Then, by virtue of [1, Theorem 23], we get that $T - \mu_0 I \in \mathcal{R}(X)$. According to Lemma 3.2 (ii) we get that

$$K(T - \mu_0 I) \subseteq K(T - \mu_0 I - \lambda I) \text{ whenever } |\lambda| < \frac{\gamma(T - \mu_0 I)}{4}. \quad (3.1)$$

As $|\mu_0| < \frac{1}{7}\gamma(T)$ then, by Lemma 3.2 (i), we get that

$$\gamma(T - \mu_0 I) > 7|\mu_0| - 3|\mu_0| > 4|\mu_0|.$$

Hence, for $\lambda = -\mu_0$ in (3.1), we get that $K(T - \mu_0 I) \subseteq K(T)$. Using again Lemma 3.2, we get that $K(T) \subseteq K(T - \mu_0 I)$. This completes the proof. ■

Now, we recall some useful tools for the sequel.

DEFINITION 3.1. Let M and N be two closed subspaces of X . Then the gap between M and N is defined by

$$g(M, N) = \max(\delta(M, N), \delta(N, M)),$$

with $\delta(M, N) = \sup\{d(x, N) : x \in M \text{ and } \|x\| = 1\}$.

The following lemma is fundamental to the proof of the main result of this section.

LEMMA 3.4. *Let $T \in \mathcal{CR}(X)$, Ω be a connected component of $\text{reg}(T)$ and $\lambda_0 \in \text{reg}(T)$. Then:*

- (i) *The mapping $\lambda \rightarrow \text{Ker}(T - \lambda I)$ is continuous at λ_0 in the gap metric.*
- (ii) *$\text{Im}(T - \lambda_0 I)$ is closed, $\text{Im}(T - \lambda I)$ is closed in a neighborhood of λ_0 and the mapping $\lambda \rightarrow \text{Im}(T - \lambda I)$ is continuous at λ_0 in the gap metric.*

- (iii) $\bigcap_{i \geq 0} \text{Im}(T - \lambda_i I) = \bigcap_{i > 0} \text{Im}(T - \lambda_i I)$, with $(\lambda_i)_{i > 0}$ a sequence of distinct points of Ω which converges to λ_0 .
- (iv) $K(T - \lambda_0 I) = \bigcap_{i > 0} \text{Im}(T - \lambda_i I)$ with $(\lambda_i)_{i > 0}$ a sequence of distinct points of Ω which converges to λ_0 .

Proof. (i) and (ii) are proved in [3, Theorem 3.3].

- (iii) Let $0 \neq x_0 \in \bigcap_{i > 0} \text{Im}(T - \lambda_i I)$. Then, for all $i > 0$, $\frac{x_0}{\|x_0\|} \in \text{Im}(T - \lambda_i I)$. First, we note that

$$d\left(\frac{x_0}{\|x_0\|}, \text{Im}(T - \lambda_0 I)\right) = \frac{1}{\|x_0\|} d(x_0, \text{Im}(T - \lambda_0 I)).$$

From the last equality and the definitions mentioned above, we deduce that

$$\begin{aligned} g(\text{Im}(T - \lambda_i I), \text{Im}(T - \lambda_0 I)) &\geq \delta(\text{Im}(T - \lambda_i I), \text{Im}(T - \lambda_0 I)) \\ &\geq \sup\{d(x, \text{Im}(T - \lambda_0 I)) : x \in \text{Im}(T - \lambda_i I), \|x\| = 1\} \\ &\geq d(x, \text{Im}(T - \lambda_0 I)), \forall x \in \text{Im}(T - \lambda_i I), \|x\| = 1 \\ &\geq d\left(\frac{x_0}{\|x_0\|}, \text{Im}(T - \lambda_0 I)\right) \\ &\geq \frac{1}{\|x_0\|} d(x_0, \text{Im}(T - \lambda_0 I)). \end{aligned}$$

Hence,

$$d(x_0, \text{Im}(T - \lambda_0 I)) \leq g(\text{Im}(T - \lambda_i I), \text{Im}(T - \lambda_0 I)) \|x_0\|.$$

This implies, by the use of (ii), that $x_0 \in \overline{\text{Im}(T - \lambda_0 I)} = \text{Im}(T - \lambda_0 I)$. Thus, $x_0 \in \bigcap_{i \geq 0} \text{Im}(A - \lambda_i I)$, as required.

- (iv) Using Lemma 3.3 and Lemma 2.1 (v), we get that for all $i \geq 0$,

$$K(T - \lambda_0 I) = K(T - \lambda_i I) = \bigcap_{n \geq 0} \text{Im}(T - \lambda_i I)^n \subseteq \text{Im}(T - \lambda_i I).$$

Then, the lefthanded side is contained in the righthanded side. So, it is sufficient to show that $\bigcap_{i > 0} \text{Im}(T - \lambda_i I) \subseteq K(T - \lambda_0 I)$. To do this, let $x \in \bigcap_{i > 0} \text{Im}(T - \lambda_i I)$. Then, by (iii), $x \in \bigcap_{i \geq 0} \text{Im}(T - \lambda_i I)$. Whence, $x \in \text{Im}(T - \lambda_0 I)$ and so, there exists $y \in \overline{D}(T)$ such that $x \in (T - \lambda_0 I)y$. Which implies that for all $i \geq 1$, $x + (\lambda_0 - \lambda_i)y \in$

$(T - \lambda_i I)y$. On the other hand, we have for all $i \geq 1$, $x \in \text{Im}(T - \lambda_i I)$. Therefore, $(\lambda_0 - \lambda_i)y \in \bigcap_{i>0} \text{Im}(T - \lambda_i I)$. As $\lambda_i \neq \lambda_0$, then $y \in \bigcap_{i>0} \text{Im}(T - \lambda_i I)$. Consequently, for all $x \in \bigcap_{i>0} \text{Im}(T - \lambda_i I)$, there exists $y \in \bigcap_{i>0} \text{Im}(T - \lambda_i I) \cap D(T)$ such that $x \in (T - \lambda_0 I)y$. Whence, $(T - \lambda_0 I) \left(\bigcap_{i>0} \text{Im}(T - \lambda_i I) \cap D(T) \right) \supseteq \bigcap_{i>0} \text{Im}(T - \lambda_i I)$. On the other hand, it is clear to see that $(T - \lambda_0 I) \left(\bigcap_{i>0} \text{Im}(T - \lambda_i I) \cap D(T) \right) \subseteq \bigcap_{i>0} \text{Im}(T - \lambda_i I)$. Thus,

$$(T - \lambda_0 I) \left(\bigcap_{i>0} \text{Im}(T - \lambda_i I) \cap D(T) \right) = \bigcap_{i>0} \text{Im}(T - \lambda_i I).$$

Using Lemma 2.1 (iv), we get the desired inclusion. ■

With all these auxiliary results behind us, we can now state our main result of this section. We establish a number of important necessary conditions for a point in the approximate point spectrum to be isolated.

THEOREM 3.2. *Let $T \in \mathcal{CR}(X)$ and let 0 be an isolated point of $\sigma_{ap}(T)$. Then:*

- (i) $H_0(T)$ and $K(T)$ are closed.
- (ii) $H_0(T) \cap K(T) = \{0\}$.
- (iii) $H_0(T) \oplus K(T)$ is closed and there exists λ_0 such that

$$H_0(T) \oplus K(T) = K(T - \lambda_0 I) = R^\infty(T - \lambda_0 I).$$

Proof. The proof follows the approach taken in [8, Proposition 9] established in the setting of bounded operators. We divide the proof into two cases.

First case: Assume that T is surjective. Then, by Lemma 1.1, there exists $\theta > 0$ such that $T - \lambda I$ is bijective for all $\lambda \in D^*(0, \theta)$. Therefore, by virtue of Theorem 3.1, we get $H_0(T)$ and $K(T)$ are closed and $X = H_0(T) \oplus K(T)$.

Second case: Assume that T is not surjective. Since 0 is isolated in $\sigma_{ap}(T)$, then there exists $\mu > 0$ such that $T - \lambda I$ is bounded below for each $0 < |\lambda| < \mu$. Using Lemma 3.3, the map $\lambda \rightarrow K(T - \lambda I)$ is locally constant on $D^*(0, \mu)$. Which implies that the map $\lambda \rightarrow K(T - \lambda I)$ is constant on $D^*(0, \mu)$. Now, fix $\lambda_0 \in D^*(0, \mu)$. Then, by virtue of Lemma 2.1 (v), we get that

$$K(T - \lambda_0 I) = R^\infty(T - \lambda_0 I) := X_0 \text{ and it is closed.}$$

Moreover, we have $T(R^\infty(T - \lambda_0 I) \cap D(T)) = (T - \lambda_0 I + \lambda_0 I)(R^\infty(T - \lambda_0 I) \cap D(T)) \subseteq R^\infty(T - \lambda_0 I) + \lambda_0 R^\infty(T - \lambda_0 I) \subseteq R^\infty(T - \lambda_0 I)$. Let $T_0 :$

$X_0 \cap D(T) \rightarrow X_0$ be the restriction of \tilde{T} , where \tilde{T} is the relation defined by $\tilde{T} : X_T \rightarrow X, x \mapsto Tx$. Now, we divide the remaining proof into four steps.

First step: Show that $K(T) = K(T_0)$. Let $x \in K(T)$. We claim that there exist $\delta > 0$ and a sequence $(x_n)_n$ such that

- $x_0 = x$,
- $x_{n+1} \in D(T)$ and $x_n \in Tx_{n+1}$ for all $n \geq 0$,
- $\|x_n\|_T \leq \delta^n \|x\|$ for all $n \geq 1$.

Indeed, if $x \in K(T)$, then there exist $c > 0$ and a sequence $(y_n)_n$ such that

- $y_0 = x$,
- for all $n \geq 0$, $y_{n+1} \in D(T)$ and $y_n \in Ty_{n+1}$,
- $d(y_n, \text{Ker}(T) \cap T(0)) \leq c^n d(x, \text{Ker}(T) \cap T(0))$.

Let $d > c$. Then, for all $n \geq 1$ there exists $\alpha_n \in T(0) \cap \text{Ker}(T) \subseteq D(T)$ such that $\|y_n - \alpha_n\| \leq d^n \|x\|$. Let $(x_n)_n$ be the sequence defined by $x_{n+1} = y_{n+1} - \alpha_{n+1}$ for all $n \geq 0$ and $x_0 = x$. Then, for all $n \geq 0$, $x_{n+1} \in D(T)$, $x_n \in Tx_{n+1}$ and $\|x_n\| \leq d^n \|x\|$. On the other hand, we have $\|x_n\|_T = \|x_n\| + \|Q_T T x_n\| = \|x_n\| + \|Q_T(x_{n-1})\|$. Then, $\|x_n\|_T = \|x_n\| + d(x_{n-1}, T(0))$, which implies that

$$\|x_n\|_T \leq d^n \|x\| + \|x_{n-1}\| \leq (d^n + d^{n-1}) \|x\|.$$

Consequently, there exists $\delta > 0$ such that $\|x_n\|_T \leq \delta^n \|x\|$, as claimed. Let g be the analytic function $g : D(\lambda_0, \frac{1}{d}) \rightarrow X_T$ defined by

$$g(\lambda) = \sum_{n=0}^{\infty} x_{n+1} (\lambda - \lambda_0)^n.$$

Using Remark 2.1, we get for all $\lambda \in D(\lambda_0, \frac{1}{d})$,

$$Q_T T \left(\sum_{n \geq 0} (\lambda - \lambda_0)^n x_{n+1} \right) = Q_T \sum_{n \geq 0} (\lambda - \lambda_0)^n x_n.$$

Which implies that

$$T \left(\sum_{n \geq 0} (\lambda - \lambda_0)^n x_{n+1} \right) - \sum_{n \geq 0} (\lambda - \lambda_0)^n x_n \subseteq T(0).$$

Thus, $T \left(\sum_{n \geq 0} (\lambda - \lambda_0)^n x_{n+1} \right) = \sum_{n \geq 0} (\lambda - \lambda_0)^n x_n + T(0)$. Whence,

$$(T - (\lambda - \lambda_0)I) \sum_{n \geq 0} (\lambda - \lambda_0)^n x_{n+1} = \sum_{n \geq 0} (\lambda - \lambda_0)^n x_n + T(0) - \sum_{n \geq 1} x_n (\lambda - \lambda_0)^n.$$

Therefore,

$$(T - (\lambda - \lambda_0)I)g(\lambda) = x + T(0), \text{ for each } |\lambda - \lambda_0| < \frac{1}{d}. \quad (3.2)$$

Particularly, $x \in \bigcap_{\lambda \in D(\lambda_0, \frac{1}{d})} \text{Im}(T - \lambda I)$. Hence, it follows from Lemma 3.4 that $x \in K(T - \lambda_0 I)$, which means that $K(T) \subseteq X_0$. Moreover, let $\epsilon > 0$ be such that $\epsilon < \inf(\frac{1}{d}, \mu)$. Then, by (3.2), we have for each $0 < |\lambda - \lambda_0| < \epsilon$, $g(\lambda) + \text{Ker}(T - (\lambda - \lambda_0)I) = (T - (\lambda - \lambda_0)I)^{-1}x + (T - (\lambda - \lambda_0)I)^{-1}(T - (\lambda - \lambda_0)I)(0)$. Then, $g(\lambda) = (T - (\lambda - \lambda_0)I)^{-1}x \in X_0$ for each $0 < |\lambda - \lambda_0| < \epsilon$ and, by continuity, we get that $x_1 = g(\lambda_0) \in X_0$. Now, let h be the analytic function $h : D(\lambda_0, \frac{1}{d}) \rightarrow X_T$ defined by

$$h(\lambda) = \sum_{n=0}^{\infty} x_{n+2} (\lambda - \lambda_0)^n.$$

Arguing as in (3.2), we get that

$$(T - (\lambda - \lambda_0)I)h(\lambda) = x_1 + T(0), \text{ for each } |\lambda - \lambda_0| < \frac{1}{d}.$$

Moreover, there exists $\epsilon > 0$ such that for each $0 < |\lambda - \lambda_0| < \epsilon$, $h(\lambda) = (T - (\lambda - \lambda_0)I)^{-1}x_1 \in X_0$ and, by continuity, we get that $x_2 = h(\lambda_0) \in X_0$. In a similar way, we prove that $x_n \in X_0$ for all $n \geq 1$. Consequently, $x \in K(T_0)$ and hence, $K(T) \subseteq K(T_0)$. Observe that $K(T_0) \subseteq K(T)$. Then, we obtain $K(T) = K(T_0)$.

Second step: Show that $H_0(T) = H_0(T_0)$. We claim that $H_0(T) \subseteq X_0$. Indeed, according to Lemma 2.1 (ii), we get that $H_0(T) \subseteq \overline{\text{Im}(\lambda I - T)} = \text{Im}(\lambda I - T)$ for each $0 < |\lambda| < \mu$. Which implies that

$$H_0(T) \subseteq \overline{(\lambda I - T)\text{Im}(\lambda I - T)} \text{ for each } 0 < |\lambda| < \mu.$$

Therefore, $H_0(T) \subseteq \overline{\text{Im}(\lambda I - T)^2}$. As the power of a bounded below linear relation is also a bounded below linear relation then, for each $\lambda \in D^*(0, \mu)$, $\text{Im}(\lambda I - T)^2$ is closed and hence, $H_0(T) \subseteq \text{Im}(\lambda I - T)^2$ for each $\lambda \in D^*(0, \mu)$. By repeating this process we get that

$$H_0(T) \subseteq R^\infty(\lambda I - T) = X_0.$$

Therefore, it follows from Lemma 2.1 (i) that $H_0(T) = H_0(T_0)$.

Third step: Show that 0 is isolated in $\sigma(T_0)$. It is easy to see that $T_0 - \lambda I$ is injective for each $0 < |\lambda| < \mu$. Furthermore, by virtue of Lemma 2.1 (iii) we get $(T_0 - \lambda I)(X_0 \cap D(T)) = (T - \lambda I)(X_0 \cap D(T)) = (T - \lambda I)(K(T - \lambda_0 I) \cap D(T)) = (T - \lambda I)(K(T - \lambda I) \cap D(T)) = K(T - \lambda I) = K(T - \lambda_0 I) = X_0$. Then, $T_0 - \lambda I$ is surjective whenever $0 < |\lambda| < \mu$ and so, $T_0 - \lambda I$ is bijective for each $0 < |\lambda| < \mu$. Hence, 0 is isolated in $\sigma(T_0)$.

Last step: Show that $H_0(T)$ and $K(T)$ are closed and $X_0 = H_0(T) \oplus K(T)$. Using the third step and Theorem 3.1, we get that $K(T_0)$ and $H_0(T_0)$ are closed in X_0 and so in X and $X_0 = H_0(T_0) \oplus K(T_0)$. But, we have, by the first step and the second step, that $K(T) = K(T_0)$ and $H_0(T) = H_0(T_0)$ then we get the desired result. ■

Remark 3.1. At this point, a natural question arises: Are necessary conditions given in Theorem 3.2 also sufficient? The answer to this question remains open. However, it is known that, in the particular case of bounded operators, the conditions (i) and (ii) of Theorem 3.2 are not sufficient to conclude that 0 is isolated in $\sigma_{ap}(T)$ (see the remark in [8, page 4]).

Remark 3.2. It is worthy to point out that the investigation of the quasinilpotent part $H_0(T)$ and the analytic core $K(T)$ of a linear relation T is convenient in the study of the spectral properties of relations. However, it is sometimes difficult to find them explicitly. Theorem 3.2 gives an alternative way to study the properties of these two subspaces without computing them.

EXAMPLE 3.1. Let consider the separable Hilbert space $l^2(\mathbb{N})$ and let then $(e_n)_{n \geq 0}$ be her canonical basis. For $k \in \mathbb{N}^*$ fixed, we define the bounded and closed linear relation T in $l^2(\mathbb{N})$ by:

$$T((x_0, x_1, \dots)) = (x_1 + \dots + x_k, x_2 + \dots + x_k, \dots, x_k, 0, 0, 0, x_{k+1}, x_{k+2}, \dots) + \langle e_{k+1} \rangle.$$

We claim that 0 is an isolated point of $\sigma_{ap}(T)$. Indeed, if we set $N = \langle (e_n)_{n=0}^k \rangle$ and $M = \langle (e_n)_{n \geq k+1} \rangle$, then we have $M \oplus N = l^2(\mathbb{N})$ and $T = T_N \oplus T_M$, where T_N is the bounded nilpotent operator on N of degree $k+1$ represented by the matrix

$$\begin{pmatrix} 0 & 1 & \dots & 1 \\ \vdots & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

and T_M is the linear relation defined on M by $T_M = S_g^{-1}S_d$, whether S_g and S_d are the left and right shift operators on M . It is clear that T_M is a bounded below linear relation and hence, by [7, Theorem 3.10], we deduce that T is a left Drazin invertible linear relation. Therefore, it follows from [4, Theorem 4.1] that 0 is isolated in $\sigma_{ap}(T)$. Thus, by virtue of Theorem 3.2, we get that $H_0(T)$ and $K(T)$ are closed, $H_0(T) \cap K(T) = \{0\}$, $H_0(T) \oplus K(T)$ is closed and there exists λ_0 such that

$$H_0(T) \oplus K(T) = K(T - \lambda_0 I) = R^\infty(T - \lambda_0 I).$$

REFERENCES

- [1] T. ÁLVAREZ, On regular linear relations, *Acta Math. Sin. (Engl. Ser.)* **28** (1) (2012), 183–194.
- [2] T. ÁLVAREZ, M. BENHARRAT, Relationship between the Kato spectrum and the Goldberg spectrum of a linear relation, *Mediterr. J. Math.* **13** (1) (2016), 365–378.
- [3] T. ÁLVAREZ, A. SANDOVICI, Regular linear relations on Banach spaces, *Banach J. Math. Anal.* **15** (1) (2021), Paper No. 4, 26 pp.
- [4] Y. CHAMKHA, M. KAMMOUN, On perturbation of Drazin invertible linear relations, to appear in *Ukrainian Math. J.*
- [5] R. CROSS, “Multivalued linear operators”, Monographs and Textbooks in Pure and Applied Mathematics 213, Marcel Dekker, Inc., New York, 1998.
- [6] A. FAVINI, A. YAGI, Multivalued linear operators and degenerate evolution equations, *Anna. Mat. Pura Appl. (4)* **163** (1993), 353–384.
- [7] A. GHORBEL, M. MNIF, Drazin inverse of multivalued operators and its applications, *Monatsh. Math.* **189** (2019), 273–293.
- [8] M. GONZÁLEZ, M. MBEKHTA, M. OUDGHIRI, On the isolated points of the surjective spectrum of a bounded operator, *Proc. Amer. Math. Soc.* **136** (10) (2008), 3521–3528.
- [9] M. LAJNEF, M. MNIF, Isolated spectral points of a linear relation, *Monatsh. Math.* **191** (2020), 595–614.
- [10] M. LAJNEF, M. MNIF, On generalized Drazin invertible linear relations, *Rocky Mountain J. Math.* **50** (4) (2020), 1387–1408.
- [11] M. MBEKHTA, Généralisation de la décomposition de Kato aux opérateurs paranormaux et spectraux, *Glasgow Math. J.* **29** (1987), 159–175.
- [12] M. MNIF, A.A. OULED-HMED, Analytic core and quasi-nilpotent part of linear relations in Banach spaces, *Filomat* **32** (7) (2018), 2499–2515.