On self-circumferences in Minkowski planes

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Abstract: This paper contains results on self-circumferences of convex figures in the frameworks of norms and (more general) also of gauges. Let $\delta(n)$ denote the self-circumference of a regular polygon with n sides in a normed plane. We will show that $\delta(n)$ is monotonically increasing from 6 to 2π if n is twice an odd number, and monotonically decreasing from 8 to 2π if n is twice an even number. Calculations of self-circumferences for the case that n is odd as well as inequalities for the self-circumference of some irregular polygons are also given. In addition, properties of the mixed area of a plane convex body and its polar dual are used to discuss the self-circumference of convex curves.

Key words: Gauge, Minkowski geometry, normed plane, polygonal gauges, Radon curve, self-circumference, self-perimeter.

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1. Introduction

The concept of Minkowski distance defined by means of a convex body centred at the origin was developed by H. Minkowski [14]. *Minkowski spaces* are finite dimensional real Banach spaces, with the planar subcase of *normed* or *Minkowski planes*, and the geometry of such spaces and planes is usually called *Minkowski geometry* (cf. [1] and [19]). The article [13] and the whole monograph [19] contain useful background material referring to Minkowski geometry and, in particular, to those parts of the theory of convex sets which are needed.

In this article we will deal with the self-circumference (or self-perimeter) of unit circles of normed or Minkowski planes and (more general) of gauges. Related inequalities for polygons can be found in [5], [9], and [10], and further results in this direction are presented in [3], [17], and [11]. We will also use properties of mixed areas of planar convex bodies and their polars to discuss self-circumferences of some types of (also non-polygonal) convex curves, including Radon curves (see [12]).

By a *planar convex body* K we mean a compact, convex subset of the Euclidean plane having non-empty interior. We shall take as *unit circle* for the considered Minkowski plane a convex body K centered at the origin. The *Minkowskian distance* $d(x, y)$ from x to y is defined by

$$d(x, y) = \frac{d_e(x, y)}{r}\,, \tag{1}$$

where $d_e(x, y)$ is the Euclidean distance from x to y, and r is the Euclidean radius of K in the direction of the vector $y - x$. We will refer to the standard plane with this new metric as *normed* or *Minkowski plane*. The Minkowskian length of a polygonal path is obtained by adding the Minkowskian lengths of the corresponding line segments. The Minkowskian length of a curve is defined by taking the supremum over all polygonal paths inscribed to that curve. The *self-circumference* of the unit circle K is the Minkowskian length of it measured with respect to K itself. In other words, the length of the boundary of K using the metric induced by K itself is called the self-circumference of K and denoted $\delta(K)$. It was first proved in [6] that

$$6 \leq \delta(K) \leq 8\,. \tag{2}$$

Equality is attained on the left if, and only if, K is the affine image of a regular hexagon, and on the right if, and only if, K is a parallelogram. Another proof is given in [16], and Chakerian and Talley [3] established a number of properties of self-circumferences and raised interesting questions. Martini and Shcherba (cf. [9] and [10]) discussed self-perimeters of quadrangles and pentagons for the more general concept of *gauges*, where the unit circle is still a convex curve, but not necessarily centered at the origin, and (1)) holds analogously; see also [11].

2. PRELIMINARIES

Let K be a plane convex body with the origin as interior point. For each angle θ, $0 \leq \theta < 2\pi$, we let $r(K, \theta)$ be the radius of K in direction $(\cos\theta, \sin\theta)$, such that the boundary of K has the equation $r = r(K, \theta)$ in polar coordinates. The distance from the origin to the supporting line of K with outward normal $(\cos\theta, \sin\theta)$ is denoted by $h(K, \theta)$. This is the restriction of the *support function* of K to the Euclidean unit circle. Since K is convex, it has a well-defined unique tangent line at all but at most a countable number of points. We let $ds(K, \theta)$ represent the element of Euclidean arclength of the boundary

of K at a point where the unit normal is given by $(\cos\theta, \sin\theta)$. Then we have for the *length of the boundary* of K

$$L(K) = \int\limits_0^{2\pi} h(K,\theta)d\theta\,, \tag{3}$$

while the Euclidean *area* of K is given by

$$A(K) = \frac{1}{2}\int\limits_0^{2\pi} h(K,\theta)ds(K,\theta)\,. \tag{4}$$

The *polar dual* of K, denoted by K^*, is another plane convex body having the origin as interior point, and it is defined by

$$h(K^*,\theta) = \frac{1}{r(K,\theta)} \quad \text{and} \quad r(K^*,\theta) = \frac{1}{h(K,\theta)}\,. \tag{5}$$

The *mixed area* $A(K_1, K_2)$ of two convex sets K_1, K_2 is defined by

$$A(K_1,K_2) = \frac{1}{2}\int\limits_0^{2\pi} h(K_1,\theta)ds(K_2,\theta)\,. \tag{6}$$

It turns out that the mixed area is symmetric in its arguments. The following result (due to Firey [4]) will be used: The mixed area of a plane convex body and its polar dual is at least π.

The unit circle K of a Minkowski plane is referred to as its *indicatrix*. We define the *isoperimetrix* of that plane to be the convex body T such that

$$h(T,\theta) = \frac{1}{r(K,\theta + \frac{\pi}{2})}\,, \tag{7}$$

cf. [1] and [19]. The boundary of a centrally symmetric set is called a *Radon curve* if it coincides with the corresponding isoperimetrix. For further properties of Radon curves we refer to [12]. We now discuss the definition of self-circumference and give some properties.

If K is a centrally symmetric convex body centered at the origin, then by (1) and the preceding discussion, the self-circumference $\delta(K)$ is given by

$$\delta(K) = \int \frac{ds(K,\theta)}{r(K,\theta + \frac{\pi}{2})}\,. \tag{8}$$

If z is any point interior to K and K is not necessarily centered at the origin (thus really yielding a *gauge*), then the positive and the negative self-circumference (depending on orientation) of K relative to z are defined by

$$\delta_+(K, z) = \int \frac{ds(K, \theta)}{r(K, \theta + \frac{\pi}{2})} \tag{9}$$

and

$$\delta_-(K, z) = \int \frac{ds(K, \theta)}{r(K, \theta - \frac{\pi}{2})}, \tag{10}$$

where the origin of the coordinate system is at z. Both $\delta_+(K, z)$ and $\delta_-(K, z)$ reduce to $\delta(K)$ in case that K is symmetric with respect to z. For K defining a gauge, Gołab [6] conjectured that $\delta(K, z) \geq 6$, for all interior points z, and $\max \delta(K, z) \leq 9$. The latter conjecture was settled by Grünbaum [7]. And in [17] the lower bound for the general case was confirmed.

If K_1 and K_2 are plane convex bodies with the origin as an interior point, then the length of the positively oriented boundary of K_1 with respect to K_2 is given by

$$\delta_+(K_1, K_2) = \int \frac{ds(K_1, \theta)}{r(K_2, \theta + \frac{\pi}{2})}, \tag{11}$$

and the length of the negatively oriented boundary by

$$\delta_-(K_1, K_2) = \int \frac{ds(K_1, \theta)}{r(K_2, \theta - \frac{\pi}{2})}. \tag{12}$$

Schäffer [16] and independently Thompson [18] proved that for K centered at the origin $\delta_+(K) = \delta_-(K^*)$ and $\delta_-(K) = \delta_+(K^*)$ hold. More generally, Chakerian [2] used the concept of mixed areas to prove that

$$\delta_+(K_1, K_2) = \delta_-(K_2^*, K_1^*) \quad \text{and} \quad \delta_-(K_1, K_2) = \delta_+(K_2^*, K_1^*). \tag{13}$$

3. Polygons

We study first the self-circumference of regular polygons. After that we give inequalities for positive and negative self-circumferences of quadrangles. In [9] and [10] results on self-circumferences of quadrangles and pentagons were obtained, and in [5] for polygons. Theorem 1 below was also proved in [5]; for the sake of completeness we include it here.

THEOREM 1. *Let $\delta(n)$ denote the self-circumference of an affine image of a regular polygon with n sides. Then $\delta(n)$ is monotonically increasing from 6 to 2π if n is twice an odd number, and monotonically decreasing from 8 to 2π if n is twice an even number.*

The formula for $\delta(n)$ in Theorem 1 is $\delta(n) = 2n\sin\frac{\pi}{n}$ for n being a doubled odd number. Since $\sin\frac{\pi}{n}$ assures rational values only for $n = 2$ and $n = 6$ (see p. 144 in [15], Problems 197.1 and 197.5), it follows that 6 is the only rational value obtained.

In case n is twice an even number, $\delta(n)$ is given by $\delta(n) = 2n\tan\frac{\pi}{n}$. The fact that $\tan\frac{\pi}{n}$ is only rational for $n = 4$ implies that in this case the only rational value in question is $\delta(n) = 8$.

In case n is an odd number, the value $\delta(n)$ with respect to the case of having a center of symmetry is given by

$$\delta(n) = 2n\tan\frac{\pi}{n}\cos\frac{\pi}{2n}\,.$$

This sequence is monotonically decreasing from 9 to 2π. The proof that $\delta(n)$ is monotonically decreasing is not trivial. We include a proof of Theorem 2.

THEOREM 2. *The sequence $\delta(n) = 2n\tan\frac{\pi}{n}\cos\frac{\pi}{2n}$, where n is odd, is monotonically decreasing from 9 to 2π, $n = 3, 5, 7, \ldots$.*

Proof. Let $g(x) = (\tan 2x\cos x)/x$. We want to show that $g'(x) > 0$. Let $g(x) = h(x)/x$. Then $g'(x) = \big(xh'(x) - h(x)\big)/x^2$; so $g'(x) > 0$ iff $xh'(x) - h(x) > 0$ for $0 < x < \frac{\pi}{4}$. It suffices to show that $(xh'(x) - h(x))' > 0$ for $0 < x < \frac{\pi}{4}$. We have $(xh' - h)' = xh'' + h' - h' = xh'' > 0$ iff $h''(x) > 0$,

$$h'(x) = 2\sec^2 2x\cos x - \tan 2x\sin x\,,$$

$$h''(x) = \frac{8\cos x\sin 2x}{\cos^3(2x)} - \frac{2\sin x}{\cos^2(2x)} - \frac{2\sin x\cos x}{\cos 2x}$$

$$= \frac{2\sin x}{\cos^3(2x)}\big(8\cos^2 x - 2\cos^2 x + 2\sin^2 x - \cos^2 x\cos^2(2x)\big)$$

$$= \frac{2\sin x}{\cos^3(2x)}\big(8\cos^2 x - 2\cos^2 x + 2\sin^2 x - \cos^2 x\cos^2(2x)\big)\,,$$

$$h''(x) = \frac{2\sin x}{\cos^3(2x)}\big(2 + 4\cos^2 x - \cos^2 x\cos^2(2x)\big)$$

$$= \frac{2\sin x}{\cos^3(2x)}\big(2 + \cos^2 x(4 - \cos^2(2x))\big) > 0\,,$$

since $|\cos x| \le 1$, and $(2 \sin x)/\cos^3(2x) > 0$ for $0 < x < \frac{\pi}{4}$.

Thus, $h''(x) > 0$ for $0 < x < \frac{\pi}{4}$, and so $g'(x) > 0$ for $0 < x < \frac{\pi}{4}$. ∎

The following two theorems give inequalities for self-circumferences of a quadrangle and a trapezoid.

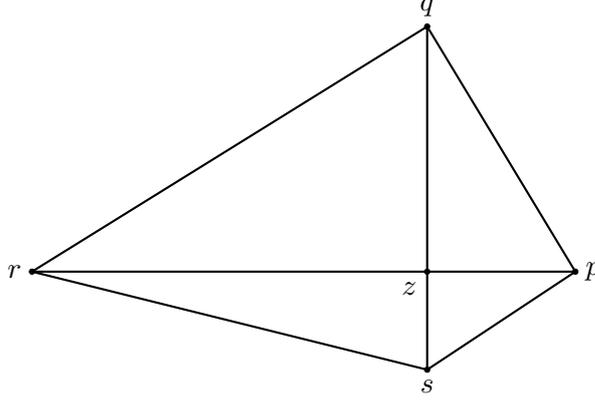Figure 1

THEOREM 3. *The self-circumference $\delta_+(P, z)$ of a convex quadrilateral P with respect to the point z of intersection of its diagonals is at least 8, with equality if and only if the quadrilateral is a parallelogram.*

Proof. Consider a quadrilateral P with vertices p, q, r, s (see Figure 1). Let z be the point of intersection of the diagonals. Then, by using similar triangles, we get

$$\delta_+(P, z) = \frac{d_e(z, p) + d_e(z, r)}{d_e(z, r)} + \frac{d_e(z, q) + d_e(z, s)}{d_e(z, s)} + \frac{d_e(z, p) + d_e(z, r)}{d_e(z, p)}$$
$$+ \frac{d_e(z, q) + d_e(z, s)}{d_e(z, s)} .$$

Thus we have

$$\delta_+(P, z) = 4 + \frac{d_e(z, p)}{d_e(z, r)} + \frac{d_e(z, r)}{d_e(z, p)} + \frac{d_e(z, q)}{d_e(z, s)} + \frac{d_e(z, s)}{d_e(z, q)} \ge 8,$$

where the last inequality follows from the arithmetic-geometric mean inequality. Equality holds if and only if $d_e(z,p) = d_e(z,r), d_e(z,q) = d_e(z,s)$, which implies that P is a parallelogram. ∎

THEOREM 4. *The self-circumference $\delta_+(P,o)$ of a trapezoid P with respect to the midpoint o of one of the diagonals is at least 8, with equality if and only if P is a parallelogram.*

Proof. Consider a trapezoid P with vertices p, q, r, s (see Figure 2). In the figure, ou is parallel to ps, ov is parallel to pq, ow is parallel to qr, and ox is parallel to rs. By using similar triangles, we get

$$\delta_+(P,o) = \frac{d_e(p,q)}{d_e(ov)} + \frac{d_e(q,r)}{d_e(ow)} + \frac{(d_e(r,s)}{d_e(ox)} + \frac{d_e(sp)}{d_e(ou)},$$

where the equalities $d_e(r,s) = 2d_e(ov), d_e(p,q) = 2d_e(o,x), d_e(q,r) = 2d_e(ow)$, and $d_e(p,s) = 2d_e(ow)$ are used. The arithmetic-geometric mean inequality implies that $\delta_+(P,o) \geq 8$, with equality if and only if $d_e(r,s) = d_e(p,q)$, yielding again a parallelogram. ∎
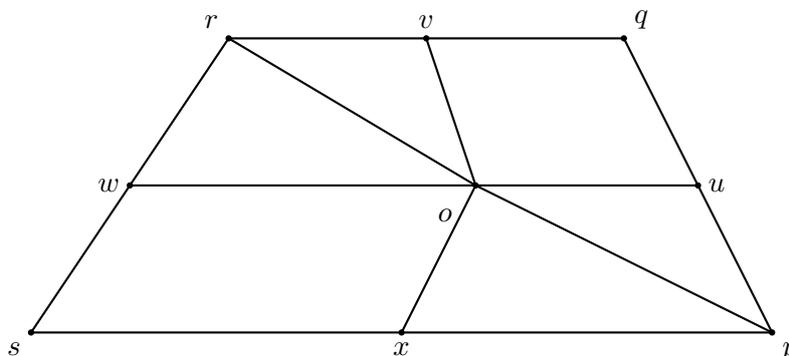

Figure 2

Chakerian and Talley [3] gave an example of a trapezoid to show that $\delta_+(K,z)$ and $\delta_-(K,z)$ do not assume their minimum at the same point, thus answering a question of Hammer posed in [8].

4. CURVES

In the following we use properties of mixed areas of a plane convex body and its polar dual to discuss self-circumferences of certain convex curves. The following theorem shows that the self-circumference of a plane convex body with four-fold symmetry is at least 2π.

THEOREM 5. *Let K be a plane convex body centered at the origin. Assume that $r(K, \theta)$ is an equation of the boundary of K in polar coordinates, and assume that $r(K, \theta) = r(K, \theta + \frac{\pi}{2})$, $0 \le \theta \le 2\pi$, i.e., K has four-fold symmetry. Then its self-circumference satisfies $\delta(K) \ge 2\pi$.*

Proof. Using the definition given in (8) and four-fold symmetry, we obtain

$$\delta(K) = \int \frac{ds(K, \theta)}{r(K, \theta + \frac{\pi}{2})} = \int \frac{ds(K, \theta)}{r(K, \theta)} \,.$$

By the property of the polar dual given in (5) and the properties of mixed areas presented in (6), it follows that

$$\delta(K) = \int \frac{ds(K, \theta)}{r(K, \theta)} = \int h(K^*, \theta) ds(K, \theta) = 2A(K^*, K) \,.$$

Firey's result from [4] states that the mixed area of a plane convex body and its polar dual is at least π. It follows that $\delta(K)$ is at least 2π. ∎

Recall that the isoperimetrix T was defined by (7). The following theorem gives the Minkowskian length of the boundary of a plane convex body with respect to the isoperimetrix.

THEOREM 6. *Let K be a plane convex body, and assume that T is the isoperimetrix, that is, the polar dual of K rotated by 90 degrees. Then $\delta_+(K, T) = 2A(K)$, where $A(K)$ is the Euclidean area of K.*

Proof. By the definition given in (11) we obtain

$$\delta_+(K, T) = \int \frac{ds(K, \theta)}{r(T, \theta + \frac{\pi}{2})} \,.$$

By the definition of the isoperimetrix, $r(T, \theta + \frac{\pi}{2}) = r(K^*, \theta)$. Thus

$$\delta_+(K, T) = \int \frac{ds(K, \theta)}{r(K^*, \theta)} = \int h(K, \theta) ds(K, \theta) = 2A(K) \,,$$

where we have used (4) and (5) giving the Euclidean area and the used property of the polar dual. ∎

If the boundary of K is a Radon curve, then it coincides with its isoperimetrix. Thus, the self-circumference of a Radon curve is equal to twice its Euclidean area. The following theorem refers to the length of the Euclidean unit circle with respect to a plane convex body K.

THEOREM 7. *Let K be a plane convex body, and assume that B is the Euclidean unit circle. Then the length of B with respect to K equals the Euclidean length of the polar dual of K. That is, $\delta_+(B, K) = L(K^*)$.*

Proof. By the result of Chakerian given in (13) we obtain

$$\delta_+(B, K) = \delta_-(K^*, B^*) = \delta_-(K^*, B).$$

Assuming that the polar dual of K is calculated at the center of the Euclidean unit circle B, it follows that $\delta_+(B, K) = L(K^*)$. ∎

In the subcase where K is a square with vertices at $(\pm 1, 0), (0, \pm 1)$, the Minkowskian distance is the same as used for the so-called Taxicab Metric. The polar dual is a square with sides parallel to the axes. Thus, the length of a Euclidean unit circle in the Taxicab Metric is the same as the Euclidean length of the circumscribed square which is 8, and thus we have finally "squared the circle".

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