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Testing Homogeneity for Poisson Processes

Prueba de homogeneidad para procesos de Poisson

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Abstract

We developed an asymptotically optimal hypothesis test concerning the homogeneity of a Poisson process over various subintervals. Under the null hypothesis, maximum likelihood estimators for the values of the intensity function on the subintervals are determined, and are used in the test for homogeneity.

***Key words*:** Poisson process, hypothesis testing, local alternatives, asymptotic distribution, asymptotically optimal, likelihood ratio test.

Resumen

Una prueba de hipótesis asintótica para verificar homogeneidad de un proceso de Poisson sobre ciertos subintervalos es desarrollada. Bajo la hipótesis nula, estimadores máximo verosímiles para los valores de la función intensidad sobre los subintervalos mencionados son determinados y usados en el test de homogeneidad.

***Palabras clave*:** proceso de Poisson, prueba de hipótesis, alternativas locales, distribución asintótica, asintóticamente óptimo, prueba de razón de verosimilitud.

1. Introduction

Poisson processes have been used to model random phenomena in areas such as communications, hydrology, meteorology, insurance, reliability, and seismology,

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among others. These processes are often appropriate for modeling a series of events over time.

Poisson processes are governed by an intensity function $\lambda(t)$, which determines the instantaneous rate of event occurrence at time t. Equivalently, a Poisson process is also governed by the cumulative intensity function $\Lambda(t) = \int_0^t \lambda(\tau)\, d\tau$. When the intensity function is a constant, the Poisson process is known as a homogeneous Poisson process. When the intensity function varies with time, the Poisson process is known as a nonhomogeneous Poisson process (NHPP). A special case of an NHPP which arises in this paper has an intensity function that is piecewise constant over various time subintervals. The main aim of this paper is to develop a hypothesis test to determine whether an observed point process is drawn from a homogeneous Poisson process or a nonhomogeneous Poisson process with a piecewise constant intensity function.

A number of authors have carried out statistical analysis on the intensity of an NHPP. For instance, Leemis (1991), Leemis (2004), Kuhl, Wilson & Johnson (1997), Arkin & Leemis (2000), Kuhl & Wilson (2000), Henderson (2003), and others have considered the nonparametric estimation of the cumulative intensity function for a NHPP, and some of these authors have devoted their attention to modeling the periodic behavior of the process.

By following ideas from Fierro (2008), and considering, as in Leemis (2004), a finite time horizon that has been partitioned into subintervals, we state a result for testing whether a Poisson process is homogeneous or not over certain time intervals. Although a nonhomogeneous Poisson process is oftentimes a more accurate model of a phenomenon occurring in a non-stationary fashion, from a statistical point of view, the modeling based on a homogeneous Poisson process is simpler due to the fact that its intensity function depends only upon a single real parameter. Even though the process could be nonhomogeneous, it is important to investigate whether the process is homogeneous at certain time intervals.

For this reason, the main aim of this paper is to develop an asymptotic likelihood test for homogeneity. It is proved that this test is asymptotically optimal. As in Neyman (1949), we study the asymptotic behavior of the log likelihood of the test, but additionally, we consider the noncentral scenario to obtain an approximation to the power of the test. A sequence of local alternative hypotheses are stated, similar to those considered in some tests, which can be found in Serfling (1980), Karr (1991) and Lehmann (1999). Under the null hypothesis, the intensity function of the process is piecewise constant and, in order to obtain sufficient information to estimate these constants, observations from each of these intervals should be considered. A maximum likelihood estimator should take this information into account, for example, when estimating the cumulative intensity function. The methods introduced here are parametric because the inference on the intensity function of a NHPP involves a finite number of parameters. This technique has been argued against by some authors because it requires the introduction of parameters by the modeler (Leemis 1991, Arkin & Leemis 2000); this technique, however, has been used by Henderson (2003) and Leemis (2004) and we believe it is appropriate in many settings. Even though, in this work, the intensity of the nonhomogeneous Poisson process is not sequentially estimated, it is worth mentioning there are other estimation methods. One of them is the Shiryayev-Roberts test which was introduced by Shiryaev (1963) and Roberts (1966). This procedure is concerned with the sequential detection of changes in distributions occurring at unknown points in time.

The article is organized as follows. In Section 2 we introduce the null hypothesis and derive maximum likelihood estimators under the null hypothesis and without restrictions on the parameters. In Section 3, the asymptotic normality of the estimators is stated with the main results introduced in Section 4. In Section 5, an example is presented. Finally a method for simulations of NHPP variables under the null hypothesis, is proposed in Section 5.

2. Preliminaries

Let T be a fixed strictly positive real number and let us partition the interval $[0, T]$ into m subintervals $[t_0, t_1], (t_1, t_2], \dots, (t_{m-1}, t_m]$, where $t_0 = 0$ and $t_m = T$. The subintervals do not necessarily have equal widths. Let us denote by $\mathcal{C}$ the class of all functions λ which are piecewise constant on each subinterval defined above. From now on, the constant value of λ on $(t_{i-1}, t_i]$ will be denoted by λ_i. Consequently,

$$\lambda(t) = \lambda_1 \mathrm{I}_{[t_0,t_1]}(t) + \sum_{i=2}^{m} \lambda_i \mathrm{I}_{(t_{i-1},t_i]}(t)$$

where I_C stands for the indicator function on a set C.

This work refers to the hypothesis test that the intensity λ is constant in certain groups of the above subintervals. To do this, we need to partition the set $J = \{1, \dots, m\}$ into r subsets $J(1), \dots, J(r)$, (r groups), that is, $J = J(1) \cup \dots \cup J(r)$, and for $u \neq v$, $J(u) \cap J(v) = \emptyset$. Let us denote by $m(u)$ the cardinality of $J(u)$. Hence $m(1) + \dots + m(r) = m$. With these notations, we are interested in finding out whether or not $\lambda(t)$ is constant on the sets $\bigcup_{i \in J(u)} (t_{i-1}, t_i]$, ($u \in \{1, \dots, r\}$). Consequently, the null hypothesis should be stated in mathematical terms as follows:

$$\mathrm{H}_0 : \forall u \in \{1, \dots, r\}, \forall i, j \in J(u), \lambda_i = \lambda_j \tag{1}$$

This hypothesis can be stated in the following simpler equivalent form:

$$\mathrm{H}_0 : \forall u \in \{1, \dots, r\}, \forall i \in J(u), \lambda_i = \lambda^u$$

where $\lambda^u = \sum_{i \in J(u)} \lambda_i / m(u)$.

Considering $r = 1$, H_0 is the hypothesis corresponding to λ is the intensity of an homogeneous Poisson process.

Assume there are $N_1, \dots, N_k$ independent realizations of a nonhomogeneous Poisson process with intensity $\lambda \in \mathcal{C}$ and as before, λ_i denotes the constant value of λ on $(t_{i-1}, t_i]$. Put $N^k = N_1 + \dots + N_k$. An estimation of λ_i can be obtained by counting the jumps of N_k into the interval $(t_{i-1}, t_i]$.

From Theorem 2.31, in Karr (1991), a likelihood function for $\lambda_1, \dots, \lambda_m$ is given on $[0, T]$ by

$$\mathcal{L}(\lambda_i; 1 \le i \le m) = \exp\left[\int_0^T \log(\lambda(t))\ \mathrm{d}N^k(t) - k\int_0^T \lambda(t)\ \mathrm{d}t\right]$$

Hence,

$$\mathcal{L}(\lambda_i; 1 \le i \le m) = \exp\left[\sum_{i=1}^{m}\{\log(\lambda_i)\triangle N_i^k - k\lambda_i\triangle t_i\}\right]$$

where $\triangle N_i^k = N^k(t_i) - N^k(t_{i-1})$ and $\triangle t_i = t_i - t_{i-1}$.

Under H_0, this likelihood function on $[0, T]$ becomes

$$\mathcal{L}_0(\lambda^u; 1 \le u \le r) = \exp\left[\sum_{u=1}^{r}\log(\lambda^u)\sum_{i\in J(u)}\triangle N_i^k - k\lambda^u\sum_{i\in J(u)}\triangle t_i\right]$$

It is easy to see that the maxima of $\mathcal{L}_0$ and $\mathcal{L}$ are attained at $\lambda^u = \widehat{\lambda^u}, 1 \le u \le r$, and $\lambda_i = \widehat{\lambda}_i, 1 \le i \le m$, respectively, where

$$\widehat{\lambda^u} = \frac{\sum_{i\in J(u)}\triangle N_i^k}{k\sum_{i\in J(u)}\triangle t_i} \qquad \text{and} \qquad \widehat{\lambda}_i = \frac{\triangle N_i^k}{k\triangle t_i}$$

For the sake of simplicity, the reference to k in these maximum likelihood estimators has been omitted.

Notice that, under H_0, $\widehat{\lambda}_i$ is not sufficient for λ_i and thus there exists information from the data which is not contained in the statistic $\widehat{\lambda}_i$. This lack of information is contained in $T = \sum_{i\in J(u)}\triangle N_i^k$, for instance. Consequently, it is prominent the convenience of using, under H_0, $\widehat{\lambda^u}$ instead of $\widehat{\lambda}_i$, in any estimation of a function of λ_i, $(i \in J(u))$. This fact is relevant in Section 5, where an estimation of the cumulative intensity function of the process is considered in variate generation by inversion and by thinning for a NHPP from event count data.

3. Asymptotic Normality of Estimators

For making inference about the parameters λ_i, $(i = 1, \dots, r)$, for instance, in order to obtain asymptotic confidence intervals for these parameters, we need the corresponding estimators to be consistent and asymptotically normal. This fact is stated in Proposition 1 below. Moreover, Corollary 1, provides us the asymptotical distribution for the parameters under the null hypothesis.

Proposition 1. *For each $i = 1, \dots, m$, $\widehat{\lambda}_i$ is consistent and asymptotically normal $\mathcal{N}(0, \lambda_i)$, which means that the following two conditions hold:*

(C1) *For each* $i = 1, \dots, m$, $\widehat{\lambda_i}$ *converges to* λ_i, *with probability* 1, *as* $k \to \infty$.

(C2) $\sqrt{k}(\widehat{\lambda_1} - \lambda_1, \dots, \widehat{\lambda_m} - \lambda_m)$ *converges in distribution to an* m*-variate normal random vector having mean zero and covariance matrix* $\mathbf{\Sigma}$ *given by*

$$\mathbf{\Sigma} = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_m \end{pmatrix}$$

***Proof*.** Conditions (C1) and (C2) directly follow from Kolmogorov's Law of Large Numbers, the independent increments property of Poisson processes and the classical Central Limit Theorem. □

Corollary 1. *Under* H_0, *for each* $u = 1, \dots, r$, $\widehat{\lambda^u}$ *converges to* λ^u *as* $k \uparrow \infty$ *and* $\sqrt{k}(\widehat{\lambda^1} - \lambda^1, \dots, \widehat{\lambda^r} - \lambda^r)$ *converges in distribution to an* r*-variate normal random vector having mean zero and covariance matrix* $\mathbf{\Sigma}$ *given by*

$$\mathbf{\Sigma} = \begin{pmatrix} \lambda^1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda^r \end{pmatrix}$$

The above corollary enables us to obtain the usual confidence-interval estimate for λ^u, and hence for λ_i where $i \in J(u)$. Indeed, an asymptotically $100(1-\alpha)\%$ confidence interval for λ^u is

$$\widehat{\lambda^u} - z_{\alpha/2}\sqrt{\widehat{\lambda^u}/k} < \lambda^u < \widehat{\lambda^u} + z_{\alpha/2}\sqrt{\widehat{\lambda^u}/k}$$

where $z_{\alpha/2}$ is the $1 - \alpha/2$ percentile of the standard normal distribution.

4. Main Result

The main result of this paper is stated in Theorem 1 below. For testing H_0 against H_0 fails to be true, we denote by $\mathcal{R}_k$ the likelihood ratio, that is

$$\mathcal{R}_k = \frac{\mathcal{L}_0(\widehat{\lambda^u}; 1 \le u \le r)}{\mathcal{L}(\widehat{\lambda_i}; 1 \le i \le m)}$$

and hence

$$\mathcal{R}_k = \exp\left[\sum_{u=1}^{r} \sum_{i \in J(u)} [\log(\widehat{\lambda^u}/\widehat{\lambda_i}) \triangle N_i^k - k(\widehat{\lambda^u} - \widehat{\lambda_i}) \triangle t_i]\right] \tag{2}$$

Even though $\mathcal{R}_k$ depends on N^k, it is worth noting $\mathcal{R}_k$ does not depend on k, i.e., $\mathcal{R}_k$ depends on k only through N^k.

In order to state the main result, for each $u \in \{1, \ldots, r\}$, we consider the following sequence of local alternatives to the null hypothesis:

$$\mathrm{H}^{(k)} : \forall u \in \{1, \ldots, r\}, \forall i \in J(u), \lambda_i = \lambda^u + \delta_i/\sqrt{k} \tag{3}$$

where $\delta = (\delta_1, \ldots, \delta_m)$ is a fixed vector in $\mathbb{R}^m$ satisfying $\sum_{i \in J(u)} \delta_i = 0$, for each $u \in \{1, \ldots, r\}$.

Theorem 1. *Under* $\mathrm{H}^{(k)}$ *as* $k \to \infty$, $-2\log(\mathcal{R}_k)$ *has noncentral asymptotically* χ^2 *distribution with* $m - r$ *degrees of freedom and noncentrality parameter*

$$\begin{aligned}
\Phi^2 &= \sum_{u=1}^{r} \frac{1}{\lambda^u} \sum_{i \in J(u)} \triangle t_i \left[\delta_i - \frac{\sum_{j \in J(u)} \delta_j \triangle t_j}{\sum_{j \in J(u)} \triangle t_j} \right]^2 \\
&= \sum_{u=1}^{r} \frac{1}{\lambda^u} \sum_{i \in J(u)} \triangle t_i \left[\delta_i^2 - \left(\frac{\sum_{j \in J(u)} \delta_j \triangle t_j}{\sum_{j \in J(u)} \triangle t_j} \right)^2 \right]
\end{aligned}$$

Proof. By taking into account that $\log(x) = (x-1) - (x-1)^2/2 + \mathrm{O}((x-1)^3)$, from (2) it is obtained

$$\begin{aligned}
-2\log(\mathcal{R}_k) &= \sum_{u=1}^{r} \sum_{i \in I(u)} \left([(\widehat{\lambda}_i - \widehat{\lambda^u})/\widehat{\lambda}_i]^2 \triangle N_i^k + \mathrm{O}([(\widehat{\lambda}_i - \widehat{\lambda^u})/\widehat{\lambda}_i]^3) \right) \\
&= \sum_{i=1}^{m} \left[(U_i^k)^2 + \mathrm{O_P}((U_i^k)^3/\sqrt{\triangle N_i^k}) \right]
\end{aligned}$$

where $U_i^k = (\widehat{\lambda}_i - \widehat{\lambda^u})\sqrt{k \triangle t_i / \widehat{\lambda}_i}$ whenever $i \in J(u)$, and in general, $A_n = \mathrm{O_P}(B_n)$ means that given any $\eta > 0$, there is a constant $M = M(\eta)$ and a positive integer $n_0 = n_0(\eta)$ such that $\Pr\{|A_n| \leq M|B_n|\} \geq 1 - \eta$ for every $n > n_0$.

For each $i = 1, \ldots, m$, let $\triangle M_i^k = (\triangle N_i^k - k\lambda_i \triangle t_i)/\sqrt{k}$. Since $\triangle M_1^k, \ldots, \triangle M_m^k$ are independent, by the classical Central Limit Theorem, $\{(\triangle M_1^k, \ldots, \triangle M_m^k)\}_{k \in \mathbb{N}}$ converges in distribution to a normal random vector having mean zero and covariance matrix $\mathbf{\Sigma}$ given by the diagonal matrix $\mathbf{\Sigma} = \mathrm{diag}(\lambda_1 \triangle t_1, \ldots, \lambda_m \triangle t_m)$. Under $\mathrm{H}^{(k)}$, for each $u = 1, \ldots, r$ and each $i \in J(u)$, $\triangle N_i^k = \lambda^u k \triangle t_i + \delta_i \sqrt{k} \triangle t_i + \sqrt{k} \triangle M_i^k$. Hence,

$$\widehat{\lambda}_i = \lambda^u + \frac{\delta_i}{\sqrt{k}} + \frac{\triangle M_i^k}{\sqrt{k} \triangle t_i} \tag{4}$$

and

$$\widehat{\lambda^u} = \lambda^u + \frac{\sum_{j \in J(u)} \delta_j \triangle t_j}{\sqrt{k} \sum_{j \in J(u)} \triangle t_j} + \frac{\sum_{j \in J(u)} \triangle M_j^k}{\sqrt{k} \sum_{j \in J(u)} \triangle t_j} \tag{5}$$

From (4) and (5), for each $i \in J(u)$ one obtains

$$\begin{aligned} U_i^k &= \sqrt{\frac{\triangle t_i}{\widehat{\lambda}_i}} \left(\delta_i + \frac{\triangle M_i^k}{\triangle t_i} - \frac{\sum_{j \in J(u)} \triangle M_j^k}{\sum_{j \in J(u)} \triangle t_j} - \frac{\sum_{j \in J(u)} \delta_j \triangle t_j}{\sum_{j \in J(u)} \triangle t_j} \right) \\ &= V_i^k - \frac{\sqrt{\triangle t_i} \sum_{j \in J(u)} \sqrt{\triangle t_j} V_j^k \sqrt{\widehat{\lambda_j} / \widehat{\lambda_i}}}{\sum_{j \in J(u)} \triangle t_j} \end{aligned}$$

where $V_i^k = \triangle M_i^k / \sqrt{\widehat{\lambda}_i \triangle t_i} + \delta_i \sqrt{\triangle t_i / \widehat{\lambda}_i}$.

Under $\mathrm{H}^{(k)}$, for each $i \in J(u)$, $\lambda_i = \lambda^u + \delta_i / \sqrt{k}$ and by Proposition 1 in Section 3, for each $i, j \in J(u)$, $\sqrt{\widehat{\lambda_j} / \widehat{\lambda_i}} \to 1$, with probability 1, as $k \to \infty$. Consequently, from a slight modification of Proposition 1 in Section 3 and Slutzky's theorem, $\{(U_1^k, \ldots, U_m^k)\}_{k \in \mathbb{N}}$ converges in distribution to $\mathbf{U} = (U_1, \ldots, U_m)$, where for each $i \in J(u)$, $(u = 1, \ldots, r)$,

$$U_i = V_i - \frac{\sqrt{\triangle t_i} \sum_{j \in J(u)} \sqrt{\triangle t_j} V_j}{\sum_{j \in J(u)} \triangle t_j}$$

and $\mathbf{V} = (V_1, \ldots, V_m)^{\mathrm{t}}$ is a vector of m independent normal random variables with variance one, and such that for each $u \in \{1, \ldots, r\}$ and each $j \in J(u)$, V_j has mean $\delta_j \sqrt{\triangle t_j / \lambda^u}$. Hence, $\{-2 \log(\mathcal{R}_k))\}_{k \in \mathbb{N}}$ converges in distribution to $\|\mathbf{U}\|^2 = \sum_{i=1}^m U_i^2$, where $\| \cdot \|$ stands for the Euclidean norm in $\mathbb{R}^m$.

Let

$$\mathbf{P} = \begin{pmatrix} \mathbf{P}(1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbf{P}(r) \end{pmatrix}$$

where for each $u = 1, \ldots, r$, $\mathbf{P}(u) = (p_{ij}(u); i, j \in J(u))$ is the matrix defined by $p_{ij}(u) = \sqrt{\triangle t_i \triangle t_j} / \sum_{j \in J(u)} \triangle t_j$. Hence, $\mathbf{U} = (\mathbf{I} - \mathbf{P})\mathbf{V}$, and since for each $u = 1, \ldots, r$, $\mathbf{P}(u)$ is an idempotent matrix having rank 1, the matrix $\mathbf{P}$ is idempotent as well and has rank r. Consequently, $\mathbf{I} - \mathbf{P}$ is idempotent and has rank $m - r$. It follows from Theorem 3.5.1 in Serfling (1980) that $\|\mathbf{U}\|^2$ has χ^2 distribution with $m - r$ degrees of freedom and non-centrality parameter $\mu(\mathbf{I} - \mathbf{P})\mu^{\mathrm{t}}$, where

$$\mu = (\delta_1 \sqrt{\triangle t_1 / \lambda^{u(1)}}, \ldots, \delta_m \sqrt{\triangle t_m / \lambda^{u(m)}})$$

and for each $i \in \{1, \ldots, m\}$, $u(i)$ is the unique integer in $\{1, \ldots, r\}$ such that $i \in J(u(i))$. Since $\mu^{\mathrm{t}}(\mathbf{I} - \mathbf{P})\mu = \|(\mathbf{I} - \mathbf{P})\mu\|^2 = \|\mu\|^2 - \|\mathbf{P}\mu\|^2$, the proof is complete. □

The corollary below is useful to test the hypothesis whether a Poisson process is homogeneous or not.

Corollary 2. *Let* $\widetilde{\lambda} = \sum_{j=1}^{m} \lambda_j/m$, $(\delta_1, \dots, \delta_m) \in \mathbb{R}^m$ *such that* $\sum_{j=1}^{m} \delta_i = 0$ *and* $\mathrm{H}^{(k)}$ *be the statistical hypothesis defined as*

$$\mathrm{H}^{(k)} : \forall i \in \{1, \dots, m\}, \lambda_i = \widetilde{\lambda} + \delta_i/\sqrt{k}$$

Then, under $\mathrm{H}^{(k)}$, $-2\log(\mathcal{R}_k)$ *has noncentral asymptotically* χ^2 *distribution with* $m-1$ *degrees of freedom and noncentrality parameter*

$$\Phi^2 = \frac{1}{\widetilde{\lambda}} \sum_{i=1}^{m} \triangle t_i \left[\delta_i - \frac{\sum_{j=1}^{m} \delta_j \triangle t_j}{\sum_{j=1}^{m} \triangle t_j}\right]^2$$

***Note* 1.** A natural application of the foregoing theorem is to calculate the approximate power of the test relative to

$$\mathrm{H}_0 : \forall u \in \{1, \dots, r\}, \forall i \in J(u), \lambda_i = \lambda^u$$

against the simple alternative

$$\mathrm{H}_1 : \forall u \in \{1, \dots, r\}, \forall i \in J(u), \lambda_i = \lambda_i^*$$

Suppose that the critical region is $\{-2\log(\mathcal{R}_k) > t_0\}$, where t_0 has been calculated for a level of significance α based upon the null hypothesis asymptotic χ^2-distribution of $-2\log(\mathcal{R}_k)$.

We interpret δ_i in $\mathrm{H}^{(k)}$ as $\sqrt{k}(\lambda_i^* - \lambda^u)$ and approximate the power of the test by means of the probability of $\{\chi^2 > t_0\}$, where χ^2 is a random variable having χ^2-distribution with $m-r$ degrees of freedom and noncentrality parameter

$$\Phi^2 = k \sum_{u=1}^{r} \frac{1}{\lambda^u} \sum_{i \in J(u)} \triangle t_i \left[\lambda_i^* - \frac{\sum_{j \in J(u)} \triangle t_j \lambda_j^*}{\sum_{j \in J(u)} \triangle t_j}\right]^2$$

***Note* 2.** By the standard Central Limit Theorem, for $m-r$ and k large enough, $-2\log(\mathcal{R}_k)$ has approximate normal distribution with mean $m-r$ and variance $2(m-r)$.

Based on Theorem 1, an asymptotically maximum likelihood test, for testing H_0, according to (1), against local alternatives, can be stated. An important property of a test is its power optimality. The following proposition allows to conclude the above-mentioned test is asymptotically uniformly most powerful.

Proposition 2. *Let* $\mathcal{B}(\mathbb{R})$ *be the Borel* σ*-algebra of subsets of* $\mathbb{R}$ *and for each* $\nu \geq 0$, P_ν *be the probability distribution on* $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ *corresponding to the* χ^2-*distribution with* g *degrees of freedom and noncentrality parameter* ν*. For testing* $H : \nu = 0$ *against* $K : \nu > 0$*, the test defined by the critical region* $[t_\alpha, \infty)$*, where* $P_0([t_\alpha, \infty)) = \alpha$*, is uniformly most powerful.*

***Proof*.** The probability density function corresponding to the χ^2-distribution with g degrees of freedom and noncentrality parameter ν is given by

$$f(x, g, \nu) = \frac{x^{g/2-1}\, \mathrm{e}^{-(x+\nu)/2}}{2^{g/2}\Gamma(g/2)} \left(1 + \sum_{j=1}^{\infty} \frac{(\nu x/4)^j \Gamma(g/2)}{j!\Gamma(j+g/2)}\right) \mathrm{I}_{]0,\infty[}(x)$$

Hence, for each $x \geq 0$, we have $f(x, g, \nu) = f(x, g, 0)G(x, g, \nu)$, where

$$G(x, g, \nu) = \mathrm{e}^{-\nu/2}\left(1 + \sum_{j=1}^{\infty} \frac{(\nu x/4)^j \Gamma(g/2)}{j!\Gamma(j + g/2)}\right)$$

Let $A = [t_\alpha, \infty[$ and B be a Borelian subset of $\mathbb{R}$ such that $P_0(B) = \alpha$. We need to prove, for each $\nu \geq 0$, $P_\nu(A) \geq P_\nu(B)$.

We have

$$P_\nu(A) - P_\nu(B) = \int_{A \cap B^{\,c}} f(x, g, 0)G(x, g, \nu)\ \mathrm{d}x - \int_{A^{\,c} \cap B} f(x, g, 0)G(x, g, \nu)\ \mathrm{d}x$$

and since G is increasing at the first variable, we derive

$$\begin{aligned} P_\nu(A) - P_\nu(B) &\geq G(t_\alpha, g, \nu)\left(\int_{A \cap B^{\,c}} f(x, g, 0)\ \mathrm{d}x - \int_{A^{\,c} \cap B} f(x, g, 0)\ \mathrm{d}x\right) \\ &= G(t_\alpha, g, \nu)(P_0(A) - P_0(B)) \\ &= 0 \end{aligned}$$

Therefore, the proof is complete. □

Corollary 3. *Let* H_0 *and* $\mathrm{H}^{(k)}$ *be the statistical hypotheses defined by (1) and (3), respectively. For testing* H_0 *against* $\mathrm{H}^{(k)}$ *with a significance level* α, *(*$0 < \alpha < 1$*), the test defined by the critical region* $\{-2\log(\mathcal{R}_k) > t_\alpha\}$ *is asymptotically uniformly most powerful, where* $t_\alpha > 0$ *is determined by*

$$\lim_{k \to \infty} \Pr(-2\log(\mathcal{R}_k) > t_\alpha) = \alpha.$$

5. An Example

Let us suppose in a call center there are k employees in charge of state connections. The call number is recorded from 10.00 am to 1.00 pm every day during a seven-day period. It is possible to assume the number of phone connections made for each server follows a Poisson process and that these k Poisson processes are independent. For this purpose, it is assume the end of a working day agrees with the beginning of the next day. Even though people usually go for a walk on Saturday and Sunday, it is suspected that on weekends this number decreases due to people are not working. To find out this circumstance, the null hypothesis is defined as "H_{01}: the call rates are every day the same". From Corollary 4.1, in this case, $-2\log(\mathcal{R}_k)$ has asymptotically χ^2-distribution with six degrees of freedom. Another test could be stated by defining the null hypothesis as "H_{02}: the call rates from Monday to Friday are the same, and the Saturday call rate equals the Sunday one". In this last case, there are two groups $J(1)$ and $J(2)$ and consequently, $-2\log(\mathcal{R}_k)$ has asymptotically χ^2-distribution with five degrees of freedom.

In order to compare both tests, we simulate $k = 10$ copies of Poisson process and test H_{01} and H_{02} with the same data set. The call rate corresponding to the working days was assumed to be 50 calls/hour, and the call rate on Saturday and Sunday was assumed to be 45 calls/hour. Both hypothesis tests were performed 10^5 times with a level of significance $\alpha = 0.05$, and as result of this simulation, H_{01} and H_{02} were rejected 96.6% and 4.95%, respectively, which illustrates the test and gives an insight of its power.

6. Variate Generation

The cumulative intensity function for a NHPP is often estimated for simulating the NHPP. A number of methods for carrying out this simulation are described in Lewis & Shedler (1979), where the simulation method for a NHPP by thinning is stated. In this section, our purpose is not to give detailed variate generation algorithms, but to give an estimation of the cumulative intensity function based on $\widehat{\lambda^u}$ $(u = 1, \dots, r)$, which are, under H_0, estimators containing sufficient information for the parameters $\lambda_1, \dots, \lambda_m$. To this end, an estimator for the cumulative intensity function is defined and the basis of variate generation by inversion is recalled.

For each $t \geq 0$, let $i(t)$ denote the unique $i \in \{1, \dots, m\}$ satisfying $t \in Q_{i(t)}$, where $Q_1 = [t_0, t_1]$ and $Q_i = (t_{i-1}, t_i]$ for $j \in \{2, \dots, m\}$. By writing $U(t) = \{u : \exists i \leq i(t), i \in J(u)\}$ and $V(u,t) = \{j \in J(u) : j < i(t)\}$, the cumulative intensity function $\Lambda : [0, T] \to \mathbb{R}$ defined as $\Lambda(t) = \int_0^t \lambda(u)\, \mathrm{d}u$ satisfies

$$\Lambda(t) = \sum_{u \in U(t)} \sum_{j \in V(u,t)} \lambda_j \triangle t_j + \lambda_{i(t)}(t - t_{i(t)-1})$$

Let u_i denote the unique $u \in \{1, \dots, r\}$ such that $i \in J(u)$ and define $u(t) = u_{i(t)}$. Under H_0, we have

$$\Lambda(t) = \sum_{u \in U(t)} \lambda^u \sum_{i \in V(u,t)} \triangle t_i + \lambda^{u(t)}(t - t_{i(t)})$$

Consequently, Λ can be estimated by $\widehat{\Lambda}$, where for $t \geq 0$,

$$\widehat{\Lambda}(t) = \sum_{u \in U(t)} \widehat{\lambda^u} \sum_{i \in V(u,t)} \triangle t_i + \widehat{\lambda^{u(t)}}(t - t_{i(t)})$$

Following Leemis (2004), a realization of a Poisson process for modeling in a discrete-event simulation can be generated, under H_0, by inversion. Let

$$\widehat{\Psi}(u) = \begin{cases} \inf\{t > 0 : \widehat{\Lambda}(t) \geq u\} & \text{if} \quad u \leq \widehat{\Lambda}(T) \\ +\infty & \text{if} \quad u > \widehat{\Lambda}(T) \end{cases}$$

Note that for each $u \geq 0$, $\widehat{\Lambda}(\widehat{\Psi}(u)) = u$, almost everywhere, and consequently, if $S_1, S_2, \dots$ are the points in a homogeneous Poisson process of rate one (which have been chosen independently of $\widehat{\Lambda}$), then $\widehat{\Psi}(S_1), \widehat{\Psi}(S_2), \ldots$ are the points in a nonhomogeneous Poisson process with cumulative intensity function $\widehat{\Lambda}$. This fact enables us to generate NHPP event times starting from standard Poisson random variate generation.

According to Henderson (2003), at the beginning of Section 3, pages 379-380, for a general rate function, a faster generation procedure of NHPP event times is obtained by thinning. This method for simulating the NHPP was introduced by Lewis & Shedler (1979) and it is based on an estimator of the rate function λ. Under H_0, a maximum likelihood estimator for λ is given by $\widehat{\lambda}$, which is defined for $t \geq 0$ as

$$\widehat{\lambda}(t) = \sum_{u=1}^{r} \widehat{\lambda^u} \sum_{i \in J(u)} \mathrm{I}_{(t_{i-1}, t_i]}(t).$$

Recall that thinning first generates a candidate event time T^*, and then accepts the event time with probability $\widehat{\lambda}(T^*)/\lambda^*$, where λ^* is an upper bound of λ. The novelty here is that in this case, thinning is based on the estimators $\widehat{\lambda^1}, \ldots, \widehat{\lambda^r}$, which, as pointed out before, are sufficient statistics for $\lambda^1, \ldots, \lambda^r$.

7. Conclusions and Recommendations

In this paper we carry out a hypothesis test that allows us to find out whether or not a NHPP could be considered homogeneous in certain time intervals. Such an inquiry becomes very important when it is assumed that the rate function is a piecewise constant on subintervals of the time. Indeed, when there exists a great non-homogeneity and an approximated piecewise constant rate function has to be defined, it is necessary to partition the time interval in many subintervals. However, if homogeneity is observed in a large subset (which need not be connected) of the time horizon, a lesser number of subintervals will be necessary and an economy of computational time and/or memory to store the information could be obtained. On the other hand, under the null hypothesis, the estimators of the constant values of the intensity function are expressed in terms of sufficient statistics, which enables us to make use of the whole information provided by the data. This fact is particularly important for generating Poisson variates by inversion or thinning procedure.

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