Revista Colombiana de Estadística
No.29 y No. 30

ON A VACATION QUEUE WITH TWO PARALLEL SERVERS EACH EQUIPPED WITH A STAND-BY

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ABSTRACT. The paper studies an $M/M/2$ queueing system in which each server is equipped with a stand-by. The stand-by are employed only during the vacation periods of main severs. The vacation periods of the two severs are assumed to be independent with an identical exponential distribution. The time dependent results giving probability generating functions for the number in the system under various states have been obtained and in a particular case a known result has been derived at. The corresponding steady state results are also derived.

1. INTRODUCTION

Madan [4,5] has studied some queueing systems with server vacations. These vacations of the human servers or analogous breakdowns of the mechanical devices are a common phenomenon and have a definite effect on the efficiency of a system and on the customer's waiting time in the queue. Among the several important contributions on this aspect of queues, a few to be mentioned are those by Scholl and Kleinrock [6], Keilson and Servi [3], Doshi [1], Shanthikumar [7] and Gaver [2]. In all these papers, the system ceases to work as soon as vacation starts. In this paper, we have introduced the idea of a stand-by which operates only during the vacation periods of the main server. The commissioning of a stand-by is common and worthwhile in many

1980 *Mathematics Subject Classification* (1985 *Revision*). AMS Primary Classification : 60k, 90 B.

Key words and phrases. Poisson arrivals, exponential service, vacation periods,stand-by, probability generating funtion, Laplace transform, steady state, mean number in the system..

real life situations. Briefly, the mathematical model is described by the following assumptions:

2. The Underlying Assumptions

(i) Arrivals occur one by one in a Poisson stream with mean arrival rate $\lambda(>0)$.

(ii) There are two parallel servers which provide identical service to the incoming units one by one in the order of their arrival. The service times at each channel are assumed to be exponential with identical mean service time $\mu^{-1}(\mu > 0)$.

(iii) Both servers are subject to a random call for vacation. As soon as a server leaves for vacation, a stand-by takes over and starts operating. And as soon as the vacation of a server terminates, it instantly replaces the stand- by and inmediately starts providing service.

(iv) The service times of each stand-by are exponential with identical mean service time $\nu^{-1}.(\nu < \mu)$.

(v) Let α dt be the first order probability that a server will stop working during the interval $(t, t+dt)$ and a vacation starts. Further, we assume that the vacation periods of a server follow an exponential distribution with mean vacation time $1/\beta.(\beta > 0)$. In this case, it follows that βdt is the first order probability that a vacation will terminate durig the time interval $(t, t+dt)$.

(vi) Commencement or termination of vacations of the two servers are independent of each other also independent of all other stochastic processes involved in the system.

3. Definitions And Time - Dependent Equations

Let $P(j, n, t)$ denote the probability at time t, j servers $(j = 0, 1, 2)$ are on vacation and there are $n(\geq 0)$ customers in the system (including those being served, if any). We note that whenever any server is away on vacation, it is implied that a stand-by is available in its place and is providing service if there is a customer in the system. Clearly, when j servers are away on vacation, $2-j$ servers are available and further, the availability of a server means it is providing service, if there is any customer in the system or else, it is idle. Let $P(n,t)$ be the probability that at time t there are n customers in the system irrespective of the number of server on vacation so that $P(n,t) = \sum_{j=0}^{2} P(j,n,t)$

Initial probability arguments based on the system's underlying assumptions lead to the following set of forward difference-differential equations for the system :

$$(1) \qquad P'(0,n,t)+(\lambda+2\mu+2\alpha)P(0,n,t) = \lambda P(0,n-1,t)+2\mu P(0,n+1,t)+\beta P(1,n,t) \ (n \geq 2)$$

$$(2) \qquad P'(0,1,t)+(\lambda+\mu+2\alpha)P(0,1,t) = \lambda P(0,0,t)+2\mu P(0,2,t)+\beta P(1,1,t)$$

$$(3) \qquad P'(0,0,t)+(\lambda+2\alpha)P(0,0,t) = \mu P(0,1,t)+\beta P(1,0,t)$$

$$(4) \qquad P'(1,n,t)+(\lambda+\mu+\nu+\alpha+\beta)P(1,n,t) = \lambda P(1,n-1,t)+(\mu+\nu)P(1,n+1,t)+2\alpha P(0,n,t) + 2\beta P(2,n,t) \ (n \geq 2)$$

$$(5) \qquad P'(1,1,t)+(\lambda+\mu+\alpha+\beta)P(1,1,t) = \lambda P(1,0,t)+(\mu+\nu)P(1,2,t)+2\alpha P(0,1,t)+2\beta P(2,1,t)$$

$$(6) \qquad P'(1,0,t) + (\lambda + \alpha + \beta)P(1,0,t)$$

$$= \mu P(1,1,t) + 2\alpha P(0,0,t) + 2\beta P(2,0,t)$$

$$(7) \qquad P'(2,n,t) + (\lambda + 2\nu + 2\beta)P(2,n,t)$$

$$= \lambda P(2,n-1,t) + 2\nu P(2,n+1,t) + \alpha P(1,n,t)\,(n \geq 2)$$

$$(8) \qquad P'(2,1,t) + (\lambda + \nu + 2\beta)P(2,1,t)$$

$$= \lambda P(2,0,t) + 2\nu P(2,2,t) + \alpha P(1,1,t)$$

$$(9) \qquad P'(2,0,t) + (\lambda + 2\beta)P(2,0,t)$$

$$= \nu P(2,1,t) + \alpha P(1,0,t)$$

We assume that initially there are i customers in the system and both servers are available $(j = 0)$ so that the initial condition is

$$(10) \qquad P(j,i,0) = (\delta_{n,i})(\delta_{0,j}) \qquad j = 0,1,2$$

where $\delta_{n,i}$ and $\delta_{0,j}$ are the Kronecker's deltas.

4. The Time-Dependent Solution

Let $\overline{P}(j,n,s)$ denote the Laplace transform (L.T.) of $P(j,n,t)$ for $j = 0,1,2$ and $n(\geq 0)$. Then taking L.T. of equations (1) to (9) and using (10), we have

$$(11) \qquad (s + \lambda + 2\mu + 2\alpha)\overline{P}(0,n,s)$$

$$= \lambda\overline{P}(0,n-1,s) + 2\mu\overline{P}(0,n+1,s) + \beta\overline{P}(1,1,s)\ (n \geq 2)$$

$$(12)\qquad (s+\lambda+\mu+2\alpha)\overline{P}(0,1,s)$$
$$=\lambda\overline{P}(0,0,s)+2\mu\,\overline{P}(0,2,s)+\beta\overline{P}(1,1,s)$$

$$(13)\qquad (s+\lambda+2\alpha)\overline{P}(0,0,s)$$
$$=\mu\overline{P}(0,1,s)+\beta\overline{P}(1,0,s)$$

$$(14)\qquad (s+\lambda+\mu+\nu+\alpha+\beta)\overline{P}(1,n,s)$$
$$=\lambda\overline{P}(1,n-1,s)+(\mu+\nu)\overline{P}(1,n+1,s)+2\alpha\overline{P}(0,n,s)$$
$$+2\beta\overline{P}(2,n,s)\quad (n\geq 2)$$

$$(15)\qquad (s+\lambda+\mu+\alpha+\beta)\overline{P}(1,1,s)$$
$$=\lambda\overline{P}(1,0,s)+(\mu+\nu)\overline{P}(1,2,s)+2\alpha\overline{P}(0,1,s)+2\beta\overline{P}(2,1,s)$$

$$(16)\qquad (s+\lambda+\alpha+\beta)\,\overline{P}(1,0,s)$$
$$=\mu\overline{P}(1,1,s)+2\alpha\overline{P}(0,0,s)+2\beta\overline{P}(2,0,s)$$

$$(17)\qquad (s+\lambda+2\nu+2\beta)\overline{P}(2,n,s)$$
$$=\lambda\overline{P}(2,n-1,s)+2\nu\overline{P}(2,n+1,s)+\alpha\overline{P}(1,n,s)\quad (n\geq 2)$$

$$(18)\qquad (s+\lambda+\nu+2\beta)\overline{P}(2,1,s)$$
$$=\lambda\overline{P}(2,0,s)+2\nu\overline{P}(2,2,s)+\alpha\overline{P}(1,1,s)$$

$$(19)\qquad (s+\lambda+2\beta)\overline{P}(2,0,s)$$
$$=\nu\overline{P}(2,1,s)+\alpha\overline{P}(1,0,s)$$

Define the probability generating functions :

$$(20)\qquad \overline{P}(j,z,s)=\sum_{n=1}^{\infty}\overline{P}(j,n,s)z^n;\quad \overline{P}(z,s)=\sum_{j=0}^{2}\overline{P}(j,z,s),\quad j=0,1,2\quad |z|\leq 1$$

Multiplying equations (11) to (19) by suitable powers of z and making use of (20), we get

$$(21)\qquad \left[(s+\lambda(1-z)+2\mu+2\alpha)z-2\mu\right]\overline{P}(0,z,s)$$
$$= z^{i+1}+2\mu(z-1)\overline{P}(0,0,s)+\mu z(z-1)\overline{P}(0,1,s)+\beta z\overline{P}(1,z,s)$$

$$(22)\qquad \left[(s+\lambda(1-z)+\mu+\nu+\alpha+\beta)z-(\mu+\nu)\right]\overline{P}(1,z,s)$$
$$= (\mu+\nu)(z-1)\overline{P}(1,0,s)+\nu z(z-1)\overline{P}(1,1,s)$$
$$+\,2\alpha z\overline{P}(0,z,s)+2\beta z\overline{P}(2,z,s)$$

$$(23)\qquad \left[(s+\lambda(1-z)+2\nu+2\beta)z-2\nu\right])\overline{P}(2,z,s)$$
$$= 2\nu(z-1)\overline{P}(2,0,s)+\nu z(z-1)\overline{P}(2,1,s)+\alpha z\overline{P}(1,z,s)$$

Solving equations (21), (22) and (23) simultaneously we obtain

$$(24)\qquad \overline{P}(j,z,s)=\frac{\overline{N}_j(z,s)}{\overline{D}(z,s)}\qquad (j=0,1,2)$$

where

$$(25)\qquad \overline{N}_0(z,s)=\left\{\left[(s+\lambda(1-z)+\mu+\nu+\alpha+\beta)z-(\mu+\nu)\right]\right.$$
$$\times\left[(s+\lambda(1-z)+2\nu+2\beta)z-2\nu\right]-2\alpha\beta z^2\right\}$$
$$\times\left\{z^{i+1}+2\mu(z-1)\overline{P}(0,0,s)+\mu z(z-1)\overline{P}(0,1,s)\right\}$$
$$+\,\beta z\left[(s+\lambda(1-z)+2\nu+2\beta)z-2\nu\right]$$
$$\times\left[(\mu+\nu)(z-1)\overline{P}(1,0,s)+\nu z(z-1)\overline{P}(1,1.s)\right]$$
$$+\,4\beta^2\nu(z-1)z^2\overline{P}(0,0.s)$$

$$
\begin{aligned}
(26) \qquad \overline{N}_1(z,s) =& 2\alpha z\big[(s+\lambda(1-z)+2\nu+2\beta)z-2\nu\big] \\
&\times \big[z^{i+1}+2\mu(z-1)\overline{P}(0,0,s)+\mu z(z-1)\overline{P}(0,1,s)\big] \\
&+ \big[(s+\lambda(1-z)+2\mu+2\alpha)z-2\mu\big] \\
&\times \big[(s+\lambda(1-z)+2\nu+2\beta)z-2\nu\big] \\
&\times \big[(\mu+\nu)(z-1)\overline{P}(1,0,s)+\nu z(z-1)\overline{P}(1,1,s)\big] \\
&+ 2\beta z\big[(s+\lambda(1-z)+2\mu+2\alpha)z-2\mu\big] \\
&\times \big[2\nu(z-1)\overline{P}(2,0,s)+\nu z(z-1)\overline{P}(2,1,s)\big]
\end{aligned}
$$

$$
\begin{aligned}
(27) \qquad \overline{N}_2(z,s) =& 2\alpha^2 z^2\big[z^{i+1}+2\mu(z-1)\overline{P}(0,0,s)+\mu z(z-1)\overline{P}(0,1,s)\big] \\
&+ \alpha z\big[(s+\lambda(1-z)+2\mu+2\alpha)z-2\mu\big] \\
&\times \big[(\mu+\nu)(z-1)\overline{P}(1,0,s)+\nu z(z-1)\overline{P}(1,1,s)\big] \\
&+ \big\{\big[(s+\lambda(1-z)+2\mu+2\alpha)z-2\mu\big] \\
&\times \big[(s+\lambda(1-z)+\mu+\nu+\alpha+\beta)z-(\mu+\nu)\big]-2\alpha\beta z^2\big\} \\
&\times \big[2\nu(z-1)\overline{P}(2,0,s)+\nu z(z-1)\overline{P}(2,1,s)\big]
\end{aligned}
$$

$$
\begin{aligned}
(28) \qquad \overline{D}(z,s) =& \big[(s+\lambda(1-z)+2\mu+2\alpha)z-2\mu\big] \\
&\times \big[(s+\lambda(1-z)+\mu+\nu+\alpha+\beta)z-(\mu+\nu)\big] \\
&\times \big[(s+\lambda(1-z)+2\nu+2\beta)z-2\nu\big] \\
&- 4\alpha\beta z^2\big[(s+\lambda(1-z)\mu+\nu+\alpha+\beta)z-(\mu+\nu)\big]
\end{aligned}
$$

Now, since each of the expressions in (24) contains six unknowns namely, $\overline{P}(j,0,s)$ and $\overline{P}(j,1,s)$, for $j=0,1,2$ appearing in their numerator give by equation (25), (26) and (27), it is sufficient to consider any one of them. Let us consider equation (24) for $j=0$. It can be shown by Rouche's Theorem that the denominator give by (28) has three zeroes inside the circle $|z| = 1$. Since $\overline{P}(0, z, s)$ is regular, the numerator $\overline{N}_0(z, s)$ must vanish for each of these zeros giving 3 linear equation in 6 unknowns. Apart from these 3 equation we have 3 other equation namely, (13), (16) and (19) involving the same 6 unknowns. Thus a total number of 6 equations are sufficient to determine all the 6 unknowns. Therefore each $\overline{P}(j, z, s)$, for $j = 0, 1, 2$ can be determined and hence $\overline{P}(z, s) = \sum_{j=0}^{2} \overline{P}(j, z, s)$, can be completely determined.

5. Probabilities Of The States Of The System

We let $z = 1$ in equation (25)-(28) and have

$$\overline{N}_0(1, s) = (s + \alpha + \beta)(s + 2\beta) - 2\alpha\beta \tag{29}$$

$$\overline{N}_1(1, s) = 2\alpha(s + 2\beta) \tag{30}$$

$$\overline{N}_2(1, s) = 2\alpha^2 \tag{31}$$

$$\overline{D}(1, s) = (s + 2\alpha)(s + \alpha + \beta)(s + 2\beta) - 4\alpha\beta(s + \alpha + \beta) \tag{32}$$

Now the right hand side of equation (32) can be factored so as to have

$$\overline{D}(1, s) = s(s + \alpha + \beta)(s + 2\beta + 2\alpha) \tag{33}$$

Equations (24), on using equation (29), (30), (31) and (33) for $j = 0, 1, 2$, yield

$$\overline{P}(0, z, s) = \frac{\overline{N}_0(1, s)}{\overline{D}(1, s)} = \frac{(s + \alpha + \beta)(s + 2\beta) - 2\alpha\beta}{s(s + \alpha + \beta)(s + 2\alpha + 2\beta)} \tag{34}$$

$$\overline{P}(1, z, s) = \frac{\overline{N}_1(1, s)}{\overline{D}(1, s)} = \frac{2\alpha(s + 2\beta)}{s(s + \alpha + \beta)(s + 2\alpha + 2\beta)} \tag{35}$$

$$\overline{P}(2, z, s) = \frac{\overline{N}_2(1, s)}{\overline{D}(1, s)} = \frac{2\alpha^2}{s(s + \alpha + \beta)(s + 2\alpha + 2\beta)} \tag{36}$$

On partial fraction decomposition equations (34), (35) and (36) respectively yield

$$\overline{P}(0,z,s) = \frac{1}{(\alpha+\beta)^2}\left\{\frac{\beta^2}{s} + \frac{2\alpha\beta}{s+\alpha+\beta} + \frac{\alpha^2}{s+2\alpha+2\beta}\right\} \tag{37}$$

$$\overline{P}(1,z,s) = \frac{1}{(\alpha+\beta)^2}\left\{\frac{2\alpha\beta}{s} + \frac{2\alpha(\alpha-\beta)}{s+\alpha+\beta} - \frac{2\alpha^2}{s+2\alpha+2\beta}\right\} \tag{38}$$

$$\overline{P}(2,z,s) = \frac{\alpha^2}{(\alpha+\beta)^2}\left\{\frac{1}{s} - \frac{2}{s+\alpha+\beta} + \frac{1}{s+2\alpha+2\beta}\right\} \tag{39}$$

One can verify that on adding equations (37), (38) and (39) respectively we have $\overline{P}(1,s) = \sum_{j=0}^{2} \overline{P}(j,z,s) = \frac{1}{s}$ as it should be.

On inverting the Laplace trasforms equations (37), (38) and (39) respectively give the probabilities that 0,1 or 2 servers are away on vacation at time t. Thus, we have

$$P(0,t) = \frac{1}{(\alpha+\beta)^2}\left\{\beta^2 + 2\alpha\beta e^{-(\alpha+\beta)t} + \alpha^2 e^{-2(\alpha+\beta)t}\right\} \tag{40}$$

$$P(1,t) = \frac{1}{(\alpha+\beta)^2}\left\{2\alpha\beta + 2\alpha(\alpha-\beta)e^{-(\alpha+\beta)t} - 2\alpha^2 e^{-2(\alpha+\beta)t}\right\} \tag{41}$$

$$P(2,t) = \frac{\alpha^2}{(\alpha+\beta)^2}\left\{1 - 2e^{-(\alpha+\beta)t} + e^{-2(\alpha+\beta)t}\right\} \tag{42}$$

Letting $t \to \infty$ in equations (40)-(42), we have the corresponding steady state probabilities of the states of the system as

$$P(0) = \frac{\beta^2}{(\alpha+\beta)^2} \tag{46}$$

$$P(1) = \frac{2\alpha\beta}{(\alpha+\beta)^2} \tag{47}$$

$$P(2) = \frac{\alpha^2}{(\alpha+\beta)^2} \tag{48}$$

6. A Particular Case

If there are no server vacation, then $\overline{P}(j,n,s) = 0$ for $j = 1,2$ and $n \geq 0$ and for that mather $\overline{N}_1(z,s) = \overline{N}_2(z,s) = 0$. With these substitutions alongwith $\alpha = 0$, the foregoing results yield

$$
\begin{aligned}
(49) \qquad \overline{N}_0(z,s) = &\Big\{ \big[(s+\lambda(1-z)+\mu+\nu+\beta)z-(\mu+\nu)\big] \\
&\times \big[(s+\lambda(1-z)+2\nu+2\beta)z-2\nu\big] \Big\} \\
&\times \left\{ z^{j+1} + 2\mu(z-1)P_0^{(2)}(s) + \mu z(z-1)P_1^{(2)}(s) \right\}
\end{aligned}
$$

$$
\begin{aligned}
(50) \qquad \overline{D}(z,s) = &\big[(s+\lambda(1-z)+2\mu)z-2\mu\big] \\
&\times \big[(s+\lambda(1-z)+\mu+\nu+\beta)z-(\mu+\nu)\big] \\
&\times \big[(s+\lambda(1-z)+2\nu+2\beta)z-2\nu\big]
\end{aligned}
$$

Obviously the two common factors of each of the equations (49) and (50) cancel with each other so that finally we have

$$
\begin{aligned}
\overline{P}(z,s) &= \frac{\overline{N}_0(z,s)}{\overline{D}(z,s)} \\
&= \frac{z^{i+1} + 2\mu(z-1)\overline{P}(0,0,s) + \mu z(z-1)\overline{P}(0,1,s)}{(s+\lambda(1-z)+2\mu)z-2\mu}
\end{aligned}
$$

The result equation (51) agrees with a known result except for notations. {see SAATY [8], equation 4-105, p 111 }

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