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Some New Reverses and Refinements of Inequalities for Relative Operator Entropy

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Abstract: In this paper we obtain new inequalities for relative operator entropy $S(A|B)$ in the case of operators satisfying the condition $mA \leq B \leq MA$, with $0 < m < M$.

Key words: Inequalities for Logarithm, Relative operator entropy, Operator entropy.

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1. Introduction

Kamei and Fujii [6, 7] defined the *relative operator entropy* $S(A|B)$, for positive invertible operators A and B, by

$$S(A|B) := A^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}, \tag{1.1}$$

which is a relative version of the operator entropy considered by Nakamura-Umegaki [12].

In general, we can define for positive operators A, B

$$S(A|B) := s - \lim_{\varepsilon\to 0+} S\big(A + \varepsilon 1_H|B\big)$$

if it exists, here 1_H is the identity operator.

For the entropy function $\eta(t) = -t\ln t$, the *operator entropy* has the following expression:

$$\eta(A) = -A\ln A = S(A|1_H) \geq 0$$

for positive contraction A. This shows that the relative operator entropy (1.1) is a relative version of the operator entropy.

Following [8, pp. 149-155], we recall some important properties of relative operator entropy for A and B positive invertible operators:

(i) We have the equalities

$$S(A|B) = -A^{1/2}\Big(\ln A^{1/2}B^{-1}A^{1/2}\Big)A^{1/2} = B^{1/2}\eta\Big(B^{-1/2}AB^{-1/2}\Big)B^{1/2}. \tag{1.2}$$

(ii) We have the inequalities

$$S(A|B) \le A\big(\ln\|B\| - \ln A\big) \quad \text{and} \quad S(A|B) \le B - A\,. \tag{1.3}$$

(iii) For any C, D positive invertible operators we have that

$$S\big(A+B|C+D\big) \ge S(A|C) + S(B|D).$$

(iv) If $B \le C$ then

$$S(A|B) \le S(A|C).$$

(v) If $B_n \downarrow B$ then

$$S(A|B_n) \downarrow S(A|B).$$

(vi) For $\alpha > 0$ we have

$$S(\alpha A|\alpha B) = \alpha S(A|B).$$

(vii) For every operator T we have

$$T^*S(A|B)T \le S\big(T^*AT|T^*BT\big).$$

The relative operator entropy is *jointly concave*, namely, for any positive invertible operators A, B, C, D we have

$$S\big(tA + (1-t)B\,|\,tC + (1-t)D\big) \ge tS(A|C) + (1-t)S(B|D)$$

for any $t \in [0,1]$.

For other results on the relative operator entropy see [1, 4, 9, 10, 11, 13].

Observe that, if we replace in (1.2) B with A, then we get

$$S(B|A) = A^{1/2}\eta\big(A^{-1/2}BA^{-1/2}\big)A^{1/2} = A^{1/2}\big(-A^{-1/2}BA^{-1/2}\ln\big(A^{-1/2}BA^{-1/2}\big)\big)A^{1/2},$$

therefore we have

$$A^{1/2}\big(A^{-1/2}BA^{-1/2}\ln\big(A^{-1/2}BA^{-1/2}\big)\big)A^{1/2}=-S(B|A) \tag{1.4}$$

for positive invertible operators A and B.

It is well know that, in general $S(A|B)$ is not equal to $S(B|A)$.

In [15], A. Uhlmann has shown that the relative operator entropy $S(A|B)$ can be represented as the strong limit

$$S(A|B)=s-\lim_{t\to 0}\frac{A\sharp_t B-A}{t}\,, \tag{1.5}$$

where

$$A\sharp_\nu B:=A^{1/2}\big(A^{-1/2}BA^{-1/2}\big)^{\nu}A^{1/2},\qquad \nu\in[0,1]\,,$$

is the *weighted geometric mean* of positive invertible operators A and B. For $\nu=\frac{1}{2}$ we denote $A\sharp B$.

This definition of the weighted geometric mean can be extended for any real number ν with $\nu\neq 0$.

For $t>0$ and the positive invertible operators A, B we define the *Tsallis relative operator entropy* (see also [3]) by

$$T_t(A|B):=\frac{A\sharp_t B-A}{t}\,.$$

The following result providing upper and lower bounds for relative operator entropy in terms of $T_t(\cdot|\cdot)$ has been obtained in [6] for $0<t\leq 1$. However, it hods for any $t>0$.

Theorem 1. *Let A, B be two positive invertible operators, then for any $t>0$ we have*

$$T_t(A|B)(A\sharp_t B)^{-1}A\leq S(A|B)\leq T_t(A|B). \tag{1.6}$$

In particular, we have for $t=1$ that

$$\big(1_H-AB^{-1}\big)A\leq S(A|B)\leq B-A\,,\quad [6] \tag{1.7}$$

and for $t=2$ that

$$\frac{1}{2}\Big(1_H-\big(AB^{-1}\big)^2\Big)A\leq S(A|B)\leq\frac{1}{2}\big(BA^{-1}B-A\big). \tag{1.8}$$

The case $t = \frac{1}{2}$ is of interest as well. Since in this case we have

$$T_{1/2}(A|B) := 2(A\sharp B - A)$$

and

$$T_{1/2}(A|B)(A\sharp_{1/2}B)^{-1}A = 2\big(1_H - A(A\sharp B)^{-1}\big)A\,,$$

hence by (1.6) we get

$$2\big(1_H - A(A\sharp B)^{-1}\big)A \leq S(A|B) \leq 2(A\sharp B - A) \leq B - A\,. \tag{1.9}$$

Motivated by the above results, in this paper we obtain new inequalities for the relative operator entropy in the case of operators satisfying the condition $mA \leq B \leq MA$, with $0 < m < M$.

2. Inequalities for log-function

We have:

Theorem 2. *For any $a, b > 0$ we have the inequalities*

$$\frac{1}{2b\min\{a,b\}}(b-a)^2 \geq \ln b - \ln a - \frac{b-a}{b} \geq \frac{1}{2b\max\{a,b\}}(b-a)^2 \tag{2.1}$$

and

$$\frac{1}{2a\min\{a,b\}}(b-a)^2 \geq \frac{b-a}{a} - \ln b + \ln a \geq \frac{1}{2a\max\{a,b\}}(b-a)^2. \tag{2.2}$$

Proof. We have

$$\int_a^b \frac{b-t}{t}\,\mathrm{d}t = b\int_a^b \frac{1}{t}\,\mathrm{d}t - \int_a^b \mathrm{d}t = b(\ln b - \ln a) - (b-a)$$

giving that

$$\ln b - \ln a - \frac{b-a}{b} = \frac{1}{b}\int_a^b \frac{b-t}{t}\,\mathrm{d}t \tag{2.3}$$

for any $a, b > 0$.

Let $b > a > 0$, then

$$\frac{1}{a}\int_a^b (b-t)\,\mathrm{d}t \geq \int_a^b \frac{b-t}{t}\,\mathrm{d}t \geq \frac{1}{b}\int_a^b (b-t)\,\mathrm{d}t$$

giving that

$$\frac{1}{2a}(b-a)^2 \geq \int_a^b \frac{b-t}{t}\,\mathrm{d}t \geq \frac{1}{2b}(b-a)^2. \tag{2.4}$$

Let $a > b > 0$, then

$$\frac{1}{b}\int_b^a (t-b)\,\mathrm{d}t \geq \int_a^b \frac{b-t}{t}\,\mathrm{d}t = \int_b^a \frac{t-b}{t}\,\mathrm{d}t \geq \frac{1}{a}\int_b^a (t-b)\,\mathrm{d}t$$

giving that

$$\frac{1}{2b}(b-a)^2 \geq \int_a^b \frac{b-t}{t}\,\mathrm{d}t \geq \frac{1}{2a}(b-a)^2. \tag{2.5}$$

Therefore, by (2.4) and (2.5) we get

$$\frac{1}{2\min\{a,b\}}(b-a)^2 \geq \int_a^b \frac{b-t}{t}\,\mathrm{d}t \geq \frac{1}{2\max\{a,b\}}(b-a)^2,$$

for any $a, b > 0$.

By utilising the equality (2.3) we get the desired result (2.1). ∎

Corollary 1. *For any $y > 0$ we have*

$$\frac{1}{2y\min\{1,y\}}(y-1)^2 \geq \ln y - \frac{y-1}{y} \geq \frac{1}{2y\max\{1,y\}}(y-1)^2, \tag{2.6}$$

$$\frac{1}{2\min\{1,y\}}(y-1)^2 \geq y-1-\ln y \geq \frac{1}{2\max\{1,y\}}(y-1)^2. \tag{2.7}$$

Remark 1. Since for any $a, b > 0$ we have $\max\{a,b\}\min\{a,b\} = ab$, then (2.1) and (2.2) can also be written as

$$\begin{aligned}\frac{1}{2a}\max\{a,b\}\left(\frac{b-a}{b}\right)^2 &\geq \ln b - \ln a - \frac{b-a}{b} \\ &\geq \frac{1}{2a}\min\{a,b\}\left(\frac{b-a}{b}\right)^2\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}\frac{1}{2b}\max\{a,b\}\left(\frac{b-a}{a}\right)^2 &\geq \frac{b-a}{a} - \ln b + \ln a \\ &\geq \frac{1}{2b}\min\{a,b\}\left(\frac{b-a}{a}\right)^2\end{aligned} \tag{2.9}$$

for any $a, b > 0$.

The inequalities can also be written as

$$\frac{1}{2}\max\{1,y\}\left(\frac{y-1}{y}\right)^2 \geq \ln y - \frac{y-1}{y} \geq \frac{1}{2}\min\{1,y\}\left(\frac{y-1}{y}\right)^2 \tag{2.10}$$

and

$$\frac{1}{2y}\max\{1,y\}(y-1)^2 \geq y-1-\ln y \geq \frac{1}{2y}\min\{1,y\}(y-1)^2, \tag{2.11}$$

for any $y > 0$.

In the recent paper [2] we obtained the following inequalities that provide upper and lower bounds for the quantity $\ln b - \ln a - \frac{b-a}{b}$:

$$\frac{1}{2}\frac{(b-a)^2}{\min^2\{a,b\}} \geq \frac{b-a}{a} - \ln b + \ln a \geq \frac{1}{2}\frac{(b-a)^2}{\max^2\{a,b\}}, \tag{2.12}$$

where $a, b > 0$ and

$$\frac{(b-a)^2}{ab} \geq \frac{b-a}{a} - \ln b + \ln a \tag{2.13}$$

for any $a, b > 0$.

It is natural to ask, which of the upper bounds for the quantity

$$\frac{b-a}{a} - \ln b + \ln a$$

as provided by (2.2), (2.12) and (2.13) is better?

It has been shown in [2] that neither of the upper bounds in (2.12) and (2.13) is always best.

Consider now the difference

$$\begin{aligned} D_1(a,b) &:= \frac{1}{2a\min\{a,b\}}(b-a)^2 - \frac{1}{2}\frac{(b-a)^2}{\min^2\{a,b\}} \\ &= \frac{1}{2}\frac{(b-a)^2}{a\min^2\{a,b\}}(\min\{a,b\} - a) \leq 0\,, \end{aligned}$$

which shows that upper bound in (2.2) is always better than the upper bound in (2.12).

Consider the difference

$$\begin{aligned} D_2(a,b) &:= \frac{1}{2a\min\{a,b\}}(b-a)^2 - \frac{(b-a)^2}{ab} \\ &= \frac{1}{2ab\min\{a,b\}}(b-a)^2\big(b-2\min\{a,b\}\big), \end{aligned}$$

which can take both positive and negative values for $a, b > 0$, showing that neither of the bounds (2.2) and (2.13) is always best.

Now, consider the difference

$$\begin{aligned} d(a,b) &:= \frac{1}{2a\max\{a,b\}}(b-a)^2 - \frac{1}{2}\frac{(b-a)^2}{\max^2\{a,b\}} \\ &= \frac{1}{2a\max^2\{a,b\}}(b-a)^2(\max\{a,b\}-a) \geq 0\,, \end{aligned}$$

which shows that lower bound in (2.2) is always better than the lower bound in (2.12).

Corollary 2. *If $y \in [k,K] \subset (0,\infty)$, then we have the local inequalities*

$$\frac{1}{2\min\{1,k\}}\frac{(y-1)^2}{y} \geq \ln y - \frac{y-1}{y} \geq \frac{1}{2\max\{1,K\}}\frac{(y-1)^2}{y}\,, \tag{2.14}$$

$$\frac{1}{2\min\{1,k\}}(y-1)^2 \geq y-1-\ln y \geq \frac{1}{2\max\{1,K\}}(y-1)^2, \tag{2.15}$$

$$\frac{1}{2}\max\{1,K\}\left(\frac{y-1}{y}\right)^2 \geq \ln y - \frac{y-1}{y} \geq \frac{1}{2}\min\{1,k\}\left(\frac{y-1}{y}\right)^2, \tag{2.16}$$

$$\frac{1}{2}\max\{1,K\}\frac{(y-1)^2}{y} \geq y-1-\ln y \geq \frac{1}{2}\min\{1,k\}\frac{(y-1)^2}{y}\,. \tag{2.17}$$

Proof. If $y \in [k,K] \subset (0,\infty)$, then by analyzing all possible locations of the interval $[k,K]$ and 1 we have

$$\begin{aligned} \min\{1,k\} \leq \min\{1,y\} \leq \min\{1,K\}\,, \\ \max\{1,k\} \leq \max\{1,y\} \leq \max\{1,K\}\,. \end{aligned}$$

By using the inequalities (2.6) and (2.7) we have

$$\begin{aligned}\frac{1}{2y\min\{1,k\}}(y-1)^2 &\geq \frac{1}{2y\min\{1,y\}}(y-1)^2\\ &\geq \ln y - \frac{y-1}{y} \geq \frac{1}{2y\max\{1,y\}}(y-1)^2\\ &\geq \frac{1}{2y\max\{1,K\}}(y-1)^2\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2\min\{1,k\}}(y-1)^2 &\geq \frac{1}{2\min\{1,y\}}(y-1)^2\\ &\geq y-1-\ln y \geq \frac{1}{2\max\{1,y\}}(y-1)^2\\ &\geq \frac{1}{2\max\{1,K\}}(y-1)^2\end{aligned}$$

for any $y \in [k, K]$, that prove (2.14) and (2.15).

The inequalities (2.16) and (2.17) follows by (2.16) and (2.17). ∎

If we consider the function $f(y) = \frac{(y-1)^2}{y}$, $y > 0$, then we observe that

$$f'(y) = \frac{y^2-1}{y^2} \quad \text{and} \quad f''(y) = \frac{2}{y^3}\,,$$

which shows that f is strictly decreasing on $(0,1)$, strictly increasing on $[1,\infty)$ and strictly convex for $y > 0$. We also have $f(\frac{1}{y}) = f(y)$ for $y > 0$.

By the properties of f we then have that

$$\begin{aligned}\max_{y\in[k,K]} \frac{(y-1)^2}{y} &= \begin{cases} \frac{(k-1)^2}{k} & \text{if } K < 1\,,\\ \max\left\{\frac{(k-1)^2}{k}, \frac{(K-1)^2}{K}\right\} & \text{if } k \leq 1 \leq K\,,\\ \frac{(K-1)^2}{K} & \text{if } 1 < k\,,\end{cases}\\ &=: U(k,K)\end{aligned} \tag{2.18}$$

and

$$\min_{y\in[k,K]} \frac{(y-1)^2}{y} = \begin{cases} \frac{(1-K)^2}{K} & \text{if } K < 1\,, \\ 0 & \text{if } k \le 1 \le K\,, \\ \frac{(k-1)^2}{k} & \text{if } 1 < k\,, \end{cases} \tag{2.19}$$
$$=: u(k,K)\,.$$

We can provide now some *global bounds* as follows.

From (2.14) we then get for any $y \in [k,K]$ that

$$\frac{1}{2\min\{1,k\}} U(k,K) \ge \ln y - \frac{y-1}{y} \ge \frac{1}{2\max\{1,K\}} u(k,K), \tag{2.20}$$

while from (2.17) we get for any $y \in [k,K]$ that

$$\frac{1}{2}\max\{1,K\} U(k,K) \ge y - 1 - \ln y \ge \frac{1}{2}\min\{1,k\} u(k,K). \tag{2.21}$$

Consider

$$Z(k,K) := \max_{y\in[k,K]} (y-1)^2 \tag{2.22}$$
$$= \begin{cases} (1-k)^2 & \text{if } K < 1\,, \\ \max\{(1-k)^2, (K-1)^2\} & \text{if } k \le 1 \le K\,, \\ (K-1)^2 & \text{if } 1 < k\,, \end{cases}$$

and

$$z(k,K) := \min_{y\in[k,K]} (y-1)^2 = \begin{cases} (1-K)^2 & \text{if } K < 1\,, \\ 0 & \text{if } k \le 1 \le K\,, \\ (k-1)^2 & \text{if } 1 < k\,. \end{cases} \tag{2.23}$$

By making use of (2.15) we get

$$\frac{1}{2\min\{1,k\}} Z(k,K) \ge y - 1 - \ln y \ge \frac{1}{2\max\{1,K\}} z(k,K), \tag{2.24}$$

for any $y \in [k,K]$.

Consider the function $g(y) = \left(\frac{y-1}{y}\right)^2$, $y > 0$, then we observe that

$$g'(y) = \frac{2(y-1)}{y^2} \quad \text{and} \quad g''(y) = \frac{2(3-2y)}{y^4}\,,$$

which shows that g is strictly decreasing on $(0,1)$, strictly increasing on $[1,\infty)$ strictly convex for $y \in (0,3/2)$ and strictly concave on $(3/2,\infty)$.

Consider

$$\begin{aligned} W(k,K) :&= \max_{y\in[k,K]} \left(\frac{y-1}{y}\right)^2 \\ &= \begin{cases} \left(\frac{1-k}{k}\right)^2 & \text{if } K < 1\,, \\ \max\left\{\left(\frac{1-k}{k}\right)^2, \left(\frac{K-1}{K}\right)^2\right\} & \text{if } k \le 1 \le K\,, \\ \left(\frac{K-1}{K}\right)^2 & \text{if } 1 < k\,, \end{cases} \end{aligned} \tag{2.25}$$

and

$$w(k,K) := \min_{y\in[k,K]} \left(\frac{y-1}{y}\right)^2 = \begin{cases} \left(\frac{1-K}{K}\right)^2 & \text{if } K < 1\,, \\ 0 & \text{if } k \le 1 \le K\,, \\ \left(\frac{k-1}{k}\right)^2 & \text{if } 1 < k\,. \end{cases} \tag{2.26}$$

Then by (2.16) we get

$$\frac{1}{2}\max\{1,K\}W(k,K) \ge \ln y - \frac{y-1}{y} \ge \frac{1}{2}\min\{1,k\}w(k,K) \tag{2.27}$$

for any $y \in [k,K]$.

3. Operator inequalities

We have the following:

Lemma 1. *Let $x \in [k,K]$ and $t > 0$, then we have*

$$\begin{aligned} &\frac{1}{2\min\{1,k^t\}}\left(\frac{x^t-1}{t} - \frac{1-x^{-t}}{t}\right) \\ &\qquad \ge \ln x - \frac{1-x^{-t}}{t} \\ &\qquad \ge \frac{1}{2\max\{1,K^t\}}\left(\frac{x^t-1}{t} - \frac{1-x^{-t}}{t}\right) \ge 0 \end{aligned} \tag{3.1}$$

and

$$\frac{1}{2}\max\left\{1,K^t\right\}t\left(\frac{1-x^{-t}}{t}\right)^2 \geq \ln x - \frac{1-x^{-t}}{t} \geq \frac{1}{2}\min\left\{1,k^t\right\}t\left(\frac{1-x^{-t}}{t}\right)^2 \geq 0\,. \tag{3.2}$$

Proof. Let $y = x^t \in \left[k^t, K^t\right]$. By using the inequality (2.14) we have

$$\frac{1}{2\min\{1,k^t\}}(x^t + x^{-t} - 2) \geq t\ln x - \frac{x^t - 1}{x^t} \geq \frac{1}{2\max\{1,K^t\}}\left(x^t + x^{-t} - 2\right) \geq 0$$

that is equivalent to (3.1).

From the inequality (2.16) we have for $y = x^t$

$$\frac{1}{2}\max\left\{1,K^t\right\}\left(1 - 2x^{-t} + x^{-2t}\right) \geq t\ln x - \frac{x^t - 1}{x^t} \geq \frac{1}{2}\min\left\{1,k^t\right\}\left(1 - 2x^{-t} + x^{-2t}\right) \geq 0$$

that is equivalent to (3.2). ∎

We have:

THEOREM 3. *Let A, B be two positive invertible operators and the constants $M > m > 0$ with the property that*

$$mA \leq B \leq MA\,. \tag{3.3}$$

Then for any $t > 0$ we have

$$\frac{1}{2\min\{1,m^t\}}\,T_t(A|B)\left(A^{-1} - (A\sharp_t B)^{-1}\right)A \geq S(A|B) - T_t(A|B)(A\sharp_t B)^{-1}A \geq \frac{1}{2\max\{1,M^t\}}T_t(A|B)\left(A^{-1} - (A\sharp_t B)^{-1}\right)A \geq 0 \tag{3.4}$$

and

$$\frac{1}{2}\max\left\{1,M^{t}\right\}t\big(T_{t}(A|B)(A\sharp_{t}B)^{-1}\big)^{2}A$$
$$\geq S(A|B)-T_{t}(A|B)(A\sharp_{t}B)^{-1}A \tag{3.5}$$
$$\geq\frac{1}{2}\min\left\{1,m^{t}\right\}t\big(T_{t}(A|B)(A\sharp_{t}B)^{-1}\big)^{2}A\geq 0\,.$$

Proof. Since $mA \leq B \leq MA$ and A is invertible, then by multiplying both sides with $A^{-1/2}$ we get $m1_H \leq A^{-1/2}BA^{-1/2} \leq M$. Denote $X = A^{-1/2}BA^{-1/2}$ and by using the functional calculus for X that has its spectrum contained in the interval $[m,M]$ and the inequality (3.1), we get

$$\frac{1}{2\min\{1,m^{t}\}}\left(\frac{\big(A^{-1/2}BA^{-1/2}\big)^{t}-1_H}{t}-\frac{1_H-\big(A^{-1/2}BA^{-1/2}\big)^{-t}}{t}\right)$$
$$\geq\ln\big(A^{-1/2}BA^{-1/2}\big)-\frac{1_H-\big(A^{-1/2}BA^{-1/2}\big)^{-t}}{t} \tag{3.6}$$
$$\geq\frac{1}{2\max\{1,M^{t}\}}\left(\frac{\big(A^{-1/2}BA^{-1/2}\big)^{t}-1_H}{t}-\frac{1_H-\big(A^{-1/2}BA^{-1/2}\big)^{-t}}{t}\right)$$
$$\geq 0$$

for any $t>0$.

Now, if we multiply both sides of (3.6) by $A^{1/2}$, then we get

$$\frac{1}{2\min\{1,m^{t}\}}A^{1/2}\left(\frac{\big(A^{-1/2}BA^{-1/2}\big)^{t}-1_H}{t}-\frac{1_H-\big(A^{-1/2}BA^{-1/2}\big)^{-t}}{t}\right)A^{1/2}$$
$$\geq A^{1/2}\big(\ln\big(A^{-1/2}BA^{-1/2}\big)\big)A^{1/2}-A^{1/2}\frac{1_H-\big(A^{-1/2}BA^{-1/2}\big)^{-t}}{t}A^{1/2}$$
$$\geq\frac{1}{2\max\{1,M^{t}\}}A^{1/2}\left(\frac{\big(A^{-1/2}BA^{-1/2}\big)^{t}-1_H}{t}\right. \tag{3.7}$$
$$\left.-\frac{1_H-\big(A^{-1/2}BA^{-1/2}\big)-t}{t}\right)A^{1/2}\ \geq 0$$

for any $t>0$.

Observe that

$$A^{1/2}\ln\big(A^{-1/2}BA^{-1/2}\big)A^{1/2}=S(A|B)\,,$$

$$A^{1/2}\frac{\left(A^{-1/2}BA^{-1/2}\right)^{t}-1}{t}A^{1/2}=\frac{A\sharp_t B-A}{t}=T_t(A|B)\,,$$

$$\begin{aligned}
&A^{1/2}\frac{1_H-\left(A^{-1/2}BA^{-1/2}\right)^{-t}}{t}A^{1/2} \qquad (3.8)\\
&\quad=A^{1/2}\frac{\left(A^{-1/2}BA^{-1/2}\right)^{t}\left(A^{-1/2}BA^{-1/2}\right)^{-t}-\left(A^{-1/2}BA^{-1/2}\right)^{-t}}{t}A^{1/2}\\
&\quad=A^{1/2}\frac{\left(A^{-1/2}BA^{-1/2}\right)^{t}-1_H}{t}\left(A^{-1/2}BA^{-1/2}\right)^{-t}A^{1/2}\\
&\quad=A^{1/2}\frac{\left(A^{-1/2}BA^{-1/2}\right)^{t}-1_H}{t}A^{1/2}A^{-1/2}\left(A^{-1/2}BA^{-1/2}\right)^{-t}A^{-1/2}A\\
&\quad=T_t(A|B)(A\sharp_t B)^{-1}A
\end{aligned}$$

and then by (3.7) we get

$$\begin{aligned}
&\frac{1}{2\min\{1,m^t\}}T_t(A|B)\left(1_H-(A\sharp_t B)^{-1}A\right)\\
&\qquad\geq S(A|B)-T_t(A|B)(A\sharp_t B)^{-1}A\\
&\qquad\geq\frac{1}{2\max\{1,M^t\}}T_t(A|B)\left(1_H-(A\sharp_t B)^{-1}A\right)\;\geq\;0
\end{aligned}$$

that is equivalent to (3.4).

From the inequality (3.2) we also have

$$\begin{aligned}
&\frac{1}{2}\max\{1,M^t\}t\left(\frac{1_H-\left(A^{-1/2}BA^{-1/2}\right)^{-t}}{t}\right)^2 \qquad (3.9)\\
&\qquad\geq\ln\left(A^{-1/2}BA^{-1/2}\right)-\frac{1_H-\left(A^{-1/2}BA^{-1/2}\right)^{-t}}{t}\\
&\qquad\geq\frac{1}{2}\min\{1,m^t\}t\left(\frac{1_H-\left(A^{-1/2}BA^{-1/2}\right)^{-t}}{t}\right)^2\;\geq\;0\,.
\end{aligned}$$

Now, if we multiply both sides of (3.9) by $A^{1/2}$, then we get

$$\begin{aligned}
&\frac{1}{2}\max\{1,M^t\}tA^{1/2}\left(\frac{1_H-\left(A^{-1/2}BA^{-1/2}\right)^{-t}}{t}\right)^2A^{1/2} \qquad (3.10)\\
&\quad\geq A^{1/2}\left(\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2}-A^{1/2}\frac{1_H-\left(A^{-1/2}BA^{-1/2}\right)^{-t}}{t}A^{1/2}\\
&\quad\geq\frac{1}{2}\min\{1,m^t\}tA^{1/2}\left(\frac{1_H-\left(A^{-1/2}BA^{-1/2}\right)^{-t}}{t}\right)^2A^{1/2}\;\geq\;0\,.
\end{aligned}$$

From (3.8) we have, by multiplying both sides by $A^{-1/2}$, that

$$\frac{1_H-\left(A^{-1/2}BA^{-1/2}\right)^{-t}}{t}=A^{-1/2}T_t(A|B)(A\sharp_tB)^{-1}A^{1/2}.$$

Then

$$\begin{aligned}
&A^{1/2}\left(\frac{1_H-\left(A^{-1/2}BA^{-1/2}\right)^{-t}}{t}\right)^2A^{1/2}\\
&\quad=A^{1/2}\left(A^{-1/2}T_t(A|B)(A\sharp_tB)^{-1}A^{1/2}\right)^2A^{1/2}\\
&\quad=A^{1/2}A^{-1/2}T_t(A|B)(A\sharp_tB)^{-1}A^{1/2}A^{-1/2}T_t(A|B)(A\sharp_tB)^{-1}A^{1/2}A^{1/2}\\
&\quad=T_t(A|B)(A\sharp_tB)^{-1}T_t(A|B)(A\sharp_tB)^{-1}A\\
&\quad=(T_t(A|B)(A\sharp_tB)^{-1})^2A\,,
\end{aligned}$$

which together with (3.10) produces the desired result (3.5). ∎

There are some particular inequalities of interest as follows.
For $t=1$ we get from (3.4) and (3.5) that

$$\begin{aligned}
&\frac{1}{2\min\{1,m\}}\,(B-A)\left(A^{-1}-B^{-1}\right)A\\
&\qquad\geq S(A|B)-\left(1_H-AB^{-1}\right)A \qquad (3.11)\\
&\qquad\geq\frac{1}{2\max\{1,M\}}\,(B-A)\left(A^{-1}-B^{-1}\right)A\;\geq\;0
\end{aligned}$$

and

$$\begin{aligned} &\frac{1}{2}\max\{1,M\}\big(1_H - AB^{-1}\big)^2 A \\ &\qquad \geq S(A|B) - \big(1_H - AB^{-1}\big)A \\ &\qquad \geq \frac{1}{2}\min\{1,m\}\big(1_H - AB^{-1}\big)^2 A \ \geq\ 0\,. \end{aligned} \tag{3.12}$$

For $t = 1/2$ we get from (3.4) and (3.5) that

$$\begin{aligned} &\frac{1}{\min\{1,\sqrt{m}\}}\,(A\sharp B - A)\big(A^{-1} - (A\sharp B)^{-1}\big)A \\ &\qquad \geq S(A|B) - 2\big(1_H - A(A\sharp B)^{-1}\big)A \\ &\qquad \geq \frac{1}{\max\{1,\sqrt{M}\}}\,(A\sharp B - A)\big(A^{-1} - (A\sharp B)^{-1}\big)A \ \geq\ 0 \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} &\max\{1,\sqrt{M}\}\big(1_H - A(A\sharp B)^{-1}\big)^2 A \\ &\qquad \geq S(A|B) - 2\big(1_H - A(A\sharp B)^{-1}\big)A \\ &\qquad \geq \min\{1,\sqrt{m}\}\big(1_H - A(A\sharp B)^{-1}\big)^2 A \ \geq\ 0\,. \end{aligned} \tag{3.14}$$

For $t = 2$ we get from (3.4) and (3.5) that

$$\begin{aligned} &\frac{1}{4\min\{1,m^2\}}\big(BA^{-1}B - A\big)\big(A^{-1} - B^{-1}AB^{-1}\big)A \\ &\qquad \geq S(A|B) - \frac{1}{2}\Big(1_H - \big(AB^{-1}\big)^2\Big)A \\ &\qquad \geq \frac{1}{4\max\{1,M^2\}}\big(BA^{-1}B - A\big)\big(A^{-1} - B^{-1}AB^{-1}\big)A \ \geq\ 0 \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} &\frac{1}{4}\max\{1,M^2\}\Big(1_H - \big(AB^{-1}\big)^2\Big)^2 A \\ &\qquad \geq S(A|B) - \frac{1}{2}\Big(1_H - \big(AB^{-1}\big)^2\Big)A \\ &\qquad \geq \frac{1}{4}\min\{1,m^2\}\Big(1_H - \big(AB^{-1}\big)^2\Big)^2 A \ \geq\ 0\,. \end{aligned} \tag{3.16}$$

We have the following:

Lemma 2. *Let $x \in [m, M]$ and $t > 0$, then we have*

$$\begin{aligned}\frac{1}{2\min\{1, m^t\}}\, t\left(\frac{x^t - 1}{t}\right)^2 &\geq \frac{x^t - 1}{t} - \ln x \\ &\geq \frac{1}{2\max\{1, M^t\}}\, t\left(\frac{x^t - 1}{t}\right)^2\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}\frac{1}{2}\max\{1, M^t\}\left(\frac{x^t - 1}{t} - \frac{1 - x^{-t}}{t}\right) &\geq \frac{x^t - 1}{t} - \ln x \\ &\geq \frac{1}{2}\min\{1, m^t\}\left(\frac{x^t - 1}{t} - \frac{1 - x^{-t}}{t}\right).\end{aligned} \tag{3.18}$$

Proof. Let $y = x^t \in [m^t, M^t]$. By using the inequality (2.15) we have (3.17) and by (2.17) we have (3.18). ∎

We also have:

Theorem 4. *Let A, B be two positive invertible operators and the constants $M > m > 0$ with the property (3.3). Then for any $t > 0$ we have*

$$\begin{aligned}&\frac{1}{2\min\{1, m^t\}}\, t\, T_t(A|B)A^{-1}T_t(A|B) \\ &\qquad \geq T_t(A|B) - S(A|B) \\ &\qquad \geq \frac{1}{2\max\{1, M^t\}}\, t\, T_t(A|B)A^{-1}T_t(A|B) \geq 0\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}&\frac{1}{2}\max\{1, M^t\}T_t(A|B)\big(1_H - (A\sharp_t B)^{-1}A\big) \\ &\qquad \geq T_t(A|B) - S(A|B) \\ &\qquad \geq \frac{1}{2}\min\{1, m^t\}\, T_t(A|B)\big(1_H - (A\sharp_t B)^{-1}A\big) \geq 0\,.\end{aligned} \tag{3.20}$$

Proof. If we use the inequality (3.17) for the selfadjoint operator $X = A^{-1/2}BA^{-1/2}$ that has its spectrum contained in the interval $[m, M]$, then we get

$$\begin{aligned}\frac{1}{2\min\{1,m^t\}}\,&t\left(\frac{\left(A^{-1/2}BA^{-1/2}\right)^t-1}{t}\right)^2\\&\geq\frac{\left(A^{-1/2}BA^{-1/2}\right)^t-1}{t}-\ln\left(A^{-1/2}BA^{-1/2}\right)\\&\geq\frac{1}{2\max\{1,M^t\}}\,t\left(\frac{\left(A^{-1/2}BA^{-1/2}\right)^t-1}{t}\right)^2\geq 0\end{aligned}$$

for any $t>0$.

If we multiply both sides of this inequality by $A^{1/2}$ we get

$$\begin{aligned}&\frac{1}{2\min\{1,m^t\}}\,tA^{1/2}\left(\frac{\left(A^{-1/2}BA^{-1/2}\right)^t-1}{t}\right)^2A^{1/2}\\&\geq A^{1/2}\frac{\left(A^{-1/2}BA^{-1/2}\right)^t-1}{t}A^{1/2}-A^{1/2}\left(\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2}\\&\geq\frac{1}{2\max\{1,M^t\}}\,tA^{1/2}\left(\frac{\left(A^{-1/2}BA^{-1/2}\right)^t-1}{t}\right)^2A^{1/2}\geq 0\end{aligned}\tag{3.21}$$

for any $t>0$.

Since

$$A^{1/2}\frac{\left(A^{-1/2}BA^{-1/2}\right)^t-1}{t}A^{1/2}=T_t(A|B)\,,$$

then

$$\frac{\left(A^{-1/2}BA^{-1/2}\right)^t-1}{t}=A^{-1/2}T_t(A|B)A^{-1/2}$$

and

$$\begin{aligned}A^{1/2}&\left(\frac{\left(A^{-1/2}BA^{-1/2}\right)^t-1}{t}\right)^2A^{1/2}\\&=A^{1/2}A^{-1/2}T_t(A|B)A^{-1/2}A^{-1/2}T_t(A|B)A^{-1/2}A^{1/2}\\&=T_t(A|B)A^{-1}T_t(A|B)\end{aligned}$$

for any $t>0$.

By making use of (3.21) we then get (3.19).

By using inequality (3.18) we have

$$\frac{1}{2}\max\{1,M^t\}\left(\frac{\left(A^{-1/2}BA^{-1/2}\right)^t-1}{t}-\frac{1-\left(A^{-1/2}BA^{-1/2}\right)^{-t}}{t}\right)$$
$$\geq \frac{\left(A^{-1/2}BA^{-1/2}\right)^t-1}{t}-\ln\left(A^{-1/2}BA^{-1/2}\right)$$
$$\geq \frac{1}{2}\min\{1,m^t\}\left(\frac{\left(A^{-1/2}BA^{-1/2}\right)^t-1}{t}-\frac{1-\left(A^{-1/2}BA^{-1/2}\right)^{-t}}{t}\right)\geq 0\,,$$

for any $t>0$.

If we multiply both sides of this inequality by $A^{1/2}$ we get

$$\frac{1}{2}\max\{1,M^t\}A^{1/2}\left(\frac{\left(A^{-1/2}BA^{-1/2}\right)^t-1}{t}-\frac{1-\left(A^{-1/2}BA^{-1/2}\right)^{-t}}{t}\right)A^{1/2}$$
$$\geq A^{1/2}\frac{\left(A^{-1/2}BA^{-1/2}\right)^t-1}{t}A^{1/2}-A^{1/2}\left(\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2}$$
$$\geq \frac{1}{2}\min\{1,m^t\}A^{1/2}\left(\frac{\left(A^{-1/2}BA^{-1/2}\right)^t-1}{t}-\frac{1-\left(A^{-1/2}BA^{-1/2}\right)^{-t}}{t}\right)A^{1/2}$$
$$\geq 0$$

for any $t>0$, and the inequality (3.20) is obtained. ∎

For $t=1$ we get from (3.19) and (3.20) that

$$\frac{1}{2\min\{1,m\}}(B-A)A^{-1}(B-A)$$
$$\geq B-A-S(A|B) \tag{3.22}$$
$$\geq \frac{1}{2\max\{1,M\}}(B-A)A^{-1}(B-A)\geq 0$$

and

$$\frac{1}{2}\max\{1,M\}(B-A)\left(1_H-B^{-1}A\right)$$
$$\geq B-A-S(A|B) \tag{3.23}$$
$$\geq \frac{1}{2}\min\{1,m\}(B-A)\left(1_H-B^{-1}A\right)\geq 0\,.$$

For $t = 1/2$ we get from (3.19) and (3.20) that

$$\begin{aligned} &\frac{1}{\min\{1,\sqrt{m}\}}\,(A\sharp B - A)A^{-1}(A\sharp B - A) \\ &\qquad \geq 2(A\sharp B - A) - S(A|B) \qquad (3.24)\\ &\qquad \geq \frac{1}{\max\{1,\sqrt{M}\}}(A\sharp B - A)A^{-1}(A\sharp B - A) \geq 0 \end{aligned}$$

and

$$\begin{aligned} &\max\left\{1,\sqrt{M}\right\}(A\sharp B - A)\big(1_H - (A\sharp B)^{-1}A\big) \\ &\qquad \geq 2(A\sharp B - A) - S(A|B) \qquad (3.25)\\ &\qquad \geq \min\left\{1,\sqrt{m}\right\}(A\sharp B - A)\big(1_H - (A\sharp B)^{-1}A\big) \geq 0\,. \end{aligned}$$

For $t = 2$ we get from (3.19) and (3.20) that

$$\begin{aligned} &\frac{1}{4\min\{1,m^2\}}\,\big(BA^{-1}B - A\big)A^{-1}\big(BA^{-1}B - A\big) \\ &\qquad \geq \frac{1}{2}\big(BA^{-1}B - A\big) - S(A|B) \qquad (3.26)\\ &\qquad \geq \frac{1}{4\max\{1,M^2\}}\,\big(BA^{-1}B - A\big)A^{-1}\big(BA^{-1}B - A\big) \geq 0 \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{4}\max\left\{1,M^2\right\}\big(BA^{-1}B - A\big)\Big(1_H - \big(B^{-1}A\big)^2\Big) \\ &\qquad \geq \frac{1}{2}\big(BA^{-1}B - A\big) - S(A|B) \qquad (3.27)\\ &\qquad \geq \frac{1}{4}\min\left\{1,m^2\right\}\big(BA^{-1}B - A\big)\Big(1_H - \big(B^{-1}A\big)^2\Big) \geq 0\,. \end{aligned}$$

4. Some global bounds

For $[m, M] \subset (0,\infty)$ and $t > 0$ and by the use of (2.18) we define

$$\begin{aligned} U_t(m,M) :&= U\big(m^t, M^t\big) \qquad (4.1)\\ &= \begin{cases} \frac{(m^t-1)^2}{m^t} & \text{if } M < 1\,,\\ \max\left\{\frac{(m^t-1)^2}{m^t}, \frac{(M^t-1)^2}{M^t}\right\} & \text{if } m \leq 1 \leq M\,,\\ \frac{(M^t-1)^2}{M^t} & \text{if } 1 < m\,, \end{cases} \end{aligned}$$

and by (2.19)

$$u_t(m,M) := u\big(m^t, M^t\big) = \begin{cases} \frac{(1-M^t)^2}{M^t} & \text{if } M < 1\,, \\ 0 & \text{if } m \le 1 \le M\,, \\ \frac{(m^t-1)^2}{m^t} & \text{if } 1 < m\,. \end{cases} \tag{4.2}$$

By (2.20) and (2.21) we have for $y = x^t \in \big[m^t, M^t\big]$ and $t > 0$ that

$$\begin{aligned} \frac{1}{2t\min\{1,m^t\}}\, U_t(m,M) &\ge \ln x - \frac{1-x^{-t}}{t} \\ &\ge \frac{1}{2t\max\{1,M^t\}} u_t(m,M) \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} \frac{1}{2t}\max\big\{1,M^t\big\} U_t(m,M) &\ge \frac{x^t-1}{t} - \ln x \\ &\ge \frac{1}{2t}\min\{1,m^t\} u_t(m,M)\,, \end{aligned} \tag{4.4}$$

where $x \in [m, M]$ and $t > 0$.

Using (2.22) and (2.23) we define

$$\begin{aligned} Z_t(m,M) :&= Z(m^t, M^t) \\ &= \begin{cases} \big(1-m^t\big)^2 & \text{if } M < 1\,, \\ \max\Big\{\big(1-m^t\big)^2, \big(M^t-1\big)^2\Big\} & \text{if } m \le 1 \le M\,, \\ \big(M^t-1\big)^2 & \text{if } 1 < m\,, \end{cases} \end{aligned} \tag{4.5}$$

and

$$z_t(m,M) := z\big(m^t, M^t\big) = \begin{cases} (1-M^t)^2 & \text{if } M < 1\,, \\ 0 & \text{if } m \le 1 \le M\,, \\ \big(m^t-1\big)^2 & \text{if } 1 < m\,. \end{cases} \tag{4.6}$$

By (2.24) we have for $y = x^t \in \big[m^t, M^t\big]$ and $t > 0$ that

$$\begin{aligned} \frac{1}{2t\min\{1,m^t\}}\, Z_t(m,M) &\ge \frac{x^t-1}{t} - \ln x \\ &\ge \frac{1}{2t\max\{1,M^t\}} z_t(m,M)\,, \end{aligned} \tag{4.7}$$

where $x \in [m, M]$ and $t > 0$.

Utilising (2.25) and (2.26) we can define

$$
\begin{aligned}
W_t(m, M) : &= W\big(m^t, M^t\big) \qquad (4.8)\\
&= \begin{cases} \big(\frac{1-m^t}{m^t}\big)^2 & \text{if } M < 1\,,\\ \max\Big\{\big(\frac{1-m^t}{m^t}\big)^2, \big(\frac{M^t-1}{M^t}\big)^2\Big\} & \text{if } m \le 1 \le M\,,\\ \big(\frac{M^t-1}{M^t}\big)^2 & \text{if } 1 < m\,,\end{cases}
\end{aligned}
$$

and

$$
w_t(m, M) := W(m^t, M^t) = \begin{cases} \big(\frac{1-M^t}{M^t}\big)^2 & \text{if } M < 1\,,\\ 0 & \text{if } m \le 1 \le M\,,\\ \big(\frac{m^t-1}{m^t}\big)^2 & \text{if } 1 < m\,.\end{cases} \qquad (4.9)
$$

By (2.24) we have for $y = x^t \in \big[m^t, M^t\big]$ and $t > 0$ that

$$
\begin{aligned}
\frac{1}{2t}\max\big\{1, M^t\big\}W_t(m, M) &\ge \ln x - \frac{1 - x^{-t}}{t}\\
&\ge \frac{1}{2t}\min\big\{1, m^t\big\}w_t(m, M)\,,
\end{aligned} \qquad (4.10)
$$

where $x \in [m, M]$ and $t > 0$.

THEOREM 5. *Let A, B be two positive invertible operators and the constants $M > m > 0$ with the property (3.3). Then for any $t > 0$ we have*

$$
\begin{aligned}
\frac{1}{2t\min\{1, m^t\}}\,U_t(m, M)A &\ge S(A|B) - T_t(A|B)(A\sharp_t B)^{-1}A\\
&\ge \frac{1}{2t\max\{1, M^t\}}\,u_t(m, M)A\,,
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2t}\max\big\{1, M^t\big\}W_t(m, M)A &\ge S(A|B) - T_t(A|B)(A\sharp_t B)^{-1}A\\
&\ge \frac{1}{2t}\min\big\{1, m^t\big\}w_t(m, M)A\,,
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2t\min\{1, m^t\}}\,Z_t(m, M)A &\ge T_t(A|B) - S(A|B)\\
&\ge \frac{1}{2t\max\{1, M^t\}}\,z_t(m, M)A
\end{aligned}
$$

and

$$\frac{1}{2t}\max\left\{1,M^t\right\}U_t(m,M)A \ge T_t(A|B)-S(A|B) \ge \frac{1}{2t}\min\left\{1,m^t\right\}u_t(m,M)A\,.$$

The proof follows by the inequalities (4.4), (4.5), (4.7) and (4.10) in a similar way as the one from the proof of Theorem 3 and we omit the details.

For $t=1$, $t=1/2$ and $t=2$ one can obtain some particular inequalities of interest, however the details are not provided here.

References

[1] S.S. Dragomir, Some inequalities for relative operator entropy; preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. 145. [http://rgmia.org/papers/v18/v18a145.pdf].

[2] S.S. Dragomir, Reverses and refinements of several inequalities for relative operator entropy; preprint *RGMIA Res. Rep. Coll.* **19** (2015). [http://rgmia.org/papers/v19/].

[3] S. Furuichi, K. Yanagi, K. Kuriyama, Fundamental properties of Tsallis relative entropy, *J. Math. Phys.* **45** (2004) 4868 – 4877.

[4] S. Furuichi, Precise estimates of bounds on relative operator entropies, *Math. Inequal. Appl.* **18** (2015), 869 – 877.

[5] S. Furuichi, N. Minculete, Alternative reverse inequalities for Young's inequality, *J. Math. Inequal.* **5** (2011), Number 4, 595 – 600.

[6] J.I. Fujii, E. Kamei, Uhlmann's interpolational method for operator means, *Math. Japon.* **34** (4) (1989), 541 – 547.

[7] J.I. Fujii, E. Kamei, Relative operator entropy in noncommutative information theory, *Math. Japon.* **34** (3) (1989), 341 – 348.

[8] T. Furuta, J. Mićić Hot, J. Pečarić, Y. Seo, "Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space", Monographs in Inequalities 1, ELEMENT, Zagreb, 2005.

[9] I.H. Kim, Operator extension of strong subadditivity of entropy, *J. Math. Phys.* **53** (2012), 122204, 3 pp.

[10] P. Kluza, M. Niezgoda, Inequalities for relative operator entropies, *Electron. J. Linear Algebra* **27** (2014), 851 – 864.

[11] M.S. Moslehian, F. Mirzapour, A. Morassaei, Operator entropy inequalities, *Colloq. Math.* **130** (2013), 159 – 168.

[12] M. Nakamura, H. Umegaki, A note on the entropy for operator algebras, *Proc. Japan Acad.* **37** (1961) 149 – 154.

[13] I. NIKOUFAR, On operator inequalities of some relative operator entropies, *Adv. Math.* **259** (2014), 376 – 383.

[14] W. SPECHT, Zur Theorie der elementaren Mittel, *Math. Z.* **74** (1960), 91 – 98.

[15] A. UHLMANN, Relative entropy and the Wigner-Yanase-Dyson-Lieb concavity in an interpolation theory, *Comm. Math. Phys.* **54** (1) (1977), 21 – 32.