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Characterizations of minimal hypersurfaces immersed in certain warped products

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Abstract: Our purpose in this paper is to investigate when a complete two-sided hypersurface immersed with constant mean curvature in a Killing warped product $M^n \times_\rho \mathbb{R}$, whose Riemannian base M^n has sectional curvature bounded from below and such that the warping function $\rho \in C^\infty(M)$ is supposed to be concave, is minimal (and, in particular, totally geodesic) in the ambient space. Our approach is based on the application of the well known generalized maximum principle of Omori-Yau.

Key words: Killing warped product, constant mean curvature hypersurfaces, minimal hypersurfaces, totally geodesic hypersurfaces.

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1. Introduction

Killing vector fields are important objects which have been widely used in order to understand the geometry of submanifolds and, more particularly, of hypersurfaces immersed in Riemannian spaces. Into this branch, Alías, Dajczer and Ripoll [1] extended classical Bernstein's theorem [4] to the context of complete minimal surfaces in Riemannian spaces of nonnegative Ricci curvature carrying a Killing vector field. This was done under the assumption that the sign of the angle function between a global Gauss mapping and the Killing vector field remains unchanged along the surface. Afterwards, Dajczer, Hinojosa and de Lira [10] defined a notion of Killing graph in a class of Riemannian manifolds endowed with a Killing vector field and solved the corresponding Dirichlet problem for prescribed mean curvature under hypothesis involving

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domain data and the Ricci curvature of the ambient space. More recently, Dajczer and de Lira [7] showed that an entire Killing graph of constant mean curvature lying inside a slab must be a totally geodesic slice, under certain restrictions on the curvature of the ambient space. To prove their Bernstein type result, they used as main tool the generalized maximum principle of Omori [11] and Yau [15] for the Laplacian in the sense of Pigola, Rigoli and Setti given in [13] (see also [2] for a modern and accessible reference to the generalized maximum principle of Omori-Yau).

When the ambient space is a Riemannian product of the type $M^n \times \mathbb{R}$, it was shown by Rosenberg, Schulze and Spruck [14] that if the Ricci curvature of the base M^n is nonnegative and its sectional curvature is bounded from below, then any entire minimal graph over M^n with nonnegative height function must be a slice. This result extends the celebrated theorem due to Bombieri, De Giorgi and Miranda [5] for entire minimal hypersurfaces in the Euclidean space. In [8], the second and third authors jointly with Parente studied complete two-sided hypersurfaces immersed in $M^n \times \mathbb{R}$, whose base is also supposed to have sectional curvature bounded from below. In this setting, they extended a technique developed in [9] obtaining sufficient conditions which assure that such a hypersurface is a slice of the ambient space, provided that its angle function has some suitable behavior. We recall that a hypersurface is said to be *two-sided* if its normal bundle is trivial, that is, there exists on it a globally defined unit normal vector field.

These aforementioned works allow us to discuss a natural question:

Question. *Under what reasonable geometric restrictions on a Riemannian manifold $\overline{M}^{n+1}$ endowed with a Killing vector field must a complete two-sided hypersurface Σ^n immersed with constant mean curvature in $\overline{M}^{n+1}$ be minimal and, in particular, totally geodesic in this ambient space?*

As is well known, under suitable assumptions on such a Killing vector field, the ambient space $\overline{M}^{n+1}$ can be regarded as a *Killing warped product* $M^n \times_\rho \mathbb{R}$, for an appropriate n-dimensional Riemannian base M^n and a certain warping function $\rho \in C^\infty(M)$. Assuming that the base M^n has sectional curvature bounded from below and supposing that the warping function ρ is concave on M^n, our purpose in this paper is just to present satisfactory answers for the above stated question. For this, in order to use the generalized maximum principle of Omori-Yau, first we establish sufficient conditions to guarantee that the Ricci curvature of a complete two-sided hypersurface is bounded from below (see Proposition 1). Afterwards, in Section 4 we state and prove our main results (see Theorems 1 and 2, and Corollaries 1 and 2). Finally, we also discuss the plausibility of the assumptions assumed in our results (see Remark 1).

2. Killing warped products

Let $\overline{M}^{n+1}$ be an $(n+1)$-dimensional Riemannian manifold endowed with a Killing vector field K. Suppose that the distribution $\mathcal{D}$ orthogonal to K is of constant rank and integrable. We denote by $\Psi : M^n \times \mathbb{I} \to \overline{M}^{n+1}$ the flow generated by K, where M^n is an arbitrarily fixed integral leaf of $\mathcal{D}$ labeled as $t = 0$, which we will suppose to be connected, and $\mathbb{I}$ is the maximal interval of definition. Without loss of generality, in what follows we will also consider $\mathbb{I} = \mathbb{R}$.

In this setting, $\overline{M}^{n+1}$ can be regard as the *Killing warped product* $M^n \times_\rho \mathbb{R}$, that is, the product manifold $M^n \times \mathbb{R}$ endowed with the warping metric

$$\langle\ ,\ \rangle = \pi_M^*\left(\langle\ ,\ \rangle_M\right) + (\rho \circ \pi_M)^2 \pi_{\mathbb{R}}^*\left(dt^2\right), \tag{2.1}$$

where π_M and $\pi_{\mathbb{R}}$ denote the canonical projections from $M \times \mathbb{R}$ onto each factor, $\langle\ ,\ \rangle_M$ is the induced Riemannian metric on the base M^n and the warping function

$$\rho \in C^\infty \quad \text{is} \quad \rho = |K| > 0,$$

where $|\cdot|$ denotes the norm of a vector field on $\overline{M}^{n+1}$.

Throughout this work, we will deal with hypersurfaces $\psi : \Sigma^n \to \overline{M}^{n+1}$ immersed in a Killing warped product $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}$ and which are *two-sided*. This condition means that there exists a globally defined unit normal vector field N on Σ^n. Let $\overline{\nabla}$, ∇ and D denote the Levi-Civita connections in $\overline{M}^{n+1}$, Σ^n and M^n, respectively. Then, as in [12], the curvature tensor R of the hypersurface Σ^n is given by

$$R(X,Y)Z = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]Z,$$

where $[\ ,\ ]$ denotes the Lie bracket and $X, Y, Z \in \mathfrak{X}(\Sigma)$. A well known fact is that the curvature tensor R of the hypersurface Σ^n can be described in terms of the shape operator A and of the curvature tensor $\overline{R}$ of the ambient space $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}$ by the Gauss equation given by

$$R(X,Y)Z = \left(\overline{R}(X,Y)Z\right)^\top + \langle AX, Z\rangle AY - \langle AY, Z\rangle AX, \tag{2.2}$$

for every tangent vector fields $X, Y, Z \in \mathfrak{X}(\Sigma)$, where $(\ )^\top$ denotes the tangential component of a vector field in $\mathfrak{X}(\overline{M})$ along Σ^n.

In this paper, we will also consider two particular smooth functions on a connected two-sided hypersurface $\psi : \Sigma^n \to \overline{M}^{n+1}$ immersed in a Killing warped product $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}$, namely, the (vertical) height function $h = \pi_{\mathbb{R}} \circ \psi$ and the angle function $\Theta = \langle N, K\rangle$, where we recall that N denotes the unit normal vector field globally defined on Σ^n.

From the decomposition $K = K^\top + \Theta N$, it is easy to see that

$$\nabla h = \frac{1}{\rho^2}K^\top \quad \text{and} \quad |\nabla h|^2 = \frac{\rho^2 - \Theta^2}{\rho^4}. \tag{2.3}$$

Moreover, assuming the constancy of the mean curvature function $H = \frac{1}{n}\mathrm{trace}(A)$, from Proposition 2.12 of [3] (see also Proposition 6 of [1] or Proposition 2.1 of [6]) we have the following formula

$$\Delta\Theta = -\left(\overline{\mathrm{Ric}}(N, N) + |A|^2\right)\Theta, \tag{2.4}$$

where $\overline{\mathrm{Ric}}$ denotes the Ricci tensor of $\overline{M}^{n+1}$ and $|A|$ stands for the Hilbert-Schmidt norm of the shape operator A of Σ^n. Finally, we also recall that it holds the following algebraic relation

$$|A|^2 = nH^2 + n(n-1)(H^2 - H_2), \tag{2.5}$$

where $H_2 = \dfrac{2}{n(n-1)}S_2$ is the mean value of the second elementary symmetric function S_2 on the eigenvalues of A.

3. Auxiliary results

In order to prove our main theorems in the next section, we will need use two auxiliary results. The first one is the well known generalized maximum principle of Omori [11] and Yau [15], which is quoted below (see also [2] for a modern and accessible reference to the generalized maximum principle of Omori-Yau).

Lemma 1. *Let Σ^n be a n-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below and let $u : \Sigma^n \to \mathbb{R}$ be a smooth function satisfying $\inf_\Sigma u > -\infty$. Then, there exists a sequence of points $\{p_k\} \subset \Sigma^n$ such that*

$$\lim_k u(p_k) = \inf_\Sigma u\,, \qquad \lim_k |\nabla u(p_k)| = 0 \qquad \textit{and} \qquad \liminf_k \Delta u(p_k) \geq 0\,.$$

The next auxiliary result will give sufficient conditions to guarantee that the Ricci curvature of a two-sided hypersurface immersed in a Killing warped product $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}$ is bounded from below. In order to prove this result, we will develop some preliminaries computations. Let us consider a two-sided hypersurface $\psi : \Sigma^n \to \overline{M}^{n+1}$ immersed in Killing warped product $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}$. For vector fields U, V, W tangent to $\overline{M}^{n+1}$, we can write

$$U = U^* + \widehat{U},$$

where U^* and $\widehat{U}$ are the orthogonal projections of U onto TM and $T\mathbb{R}$, respectively. Thus,

$$\widehat{U} = \frac{\langle U, K\rangle}{\langle K, K\rangle} K = \frac{\langle U, K\rangle}{\rho^2} K,$$

where (as in the previous section) $K = \partial_t$. Thus, with a straightforward computation it is not difficult to verify that

$$\begin{aligned}\overline{R}(U, V)W = R_M(U^*, V^*)W^* &- \frac{\langle V, K\rangle}{\rho^2}\overline{R}(K, U^*)W^* \\ &+ \frac{\langle V, K\rangle\langle W, K\rangle}{\rho^4}\overline{R}(U^*, K)K + \frac{\langle U, K\rangle}{\rho^2}\overline{R}(K, V^*)W^* \\ &- \frac{\langle U, K\rangle\langle W, K\rangle}{\rho^4}\overline{R}(V^*, K)K.\end{aligned}$$

Then, from Lemma 7.34 and Proposition 7.42 of [12] we get

$$\begin{aligned}\overline{R}(U,V)W = R_M(U^*, V^*)W^* &- \frac{\langle V, K\rangle \operatorname{Hess}_M \rho(U^*, W^*)}{\rho^3} K \\ &+ \frac{\langle V, K\rangle\langle W, K\rangle\langle K, K\rangle}{\rho^5}\overline{\nabla}_{U^*}\overline{\nabla}(\rho\circ\pi_M) + \frac{\langle U, K\rangle \operatorname{Hess}_M \rho(V^*, W^*)}{\rho^3} K \\ &- \frac{\langle U, K\rangle\langle W, K\rangle\langle K, K\rangle}{\rho^5}\overline{\nabla}_{V^*}\overline{\nabla}(\rho\circ\pi_M),\end{aligned}$$

where Hess_M is the Hessian on M^n. So, we have that

$$\begin{aligned}\overline{R}(U, V)W = R_M(U^*, V^*)W^* &- \frac{\langle V, K\rangle \operatorname{Hess}_M \rho(U^*, W^*)}{\rho^3} K \\ &+ \frac{\langle V, K\rangle\langle W, K\rangle}{\rho^3} D_{U^*} D\rho + \frac{\langle U, K\rangle \operatorname{Hess}_M \rho(V^*, W^*)}{\rho^3} K \\ &- \frac{\langle U, K\rangle\langle W, K\rangle}{\rho^3} D_{V^*} D\rho\,.\end{aligned}$$

In particular, taking a local orthonormal frame $\{E_1, \ldots, E_n\}$ tangent to Σ^n and X a vector field tangent to Σ^n, we can take $U = W = X$ and $V = E_i$ in the last equation to obtain

$$\begin{aligned}\overline{R}(X, E_i)X = R_M(X^*, E_i^*)X^* &- \frac{\langle E_i, K\rangle \operatorname{Hess}_M \rho(X^*, X^*)}{\rho^3}K \\ &+ \frac{\langle E_i, K\rangle\langle X, K\rangle}{\rho^3} D_{X^*}D\rho + \frac{\langle X, K\rangle \operatorname{Hess}_M \rho(E_i^*, X^*)}{\rho^3}K \\ &- \frac{\langle X, K\rangle^2}{\rho^3} D_{E_i^*}D\rho\,.\end{aligned}$$

Hence, we conclude that

$$\begin{aligned}\left\langle \overline{R}(X, E_i)X, E_i\right\rangle &= \langle R_M(X^*, E_i^*)X^*, E_i\rangle - \frac{\langle E_i, K\rangle^2}{\rho^3} \operatorname{Hess}_M \rho(X^*, X^*) \\ &+ \frac{\langle E_i, K\rangle\langle X, K\rangle}{\rho^3}\langle D_{X^*}D\rho, E_i\rangle - \frac{\langle X, K\rangle^2}{\rho^3}\langle D_{E_i^*}D\rho, E_i\rangle \\ &+ \frac{\langle E_i, K\rangle\langle X, K\rangle}{\rho^3} \operatorname{Hess}_M \rho(E_i^*, X^*) \\ &= \langle R_M(X^*, E_i^*)X^*, E_i^*\rangle - \frac{\langle E_i, K\rangle^2}{\rho^3} \operatorname{Hess}_M \rho(X^*, X^*) \\ &+ \frac{\langle E_i, K\rangle\langle X, K\rangle}{\rho^3} \operatorname{Hess}_M \rho(X^*, E_i^*) \\ &+ \frac{\langle E_i, K\rangle\langle X, K\rangle}{\rho^3} \operatorname{Hess}_M \rho(X^*, E_i^*) - \frac{\langle X, K\rangle^2}{\rho^3} \operatorname{Hess}_M(E_i^*, E_i^*).\end{aligned}$$

Consequently, we get

$$\begin{aligned}\left\langle \overline{R}(X, E_i)X, E_i\right\rangle &= K_M(X^*, E_i^*)\left(\langle X^*, X^*\rangle\langle E_i^*, E_i^*\rangle - \langle X^*, E_i^*\rangle^2\right) \\ &- \frac{\langle E_i, K\rangle^2}{\rho^3} \operatorname{Hess}_M \rho(X^*, X^*) - \frac{\langle X, K\rangle^2}{\rho^3} \operatorname{Hess}_M \rho(E_i^*, E_i^*) \\ &+ 2\frac{\langle E_i, K\rangle\langle X, K\rangle}{\rho^3} \operatorname{Hess}_M \rho(X^*, E_i^*) \\ &= K_M(X^*, E_i^*)\left(\langle X^*, X^*\rangle\langle E_i^*, E_i^*\rangle - \langle X^*, E_i^*\rangle^2\right) \\ &- \frac{1}{\rho} \operatorname{Hess}_M \rho\Big(\widetilde{X_i^*}, \widetilde{X_i^*}\Big) + \frac{2}{\rho} \operatorname{Hess}_M \rho\Big(\widetilde{X_i^*}, \widetilde{E_i^*}\Big) - \frac{1}{\rho} \operatorname{Hess}_M \rho\Big(\widetilde{E_i^*}, \widetilde{E_i^*}\Big),\end{aligned}$$

where $\widetilde{X_i^*} = \frac{\langle E_i, K\rangle}{\rho} X^*$ and $\widetilde{E_i^*} = \frac{\langle X, K\rangle}{\rho} E_i^*$.

Hence,

$$\begin{aligned}\big\langle \overline{R}(X,E_i)X,E_i\big\rangle &= K_M(X^*,E_i^*)\left(\langle X^*,X^*\rangle\langle E_i^*,E_i^*\rangle - \langle X^*,E_i^*\rangle^2\right)\\ &\quad - \frac{1}{\rho}\,\mathrm{Hess}_M\,\rho\Big(\widetilde{X_i^*}-\widetilde{E_i^*},\widetilde{X_i^*}-\widetilde{E_i^*}\Big).\end{aligned} \tag{3.1}$$

Therefore, we obtain that

$$\begin{aligned}\sum_{i=1}^{n}\big\langle \overline{R}(X,E_i)X,E_i\big\rangle &= \sum_{i=1}^{n} K_M(X^*,E_i^*)\left(\langle X^*,X^*\rangle\langle E_i^*,E_i^*\rangle - \langle X^*,E_i^*\rangle^2\right)\\ &\quad - \sum_{i=1}^{n}\frac{1}{\rho}\,\mathrm{Hess}_M\,\rho\Big(\widetilde{X_i^*}-\widetilde{E_i^*},\widetilde{X_i^*}-\widetilde{E_i^*}\Big).\end{aligned} \tag{3.2}$$

At this point, we recall that a *concave* function defined on a Riemannian manifold M^n is a smooth function $\rho \in C^\infty(M)$ whose Hessian operator $\mathrm{Hess}_M\rho$ is negative semidefinite. Now, we are in position to establish the following result:

PROPOSITION 1. *Let $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}$ be a Killing warped product with concave warping function ρ and whose base M^n has sectional curvature satisfying $K_M \geq -\kappa$, for some constant $\kappa \geq 0$. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a two-sided hypersurface with bounded mean curvature H and H_2 bounded from below. Then, the Ricci curvature* Ric *of Σ^n is bounded from below.*

Proof. From the Gauss equation (2.2), taking a local orthonormal frame $\{E_1,\dots,E_n\}$ tangent to Σ^n, we have that the Ricci curvature Ric of Σ^n is given by

$$\mathrm{Ric}(X,X) = \sum_{i=1}^{n}\langle \overline{R}(X,E_i)X,E_i\rangle + nH\langle AX,X\rangle - \langle AX,AX\rangle, \tag{3.3}$$

for all vector field X tangent to Σ^n. Now, we observe that, for each $i = 1,\dots,n$, (2.3) implies that

$$\begin{aligned}\langle X^*,&X^*\rangle\langle E_i^*,E_i^*\rangle\\ &= \langle X - \langle X,\nabla h\rangle K, X - \langle X,\nabla h\rangle K\rangle\,\langle E_i - \langle E_i,\nabla h\rangle K, E_i - \langle E_i,\nabla h\rangle K\rangle\\ &= |X|^2 - \rho^2|X|^2\langle E_i,\nabla h\rangle^2 - \rho^2\langle X,\nabla h\rangle^2 + \rho^4\langle X,\nabla h\rangle^2\langle E_i,\nabla h\rangle^2\end{aligned}$$

and

$$
\begin{aligned}
\langle X^*, E_i^*\rangle^2 &= \langle X - \langle X, \nabla h\rangle K, E_i - \langle E_i, \nabla h\rangle K\rangle^2 \\
&= \langle X, E_i\rangle^2 - 2\rho^2\langle X, \nabla h\rangle\langle X, E_i\rangle\langle E_i, \nabla h\rangle + \rho^4\langle X, \nabla h\rangle^2\langle E_i, \nabla h\rangle^2.
\end{aligned}
$$

Consequently, we get

$$
\begin{aligned}
&\sum_{i=1}^{n}\langle X^*, X^*\rangle\langle E_i^*, E_i^*\rangle - \langle X^*, E_i^*\rangle^2 \\
&\qquad = (n-1)|X|^2 - \rho^2|X|^2|\nabla h|^2 - (n-2)\rho^2\langle X, \nabla h\rangle^2 \le (n-1)|X|^2.
\end{aligned}
$$

Hence, taking into account our constraint on the sectional curvature of M^n, we obtain

$$
\sum_{i=1}^{n} K_M(X^*, E_i^*)\left(\langle X^*, X^*\rangle\langle E_i^*, E_i^*\rangle - \langle X^*, E_i^*\rangle^2\right) \ge -(n-1)\kappa|X|^2. \quad (3.4)
$$

On the other hand, we have that

$$
nH\langle AX, X\rangle - \langle AX, AX\rangle \ge -|A|(|nH| + |A|)|X|^2
$$

for all tangent vector field $X \in \mathfrak{X}(\Sigma)$. Since ρ is concave, it follows from (3.2), (3.3) and (3.4) that

$$
\mathrm{Ric}(X, X) \ge -\big((n-1)\kappa + |A|(|nH| + |A|)\big)|X|^2.
$$

Therefore, taking into account relation (2.5), our hypothesis on H and H_2 assure that the Ricci curvature Ric of Σ^n is bounded from below. ∎

4. Main results

In this section, we present our main results concerning the characterization of minimal (and, in particular, totally geodesic) complete two-sided hypersurfaces immersed in a Killing warped product. So, we state and prove our first one.

Theorem 1. *Let $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}$ be a Killing warped product with concave warping function ρ and whose base (not necessarily complete) M^n has nonnegative sectional curvature K_M. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete two-sided hypersurface with constant mean curvature H and H_2 bounded from below. Suppose that the angle function Θ of Σ^n is bounded away from zero. Then, Σ^n is minimal. Moreover, if H_2 is constant, then Σ^n is totally geodesic.*

Proof. Firstly, since we are assuming that Θ is bounded away from zero, for an appropriated choice of N we can suppose that $\Theta > 0$ and, consequently, $\inf_\Sigma \Theta > 0$. Then, taking into account Proposition 1, we can apply Lemma 1 to guarantee the existence of a sequence of points $\{p_k\} \subset \Sigma^n$ such that

$$\lim_k \Theta(p_k) = \inf_\Sigma \Theta \qquad \text{and} \qquad \liminf_k \Delta\Theta(p_k) \geq 0\,.$$

On the other hand, from Corollary 7.43 of [12] we get

$$\begin{aligned} \overline{\mathrm{Ric}}(N,N) &= \overline{\mathrm{Ric}}(N^*,N^*) + \overline{\mathrm{Ric}}(N^\perp,N^\perp) \\ &= \mathrm{Ric}_M(N^*,N^*) - \frac{1}{\rho}\,\mathrm{Hess}_M\,\rho(N^*,N^*) - \langle N^\perp,N^\perp\rangle \frac{\Delta_M\rho}{\rho} \\ &= \mathrm{Ric}_M(N^*,N^*) - \frac{1}{\rho}\,\mathrm{Hess}_M\,\rho(N^*,N^*) - \frac{\Theta^2}{\rho^3}\Delta_M\rho\,, \end{aligned} \tag{4.1}$$

where Hess_M and Δ_M are the Hessian and the Laplacian on M^n, respectively. Thus, from (2.4) and (4.1) we obtain the following formula

$$\Delta\Theta = -\left(\mathrm{Ric}_M(N^*,N^*) - \frac{1}{\rho}\,\mathrm{Hess}_M\,\rho(N^*,N^*) - \frac{\Theta^2}{\rho^3}\Delta_M\rho + |A|^2\right)\Theta\,. \tag{4.2}$$

Since ρ is concave, from (4.2) we have that

$$\Delta\Theta \leq -\left(\mathrm{Ric}_M(N^*,N^*) + |A|^2\right)\Theta\,. \tag{4.3}$$

So, taking into account relation (2.5) jointly with the hypothesis that H is constant and H_2 is bounded from below, it follows from (4.3) that

$$\begin{aligned} 0 \leq \liminf_k \Delta\Theta(p_k) &\leq -\lim_k \left(\mathrm{Ric}_M(N^*,N^*) + |A|^2\right)\Theta(p_k) \\ &\leq -\lim_k \left(\mathrm{Ric}_M(N^*,N^*) + nH^2\right)\Theta(p_k) \leq 0\,. \end{aligned} \tag{4.4}$$

Consequently, since Ric_M is nonnegative and $\inf_\Sigma \Theta > 0$, we conclude that $H = 0$, that is, Σ^n is minimal. Finally, assuming that H_2 is constant, from relation (2.5) we obtain that $|A|$ is also constant. Therefore, returning to (4.4) we get that $|A|$ must be identically zero and, hence, Σ^n is totally geodesic. ∎

It is not difficult to verify that from the proof of Theorem 1 we obtain the following result:

Corollary 1. *Let* $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}$ *be a Killing warped product with concave warping function* ρ *and whose base (not necessarily complete)* M^n *has nonnegative sectional curvature* K_M. *Let* $\psi : \Sigma^n \to \overline{M}^{n+1}$ *be a complete two-sided hypersurface with constant mean curvature* H *and* $H_2 \geq 0$ *(not necessarily constant). Suppose that the angle function* Θ *of* Σ^n *is bounded away from zero. Then,* Σ^n *is totally geodesic.*

Proceeding, we consider the case that the sectional curvature of the Riemannian base of the ambient space can be negative. In order to obtain our next result we need to assume a suitable constraint on the norm of gradient of the height function. More precisely, we get the following result:

Theorem 2. *Let* $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}$ *be a Killing warped product with concave warping function* ρ *and whose base (not necessarily complete)* M^n *has sectional curvature satisfying* $K_M \geq -\kappa$, *for some constant* $\kappa > 0$. *Let* $\psi : \Sigma^n \to \overline{M}^{n+1}$ *be a complete two-sided hypersurface with constant mean curvature* H *and* H_2 *bounded from below. Suppose that the angle function* Θ *of* Σ^n *is bounded away from zero. If the height function of* Σ^n *satisfies*

$$|\nabla h|^2 \leq \frac{\alpha}{(n-1)\kappa\rho^2}|A|^2, \tag{4.5}$$

for some constant $0 < \alpha < 1$, *then* Σ^n *is minimal. Moreover, if* H_2 *is constant, then* Σ^n *is a slice and* M^n *is complete.*

Proof. As in the the proof of Theorem 1, we can choose N such that $\inf_\Sigma \Theta > 0$. Then, taking into account our constraint on K_M, it follows from (4.3) that

$$\Delta\Theta \leq \left((n-1)\kappa\rho^2|\nabla h|^2 - |A|^2\right)\Theta \tag{4.6}$$

Using our hypothesis (4.5), from (4.6) we have that

$$\Delta\Theta \leq (\alpha - 1)|A|^2\Theta\,. \tag{4.7}$$

Thus, in a similar way of the proof of Theorem 1, it follows from (4.7) that there exists a sequence of points $\{p_k\} \subset \Sigma^n$ such that

$$\begin{aligned} 0 \leq \liminf_k \Delta\Theta(p_k) &\leq \lim_k \left((\alpha-1)|A|^2\Theta\right)(p_k) \\ &= (\alpha - 1)\inf_\Sigma \Theta \lim_k |A|^2(p_k) \leq 0\,. \end{aligned} \tag{4.8}$$

Hence, from (4.8) we obtain that $\lim_k |A|^2(p_k) = 0$. Consequently, since H is constant and (2.5) implies that $nH^2 \leq |A|^2$, we conclude that $H = 0$, that is, Σ^n is minimal. Assuming that H_2 is constant, using again relation (2.5) we obtain that $|A|$ must be identically zero. Therefore, using once more hypothesis (4.5) we get that Σ^n is a slice and, in particular, M^n is complete. ∎

From the proof of Theorem 2 we also get the following consequence:

Corollary 2. *Let* $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}$ *be a Killing warped product with concave warping function* ρ *and whose base (not necessarily complete)* M^n *has sectional curvature satisfying* $K_M \geq -\kappa$, *for some constant* $\kappa > 0$. *Let* $\psi : \Sigma^n \to \overline{M}^{n+1}$ *be a complete two-sided hypersurface with constant mean curvature* H *and* $H_2 \geq 0$ *(not necessarily constant). Suppose that the angle function* Θ *of* Σ^n *is bounded away from zero. If the height function of* Σ^n *satisfies condition* (4.5), *then* Σ^n *is a slice and* M^n *is complete.*

We close our paper discussing the plausibility of the assumptions assumed in Theorems 1 and 2.

Remark 1. We observe that Theorem 1 does not hold when the base of the ambient space has negative sectional curvature and that hypothesis (4.5) in Theorem 2 cannot be extended for $\alpha = 1$. Indeed, let

$$\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 \,:\, y > 0\}$$

be the 2-dimensional hyperbolic space endowed with its canonical complete metric

$$\langle\ ,\ \rangle_{\mathbb{H}^2} = \frac{1}{y^2}\big(dx^2 + dy^2\big)$$

and let $u : \mathbb{H}^2 \to \mathbb{R}$ be the smooth function given by $u(x, y) = a \ln y$, where $a \in \mathbb{R} \setminus \{0\}$. Let us consider the entire vertical graph

$$\Sigma(u) = \{(x, y, u(x, y)) \,:\, y > 0\} \subset \mathbb{H}^2 \times \mathbb{R}.$$

According to Example 10 in [8], $\Sigma(u)$ has constant mean curvature $H = \dfrac{a}{2(1 + a^2)^{1/2}}$ and $H_2 = 0$. Moreover, its angle function is given by

$$\Theta = \frac{1}{(1 + a^2)^{1/2}} > 0$$

and its height function h satisfies

$$|\nabla h|^2 = \frac{|Du|^2_{\mathbb{H}^2}}{1 + |Du|^2_{\mathbb{H}^2}} = \frac{a^2}{1 + a^2} = |A|^2.$$

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