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Nonstandard definition of the Stratonovich integral

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ABSTRACT. By using the relation between the Ito integral and the Stratonovich integral, a nonstandard definition of the Stratonovich integral is given.

For a good introduction of nonstandard analysis we can see Albeverio et al,(1986). The main features needed in this work paper can been seen in M. Muñoz de Özak (1997).

1. Introduction

The Stratonovich integral obeys the same rules as the Newton-Leibniz Calculus. By using nonstandard analysis the Ito integral can be regarded as a Riemann-Stieltjes sum (Anderson 1976). We used this result to give an easy representation of the Stratonovich integral.

Starting with two continuous, real d-dimensional semi-martingales we give a nonstandard representation of the Stratonovich integral as

$$\overline{S}_{\underline{t}} = \int_0^{\underline{t}} Y\partial X = \sum_{\underline{s}<\underline{t}} 1/2[Y(\underline{s}+\Delta t) + Y(\underline{s})]\Delta X(\underline{s})$$

for X, Y S-continuous internal semi-martingales and establish the existence of its standard part which is then shown to correspond to the usual Stratonovich integral

$$S_t = st(S_{\underline{t}}) = \int_0^t y\partial x = st\left(\sum_{\underline{s}<\underline{t}} 1/2[Y(\underline{s}+\Delta t) + Y(\underline{s})]\Delta X(\underline{s})\right)$$

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where x and y are continuous $\mathfrak{F}_t$-semi-martingales, and X and Y are internal semi-martingale liftings of x and y, respectively.

2. Stratonovich integral

Let $N \in {}^*\mathbb{N} - \mathbb{N}$, define $\Delta t = 1/N \approx 0$. Let T be the hyperfinite time line,

$$T = \{0, \Delta t, 2\Delta t, \cdots, k\Delta t, \cdots\} = \{k\Delta t : k \in {}^*\mathbb{N}_0\}.$$

Let $\mathfrak{B}$ the class of internal hyperfinite subsets of T and let $\overline{\lambda}$ be the counting measure on $\mathfrak{B}$. Then $(T, \mathfrak{B}, \overline{\lambda})$ is an internal measurable space. Also, T is an internal S-dense subset of ${}^*[0, \infty)$. We will also require another measurable space $(\Omega, \mathfrak{A}, \overline{P})$, and will denote with $(\Omega, L(\mathfrak{A}), L(\overline{P}))$ the corresponding Loeb space.

Let (M, ρ) be a complete metric space and denote with $D(M)$ the set of càdlàg functions $f : [0, \infty) \to M$. We know that there is a unique topology, J_1, for which $D(M)$ is polish space. In general we will denote with D this space, when $M = \mathbb{R}$.

Following Hoover and Perkins (1983), we find that the nearstandard points in ${}^*D$ are of three kinds: SD, SDJ and S-continuous (SC).

1. Definition. Let $F \in {}^*D$ be such that $F(\underline{t}) \in ns({}^*\mathbb{R})$ for $\underline{t} \in ns({}^*[0, \infty))$. then:

(a) F is of class SD if for each $t \in [0, \infty)$ there are points $\underline{t}_1 \approx \underline{t}_2 \approx t$ such that

(i) If $\underline{u}_1 \approx t, \underline{u}_1 \geq \underline{t}_1$, then $F(\underline{u}_1) \approx F(\underline{t}_1)$.

(ii) If $\underline{u}_2 \approx t, \underline{u}_2 < \underline{t}_2$, then $F(\underline{u}_2) \approx F(\underline{t}_2^-)$.

(b) F is of class SDJ or a larc lift, if (a) holds with $\underline{t}_1 = \underline{t}_2$ and $F(\underline{t}) \approx F(0)$ for all $\underline{t} \approx 0$ in ${}^*[0, \infty)$.

(c) F is S-continuous (SC) if $F(\underline{t}) \approx F(\underline{u})$ whenever $\underline{t} \approx \underline{u}$, $\underline{t}, \underline{u} \in ns(T)$.

If $F : T \to {}^*M$, F is SD (SDJ, SC) on T if it is the restriction to T of an SD (SDJ, SC) function on ${}^*[0, \infty)$. For a function on T we can define a real valued function $st(F)$ by

$$st(F)(t) = \lim_{\substack{{}^o\underline{t} \downarrow t \\ \underline{t} \in T}} {}^oF(\underline{t}).$$

In Hoover and Perkins (1983) it is shown that the class of functions in ${}^*D$ which are nearstandard in the J_1 topology is SDJ, and that the function $st|_{SDJ}$ is the nearstandard part for the J_1 topology.

2. Definition. An internal stochastic process X is of class SD (SDJ, SC) if for almost all $w \in \Omega$, the mapping

$$X(\cdot, w) : T \to {}^*M$$

is of class SD (SDJ, SC).

If X is SD, we can define a standard stochastic process with sample paths in D as follows: fix $x_o \in M$ and define

$$st(X)(t,w)\begin{cases} st(X(\cdot,w))(t), & \text{if} \quad X(\cdot,w) \quad \text{is} \quad SD, \\ x_o, & \text{otherwise.} \end{cases}$$

3. Definition. An SD (SDJ, SC) lifting of a stochastic process $x : [0, \infty) \times \Omega \to M$ is an internal stochastic process X of class SD (SDJ, SC), $X : T \times \Omega \to {}^*M$, such that $st(X)$ and x are indistinguishable.

Remark 1. We can replace T by an S-dense set of ${}^*[0, \infty)$ in the above definitions. An internal filtration on Ω indexed by T is an internal increasing collection of *sub-σ-fields of $\mathfrak{A}$, $\{\mathfrak{B}_{\underline{t}} : \underline{t} \in T\}$. The standard part of $\{\mathfrak{B}_{\underline{t}}\}$ is the filtration defined by

$$\mathfrak{F}_t = \left(\bigcap_{\substack{{}^\circ\underline{t}>t \\ \underline{t}\in T}} \sigma(\mathfrak{B}_{\underline{t}}) \right) \vee \mathfrak{N},$$

where $\mathfrak{N}$ is the class of $L(\overline{P})$ null sets of $L(\mathfrak{A})$. The set $\{\mathfrak{F}_t : t \geq 0\}$ satisfies the usual conditions (Albeverio et al.(1986) Corollary (4.3.2)).

4. Definition. A stopping time with respect to a filtration $\{\mathfrak{F}_t : t \in [0, \infty)\}$, is a mapping $U : \Omega \to [0, \infty)$ such that $\{U \leq t\} \in \mathfrak{F}_t$ for all $t \in [0, \infty)$, with $\mathfrak{F}_\infty = L(\mathfrak{A})$.

A *stopping time with respect to an internal filtration $\{\mathfrak{B}_{\underline{t}} : \underline{t} \in T\}$, or a $\mathfrak{B}_{\underline{t}}$-stopping time, is an internal mapping $V : \Omega \to T \cup \{\infty\}$ such that $\{V \leq \underline{t}\} \in \mathfrak{B}_{\underline{t}}$ for all $\underline{t} \in T \cup \{\infty\}$, with $\mathfrak{B}_\infty = \mathfrak{A}$.

Let

$$\mathfrak{B}_V = \{A \in \mathfrak{A} : A \cap \{V = \underline{t}\} \in \mathfrak{B}_{\underline{t}}, \quad \forall \underline{t} \in T\}.$$

5. Definition.

(i) A stochastic process $x : [0, \infty) \times \Omega \to \mathbb{R}^d$ is a d dimensional local martingale with respect to the filtration $\{\mathfrak{F}_t\}$, if x is an $\mathfrak{F}_t$ adapted process with sample paths a.e. in $D(\mathbb{R}^d)$ and there is a sequence of stopping times $\{U_n\}_{n\in\mathbb{N}}$ increasing to ∞ a.e., such that $x(t\wedge U_n)$ is uniformly integrable $\mathfrak{F}_t$-martingale for all n. $\{U_n\}$ is said to reduce x.

(ii) An internal stochastic process $X : T\times\Omega \to {}^*\mathbb{R}^d$ is an S-local martingale with respect to $\{\mathfrak{B}_{\underline{t}}\}$, if there is a nondecreasing sequence of $\mathfrak{B}_{\underline{t}}$ *stopping times $\{V_n\}$ such that

$$\lim_{n\to\infty} {}^oV_n = \infty \quad \text{a.s,} \tag{1}$$

$$\|X(\underline{t}\wedge V_n)\| \quad \text{is S-integrable for each} \quad \underline{t}\in T\cup\{\infty\} \quad \text{and for all } n. \tag{2}$$

$${}^oX(V_n) = st(X)({}^oV_n), \qquad a.s., \tag{3}$$

and

$$X(\underline{t}\wedge V_n) \text{ is a } {}^* - \text{martingale.} \tag{4}$$

$\{V_n\}$ is said to reduce X.

6. Definition. If x is an $\mathfrak{F}_t$ local martingale and $\{\mathfrak{B}_{\underline{t}} : \underline{t}\in T\}$ is an internal filtration, then a $\mathfrak{B}_{\underline{t}}$-local martingale lifting of x is an SDJ lifting X such that X is an S-local martingale.

Notation: For $Y_i : T\times\Omega \to {}^*\mathbb{R}^d$ $(i=1,2)$ internal, we write

$$|Y_i|(\underline{t},w) = \sum_{\underline{s}<\underline{t}} \|\Delta Y_i(\underline{s},w)\| \tag{5}$$

and

$$[Y_1,Y_2]_{\underline{t}} = \sum_{\underline{s}<\underline{t}} \Delta Y_1(\underline{s})\cdot\Delta Y_2(\underline{s}), \tag{6}$$

where $\Delta Y_i(\underline{s}) = Y_i(\underline{s}+\Delta t) - Y_i(\underline{s})$ and $\cdot$ denotes the scalar product.

Let $x : [0,\infty)\times\Omega \to \mathbb{R}^d$ be a local martingale. If $t>0$ is fixed and $Q=\{t_o,\cdots,t_l\}$ is a finite subset of $[0,t]$ with $0=t_o<\cdots<t_l=t$, let $\|Q\| = \sup_{i\le l}|t_i-t_{i-1}|$ and $S_t(x,Q) = \sum_{i=1}^l \|x(t_i)-x(t_{i-1})\|^2$. $S_t(x,Q)$ converges in probability to a limit $[x,x]_t$ as $\|Q\|\to 0$ and we may choose a version of $[x,x]$ with sample paths in D. If y also is a local martingale, then

$$[x,y] = \frac{1}{2}([x+y,x+y]-[x,x]-[y,y]).$$

If X is an internal local martingale lifting of x, $[X,X]$ is an SDJ lifting of $[x,x]$ and $S_t(x,Q)$ converges in probability to $st([X,X])(t)$ (Hoover and Perkins (1983) lemma 6.7).

7. Definition. A process of bounded variation $a : [0,\infty)\times\Omega\to\mathbb{R}^d$ is an $\mathfrak{F}_t$-adapted process with $a(0)=0$, whose sample paths belong to D and are of bounded variation on bounded intervals. With $|a|(t)$ we denote the variation of a on $[0,t]$.

8. Definition. Let $\{\mathfrak{B}_{\underline{t}}\}$ be an internal filtration. If a is a process of bounded variation, a $\mathfrak{B}_{\underline{t}}$-BV lifting of a is a $\mathfrak{B}_{\underline{t}}$ adapted process A such that A and $|A|$ (defined by (5)) are SDJ liftings of a and $|a|$ respectively.

9. Definition. A d-dimensional semi-martingale z is an $\mathfrak{F}_t$-adapted process, $\mathbb{R}^d$ valued, and with sample paths in D, such that $z(t) - z(0) = x(t) + a(t)$, where x is a local martingale with $x(0) = 0$ and a is a process of bounded variation with $a(0) = 0$. A $\mathfrak{B}_{\underline{t}}$- semi-martingale lifting of (a, x) is a pair (A, X) such that X is a $\mathfrak{B}_{\underline{t}}$ local martingale lifting of x, A is a $\mathfrak{B}_{\underline{t}}$- BV lifting of a and (A, X) is SDJ.

10. Definition. A predictable rectangle with respect to the filtration $\{\mathfrak{F}_t\}_{t\in[0,\infty)}$ is a set of the form $(s, t] \times F_s$, where $F_s \in \mathfrak{F}_s$ or $[0, t] \times F_o$, where $F_o \in \mathfrak{F}_o$. A set is called predictable if it is in the σ-algebra generated by the predictable rectangles. A process $x : [0, \infty) \times \Omega \to \mathbb{R}$ is predictable if it is measurable with respect to the σ-algebra of predictable sets.

Suppose M a normed linear space with norm $\|\cdot\|$. If x is a local martingale with $x(0) = 0$, a is a process of bounded variation, and $z = x + a$ is a semi-martingale, denote with $\mathfrak{L}(z, M)$ the space of functions $h : [0, \infty) \times \Omega \to M$ such that h is predictable, and we have

(a) $E\left(\left(\int_0^{R_n} \|h(s)\|^2 d[x, x]_s\right)^{1/2}\right) < \infty$ for some sequence of stopping times $\{R_n\}$ increasing to ∞ a.s.

(b) $\int_0^t \|h(s)\| d|a|_s < \infty$ for all $t \geq 0$ a.s.

If $H : T \times \Omega \to {}^*\mathbb{R}^{k+d}$ and $Z : T \times \Omega \to {}^*\mathbb{R}^d$ are internal processes, define $H \circ Z : T \times \Omega \to {}^*\mathbb{R}^k$ by

$$H \circ Z = \sum_{\underline{s}<\underline{t}} H(\underline{s}) \Delta Z(\underline{s}).$$

For appropiate functions $h \in \mathfrak{L}(z, M)$, z a semi-martingale and appropiate liftings H, Z of h and z, we may define the stochastic integral $\int_0^t h(s)dz(s)$ as $st(H \circ Z)(t)$.

Notation. If $t \in {}^*[0, \infty)$, $[t]$ is the greatest element of T satisfying $[t] \leq t$. More generally if $T' \subseteq T$, let

$$[t]^{T'} = \begin{cases} \max\{\underline{t} \in T : \underline{t} \leq t\}, & \text{if this set is nonempty,} \\ \min T', & \text{otherwise.} \end{cases}$$

11. Proposition. *Suppose $\{\mathfrak{B}_{\underline{t}} : \underline{t} \in T\}$ is an internal filtration and $A : T \times \Omega \to {}^*\mathbb{R}^d$ is a $\mathfrak{B}_{\underline{t}}$-adapted, SD lifting of a, a bounded variation process with $a(0) = 0$ a.s., then there is a positive infinitesimal $\Delta' t \in T$ such that if*

T'' is an internal S-dense subset of $T' = \{k\Delta' t : k \in {}^*\mathbb{N}_0\}$, such that $A([\underline{t}]^{T''})$ is a $\mathfrak{B}_{[\underline{t}]^{T''}}$-BV lifting of a.

For the proof, see Lemma 7.5 in Hoover and Perkins (1983).

Remark 2. If $a = st(A)$ is continuous, by Proposition 2.5 in Hoover and Perkins (1983), $A|_{T''\times\Omega}$ is S-continuous.

Remark 3. In order to define the Stratonovich stochastic integral we will only use continuous semi-martingales. So, if z is a continuous semi-martingale, we have a canonical representation of z as $z = z_o + m + a$, where m is a continuous local semi-martingale with $m(0) = 0$, a is a continuous process of bounded variation, and z_o is an $\mathfrak{F}_o$-measurable random variable.

12. Theorem. *If $y_1 = y_1(0) + m_1 + a_1$ and $y_2 = y_2(0) + m_2 + a_2$ are continuous semi-martingales with respect to the filtration $\mathfrak{F}_t$, there exist an internal filtration $\mathfrak{B}'_{\underline{t}}$ and S-continuous $\mathfrak{B}_t$-semi-martingale liftings Y_1 and Y_2 of y_1 and y_2 respectively.*

Proof. By Theorem 7.6 in Hoover and Perkins (1983), taking at the same time (a_1, m_1) and (a_2, m_2) it follows the existence of the desired SDJ internal semi-martingale liftings, by the continuity of the semi-martingales y_1 and y_2 it follows from Remark 2 that Y_1 and Y_2 are S-continuous. □

13. Corollary. *If y_1 and y_2 are continuous $\mathfrak{B}_{\underline{t}}$ semi-martingales, $y_i = y_i(0) + m_i + a_i$, $i = 1, 2$, then $[y_1, y_2] = st([M_1, M_2])$, where M_1 and M_2 are liftings of m_1, and m_2 respectively.*

Proof. From Theorem 1.2.12 there exist S-continuous semi-martingale liftings $Y_i = Y_i(0) + M_i + A_i$ of y_i, $i = 1, 2$. We have, for $Y_1(0) = Y_2(0) = 0$, that

$$[Y_1, Y_2](\underline{t}) = \sum_{\underline{s}<\underline{t}} \Delta Y_1(\underline{s}) \cdot \Delta Y_2(\underline{s}),$$

so that

$$\sum_{\underline{s}<\underline{t}} \Delta Y_1(\underline{s}) \cdot \Delta Y_2(\underline{s}) = \sum_{\underline{s}<\underline{t}} \Delta M_1(\underline{s}) \cdot \Delta M_2(\underline{s}) + \sum_{\underline{s}<\underline{t}} \Delta A_1(\underline{s}) \cdot \Delta M_2(\underline{s}) + \sum_{\underline{s}<\underline{t}} \Delta M_1(\underline{s}) \cdot \Delta A_2(\underline{s}) + \sum_{\underline{s}<\underline{t}} \Delta A_1(\underline{s}) \cdot \Delta A_2(\underline{s}).$$

Since M_i and A_i are continuous, then

$$\left| \sum_{\underline{s}<\underline{t}} \Delta Z(\underline{s}) \cdot \Delta A_i(\underline{s}) \right| \leq \max_{\underline{s}\in T} |\Delta Z(\underline{s})| |A_i| \approx 0,$$

if we replace Z by M_i or by A_i. Then the last three sums in the formula are infinitesimal, and then we have

$$[Y_1, Y_2] \approx \sum_{\underline{s}<\underline{t}} \Delta M_1(\underline{s}) \cdot \Delta M_2(\underline{s}),$$

and then

$$st([Y_1, Y_2]) = st([M_1, M_2]) = [m_1, m_2] = [y_1, y_2]. \quad \square$$

If x and y are continuous semi-martingales, the Stratonovich integral is defined as

$$S_t = \int_0^t y \partial x = \int_0^t y dx + \frac{1}{2}[x, y],$$

where the right side integral is the Ito integral.

14. Theorem. *Let x and y be continuous $\mathfrak{F}_t$-semi-martingales. Then there exist a $\{\mathfrak{B}_{\underline{t}}\}$ internal filtration and internal semi-martingale liftings X and Y of x and y, respectively, such that*

$$S_t = \int_0^t y \partial x = st\left(\sum_{\underline{s}<\underline{t}} \frac{1}{2}[Y(\underline{s} + \Delta t) + Y(\underline{s})] \Delta X(\underline{s})\right).$$

Proof. Let (x, y) be continuous $\mathfrak{F}_t$-semi-martingales, with canonical representation $x_t = x(0) + m_t + a_t$ for x_t (continuity implies predictability). From Theorem 12 there exists an internal filtration $\{\mathfrak{B}_{\underline{t}}\}$ and an S-continuous semi-martingale lifting (X, Y) of (x, y). Let the canonical decomposition of $X_{\underline{t}}$ be $X_{\underline{t}} = X(0) + M_{\underline{t}} + A_{\underline{t}}$, where $X(0)$ is an internal random variable $\mathfrak{B}_0$ measurable, M is an internal local-martingale and A is an internal process of bounded variation. Also assume that $X(0)$, M and A are liftings of $x(0)$, m and a, respectively. Let $\{V_n\}$ be the internal stopping time reducing X. For $\underline{s} \in {}^*[0, \infty)$ define, for an internal stochastic process H,

$$T_H(\underline{s}) = \min\{\underline{t} \in T : \|H(\underline{t})\| > \underline{s}\}$$

with $\min \emptyset = \infty$. For $\underline{n} \in T$, $\underline{n} \approx n$, define an internal stopping time

$$R_n = V_n \wedge T_M(n) \wedge T_Y(n) \wedge \underline{n}.$$

For $M^*(\underline{t}, w) = \max_{\underline{s} \leq \underline{t}} \|M(\underline{s}, w)\|$ and V an internal stopping time, we have, from Lemma 6.3 (b) in Hoover and Perkins (1983), that $M^*(V)^p$ is S-integrable if and only if $([M, M]_V^{p/2})$ is.

Now, if $R_n > n$ then $M^*(R_n) = \|M(R_n)\|$, and so, for $\gamma \in {}^*\mathbb{N} - \mathbb{N}$, we have

$$^o\int M^*(R_n) I_{\{M^*(R_n)>\gamma\}} d\overline{P} = {}^o\int \|M(R_n)\| I_{\{\|M(R_n)\|>\gamma\}} d\overline{P}.$$

By the Optional Sampling Theorem, the S-integrability of $M(V_n \wedge \underline{n})$ implies the S-integrability of $M(R_n)$, and then we have

$$^o\int M^*(R_n) I_{\{M^*(R_n)>\gamma\}} d\overline{P} = 0$$

so that $M^*(R_n)$ is S-integrable, which is equivalent to say that $([M,M]_{R_n})^{1/2})$ is S-integrable.

Since $[x,x] = st([X,X]) = st([M,M])$ we have, from Theorem 3.2.9. in Albeverio et al (1986) and the above results, that $([x,x])^{1/2}({}^oR_n)$ is integrable. Thus, we obtain

$$E\left(\left(\int_o^{{}^oR_n} \|y(s)\|^2 d[x,x](s)\right)^{1/2}\right) \leq E\left(n([x,x]_{R_n})^{1/2}\right) =$$

$$nE([x,x]^{1/2}({}^oR_n)) < \infty. \qquad (*)$$

On the other hand,

$$\int_0^t \|y(s)\| d|a|_s < \infty, \qquad (**)$$

which holds by the continuity of the integrand. Then $y \in \mathfrak{L}(x,M)$, the space of functions of Definition 1.2.10, and from the Remarks 2 and 3, we finally have that

$$\int_0^t y dx = st\left(\sum_{\underline{s}<\underline{t}} Y(\underline{s}) \cdot \Delta X(\underline{s})\right).$$

Now, from Corollary 13.,

$$[x,y] = st\left(\sum_{\underline{s}<t} \Delta X(\underline{s}) \cdot \Delta Y(\underline{s})\right).$$

Thus

$$S_t = st\left(\sum_{\underline{s}<\underline{t}} Y(\underline{s})\cdot\Delta X(\underline{s})\right) + \frac{1}{2}st\left(\sum_{\underline{s}<\underline{t}}\Delta X(\underline{s})\cdot\Delta Y(\underline{s})\right)$$

$$= st\left(\sum_{\underline{s}<\underline{t}}\left[Y(\underline{s})\cdot\Delta X(\underline{s})) + \frac{1}{2}\Delta X(\underline{s})\cdot\Delta Y(\underline{s})\right]\right)$$

$$= st\left(\sum_{\underline{s}<\underline{t}}\left[Y(\underline{s}) + \frac{1}{2}Y(\underline{s}+\Delta t) - \frac{1}{2}Y(\underline{s})\right]\Delta X(\underline{s})\right)$$

$$= st\left(\sum_{\underline{s}<\underline{t}}\frac{1}{2}\left[Y(\underline{s}) + Y(\underline{s}+\Delta t)\right]\Delta X(\underline{s})\right).$$

Observe that if X and Y are S-continuous semi-martingales we can always define
$st(X\circ Y + \frac{1}{2}[X,Y])$, because $[X,Y]$ also is S-continuous (Theorem 14 in Lindstrom (1980)). □

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