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Cointegration Vector Estimation by DOLS for a Three-Dimensional Panel

Estimación de un modelo de cointegración utilizando DOLS para un panel de tres dimensiones

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Abstract

This paper extends the results of the dynamic ordinary least squares cointegration vector estimator available in the literature to a three-dimensional panel. We use a balanced panel of N and M lengths observed over T periods. The cointegration vector is homogeneous across individuals but we allow for individual heterogeneity using different short-run dynamics, individual-specific fixed effects and individual-specific time trends. We also model cross-sectional dependence using time-specific effects. The estimator has a Gaussian sequential limit distribution that is obtained by first letting $T \to \infty$ and then letting $N \to \infty$, $M \to \infty$. The Monte Carlo simulations show evidence that the finite sample properties of the estimator are closely related to the asymptotic ones.

***Key words*:** Cointegration, Multidimensional, Panel Data.

Resumen

Este documento extiende los resultados de los estimadores mínimos cuadrados dinámicos para series cointegradas disponible en la literatura a un panel de tres dimensiones. Se utiliza un panel balanceado de longitudes N y M para un periodo de tiempo de longitud T. El vector de cointegración es homogéneo a través de los individuos; sin embargo, el modelo permite cierto grado de heterogeneidad al usar diferentes dinámicas de corto plazo, efectos fijos y tendencias a niveles individuales. También se utilizan efectos en el

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tiempo para incluir dependencias cruzadas entre los individuos. El estimador tiene una distribución secuencial límite gausiana en la cual primero $T \to \infty$ y posteriormente $N \to \infty$, $M \to \infty$. Simulaciones Monte Carlo muestran evidencia de que las propiedades de muestra finita del estimador son cercanas a las asintóticas.

***Palabras clave*:** cointegración, modelos panel, multidimensional.

1. Introduction

This paper proposes an extension of the dynamic ordinary least squares (DOLS) cointegration panel estimators of Mark & Sul (2003) to a three-dimensional panel. The single equation DOLS for estimating and testing the cointegration hypothesis was proposed by Phillips & Loretan (1991), Saikkonen (1991), and generalized by Stock & Watson (1993).

DOLS is a single-equation cointegration technique that overcomes the common problems of the static and modified OLS. The static OLS finite sample estimates of long-run relationships are potentially biased and inferences cannot be drawn using t-statistics (Banerjee, Hendry & Smith 1986, Kremers, Ericsson & Dolado 1992). DOLS methodology is based on an equation that includes lags and leads of right-hand side variables, which eliminates the effect of the endogeneity of these variables. Therefore, it is possible to construct asymptotically-valid test statistics and also to estimate the long-run relationships.

Panel DOLS (PDOLS) has been analyzed by Kao & Chiang (2000) and Mark & Sul (2003). Kao & Chiang (2000) study the properties of panel DOLS when there are fixed effects in the cointegration regressions. Mark & Sul (2003) allow for individual heterogeneity through different short-run dynamics, individual-specific fixed effects and individual-specific time trends. They also permit a limited degree of cross-sectional dependence through the presence of time-specific effects.

Panel analysis usually employs two dimensions, being time one of them. However, given the great availability of data nowadays two dimensions are not always enough, in these cases, a panel in three dimensions is a relevant option. These methodologies are very useful as they model the heterogeneity of the data in a more rigorous way. Some empirical application of panels in three dimensions can be found in Eilat & Einav (2004), Davies (2006) and Davies, Lahiri & Sheng (2011), among others[1].

For extending the results of Mark & Sul (2003) to a three dimensions setup, we use a balanced panel of three dimensions with lengths N, M and T. The cointegration vector is homogeneous across individuals but we allow for individual heterogeneity using different short-run dynamics, individual-specific fixed effects and individual-specific time trends. Both individual effects are considered in the first two dimensions. As in Mark & Sul (2003), we also model some degree of cross-sectional dependence using time-specific effects. After obtaining the Panel

[1]The three dimensions in Eilat & Einav (2004) are time, country of origin and country of destination, in Davies (2006) and Davies et al. (2011) are individual, forecast horizon and forecast target (period of time).

DOLS estimator in three dimensions, PDOLS-3D, we present the sequential limit distribution of the estimator by first letting $T \to \infty$, then letting $N \to \infty$, $M \to \infty$. Finally, to evaluate the small sample properties of the PDOLS-3D t-tests, Monte Carlo experiments are implemented.

The remainder of the paper is organized as follows. Section 2 describes the cointegration representation for a three-dimensional panel. Section 3 describes the PDOLS-3D estimator. The asymptotic distribution of this estimator is presented in Section 4. A Monte Carlo experiment is implemented in Section 5. Section 6 shows an empirical application of this methodology. Finally, some concluding remarks are presented in Section 7.

2. Representation of a Cointegrated Model in Panel Data in Three Dimensions

Consider the following triangular representation of a cointegrated system for a panel with individuals indexed by $i = 1, \ldots, N$ and $j = 1, \ldots, M^\star$ over time periods $t = 1, \ldots, T$

$$\begin{aligned} y_{ijt} &= \alpha_i^{(N)} + \alpha_j^{(M)} + \lambda_i^{(N)} t + \lambda_j^{(M)} t + \theta_t + \boldsymbol{\gamma}' \boldsymbol{x}_{ijt} + u_{ijt} \\ \boldsymbol{x}_{ijt} &= \boldsymbol{x}_{ijt-1} + \boldsymbol{v}_{ijt} \end{aligned} \tag{1}$$

where $\{y_{ijt}\}$ is the dependent variable integrated of order one, $\{\boldsymbol{x}_{ijt}\}$ is a k-dimensional vector of integrated series of order one and $\{u_{ijt}, \boldsymbol{v}'_{ijt}\}'$ is a covariance stationary error process independent across i and j but possibly dependent across t. In this case, the variables are said to be cointegrated for each member of the panel, with cointegrated vector $\boldsymbol{\gamma}$. Individual heterogeneity is considered through different short-run dynamics, individual-specific fixed effects of the first two dimensions, $\alpha_i^{(N)}$ and $\alpha_j^{(M)}$, and individual-specific time trends in those dimensions, $\lambda_i^{(N)}$ and $\lambda_j^{(M)}$. A limited degree of cross-sectional dependence is also permitted by the presence of time-specific effects, θ_t. In this notation, (N) and (M) indicate the first and second dimension, respectively. On the other hand, N and $M^\star$ indicate the number of individuals in the the first and second dimension, respectively.

3. Panel DOLS Estimator in Three Dimensions

PDOLS methodology is based on the estimation of the following equation

$$y_{ijt} = \alpha_i^{(N)} + \alpha_j^{(M)} + \lambda_i^{(N)} t + \lambda_j^{(M)} t + \theta_t + \boldsymbol{\gamma}' \boldsymbol{x}_{ijt} + \boldsymbol{\delta}_i' \boldsymbol{z}_{ijt} + u_{ijt} \tag{2}$$

where $\boldsymbol{z}_{ijt} = (\Delta \boldsymbol{x}'_{ijt-p}, \ldots, \Delta \boldsymbol{x}'_{ijt}, \ldots, \Delta \boldsymbol{x}'_{ijt+p})'$ is a $(2p+1)k$-dimensional vector of leads and lags of the first differences of the variables $\boldsymbol{x}_{ijt}$. The inclusion of lags and leads eliminates the effect of the endogeneity of these variables. To avoid perfect collinearity, $\alpha_{M^\star}^{(M)} = \lambda_{M^\star}^{(M)} = 0$.

Equation (2) can be expressed as follows,

$$y^\ddagger_{ijt} = \alpha_i^{(N)} + \alpha_j^{(M)} + \lambda_i^{(N)} t + \lambda_j^{(M)} t + \theta_t + \boldsymbol{\gamma}' \boldsymbol{x}^\ddagger_{ijt} + u_{ijt} \tag{3}$$

where $y^{\ddagger}_{ijt}$ and $\boldsymbol{x}^{\ddagger}_{ijt}$ represent the linear projection of the dependent variable and the variables $\boldsymbol{x}_{ijt}$ with respect to the short run components, $\boldsymbol{z}_{ijt}$.

Taking average of (3) in the time dimension gives

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T} y^{\ddagger}_{ijt} &= \alpha_i^{(N)} + \alpha_j^{(M)} + \left(\lambda_i^{(N)} + \lambda_j^{(M)}\right)\left[\frac{(T+1)}{2}\right] \\ &+ \boldsymbol{\gamma}'\left[\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{x}^{\ddagger}_{ijt}\right] + \frac{1}{T}\sum_{t=1}^{T}\theta_t + \frac{1}{T}\sum_{t=1}^{T}u_{ijt}\end{aligned} \tag{4}$$

Subtracting (4) from (3) eliminates $\alpha_i^{(N)}$ and $\alpha_j^{(M)}$, and gives

$$\begin{aligned}y^{\ddagger}_{ijt} - \frac{1}{T}\sum_{s=1}^{T} y^{\ddagger}_{ijs} &= \left(\lambda_i^{(N)} + \lambda_j^{(M)}\right)\left[t - \frac{(T+1)}{2}\right] + \boldsymbol{\gamma}'\left[\boldsymbol{x}^{\ddagger}_{ijt} - \frac{1}{T}\sum_{s=1}^{T}\boldsymbol{x}^{\ddagger}_{ijs}\right] \\ &+ \left[\theta_t - \frac{1}{T}\sum_{s=1}^{T}\theta_s\right] + \left[u_{ijt} - \frac{1}{T}\sum_{s=1}^{T}u_{ijs}\right]\end{aligned} \tag{5}$$

Taking double average of equation (5) in the first two dimensions gives the following result

$$\begin{aligned}&\frac{1}{NM^\star}\sum_{i=1}^{N}\sum_{j=1}^{M^\star} y^{\ddagger}_{ijt} - \frac{1}{NM^\star T}\sum_{i=1}^{N}\sum_{j=1}^{M^\star}\sum_{s=1}^{T} y^{\ddagger}_{ijs} \\ &\quad = \left[t - \frac{(T+1)}{2}\right]\frac{1}{NM^\star}\sum_{i=1}^{N}\sum_{j=1}^{M^\star}\left(\lambda_i^{(N)} + \lambda_j^{(M)}\right) \\ &\quad + \frac{\boldsymbol{\gamma}'}{NM^\star}\sum_{i=1}^{N}\sum_{j=1}^{M^\star}\left[\boldsymbol{x}^{\ddagger}_{ijt} - \frac{1}{T}\sum_{s=1}^{T}\boldsymbol{x}^{\ddagger}_{ijs}\right] + \theta_t - \frac{1}{T}\sum_{s=1}^{T}\theta_s \\ &\quad + \frac{1}{NM^\star}\sum_{i=1}^{N}\sum_{j=1}^{M^\star}\left[u_{ijt} - \frac{1}{T}\sum_{s=1}^{T}u_{ijs}\right]\end{aligned} \tag{6}$$

Subtracting (6) from (5) eliminates the common time effects, then the model can be rewritten as

$$y^{\ddagger *}_{ijt} = \left(\widetilde{\lambda}_i^{(N)} + \widetilde{\lambda}_j^{(M)}\right)\widetilde{t} + \boldsymbol{\gamma}'\boldsymbol{x}^{\ddagger *}_{ijt} + u^{*}_{ijt} \tag{7}$$

Where the superscripts '$*$' and '$\sim$' denote the following deviations

$$
\begin{aligned}
y_{ijt}^{\ddagger *} &= \left(y_{ijt}^{\ddagger} - \frac{1}{T}\sum_{s=1}^{T} y_{ijs}^{\ddagger} \right) \\
&\quad - \left(\frac{1}{NM^{\star}}\sum_{n=1}^{N}\sum_{m=1}^{M^{\star}} y_{nmt}^{\ddagger} - \frac{1}{NM^{\star}T}\sum_{n=1}^{N}\sum_{m=1}^{M^{\star}}\sum_{s=1}^{T} y_{nms}^{\ddagger} \right), \\
\boldsymbol{x}_{ijt}^{\ddagger *} &= \left(\boldsymbol{x}_{ijt}^{\ddagger} - \frac{1}{T}\sum_{s=1}^{T} \boldsymbol{x}_{ijs}^{\ddagger} \right) \\
&\quad - \left(\frac{1}{NM^{\star}}\sum_{n=1}^{N}\sum_{m=1}^{M^{\star}} \boldsymbol{x}_{nmt}^{\ddagger} - \frac{1}{NM^{\star}T}\sum_{n=1}^{N}\sum_{m=1}^{M^{\star}}\sum_{s=1}^{T} \boldsymbol{x}_{nms}^{\ddagger} \right), \\
u_{ijt}^{*} &= \left(u_{ijt} - \frac{1}{T}\sum_{s=1}^{T} u_{ijs} \right) \\
&\quad - \left(\frac{1}{NM^{\star}}\sum_{n=1}^{N}\sum_{m=1}^{M^{\star}} u_{nmt} - \frac{1}{NM^{\star}T}\sum_{n=1}^{N}\sum_{m=1}^{M^{\star}}\sum_{s=1}^{T} u_{nms} \right), \\
\widetilde{\lambda}_i^{(N)} &= \lambda_i^{(N)} - \frac{1}{N}\sum_{n=1}^{N} \lambda_n^{(N)}, \\
\widetilde{\lambda}_j^{(M)} &= \lambda_j^{(M)} - \frac{1}{M^{\star}}\sum_{m=1}^{M^{\star}} \lambda_m^{(M)}, \\
\widetilde{t} &= t - \frac{(T+1)}{2}.
\end{aligned}
$$

Let us define the grand coefficient vector of the model as $\boldsymbol{\beta}' = \left(\boldsymbol{\gamma}', \widetilde{\boldsymbol{\lambda}}'_N, \widetilde{\boldsymbol{\lambda}}'_M\right)$ where $\widetilde{\boldsymbol{\lambda}}'_N = \left(\widetilde{\lambda}_1^{(N)}, \widetilde{\lambda}_2^{(N)}, \dots, \widetilde{\lambda}_N^{(N)}\right)$, $\widetilde{\boldsymbol{\lambda}}'_M = \left(\widetilde{\lambda}_1^{(M)}, \widetilde{\lambda}_2^{(M)}, \dots, \widetilde{\lambda}_M^{(M)}\right)$, $M = M^{\star} - 1$, and the following matrices

$$
\begin{aligned}
\boldsymbol{q}_{11t}^{\ddagger *} &= \left(\boldsymbol{x}_{11t}^{\ddagger *\prime}, \quad \widetilde{t}, 0, \dots, 0, \widetilde{t}, 0, \dots, 0 \right)' \\
\boldsymbol{q}_{12t}^{\ddagger *} &= \left(\boldsymbol{x}_{12t}^{\ddagger *\prime}, \quad \widetilde{t}, 0, \dots, 0, 0, \widetilde{t}, \dots, 0 \right)' \\
&\ \ \vdots \qquad\qquad\qquad \vdots \\
\boldsymbol{q}_{1Mt}^{\ddagger *} &= \left(\boldsymbol{x}_{1Mt}^{\ddagger *\prime}, \quad \widetilde{t}, 0, \dots, 0, 0, 0, \dots, \widetilde{t} \right)' \\
\boldsymbol{q}_{21t}^{\ddagger *} &= \left(\boldsymbol{x}_{21t}^{\ddagger *\prime}, \quad 0, \widetilde{t}, \dots, 0, \widetilde{t}, 0, \dots, 0 \right)' \\
&\ \ \vdots \qquad\qquad\qquad \vdots \\
\boldsymbol{q}_{2Mt}^{\ddagger *} &= \left(\boldsymbol{x}_{2Mt}^{\ddagger *\prime}, \quad 0, \widetilde{t}, \dots, , 0, 0, 0, \dots, \widetilde{t} \right)' \\
&\ \ \vdots \qquad\qquad\qquad \vdots \\
\boldsymbol{q}_{NMt}^{\ddagger *} &= \left(\boldsymbol{x}_{NMt}^{\ddagger *\prime}, \quad 0, 0, \dots, \widetilde{t}, 0, 0, \dots, \widetilde{t} \right)'
\end{aligned}
\tag{8}
$$

Then, the model can finally be expressed as

$$y_{ijt}^{\ddagger *} \;=\; \boldsymbol{\beta}' q_{ijt}^{\ddagger *} \;+\; u_{ijt}^{\ddagger *} \tag{9}$$

And the PDOLS-3D estimator of $\boldsymbol{\beta}$ is

$$\hat{\boldsymbol{\beta}}_{NMT} = \left[\sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{t=1}^{T} \boldsymbol{q}_{ijt}^{\ddagger *}\boldsymbol{q}_{ijt}^{\ddagger *'}\right]^{-1} \left[\sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{t=1}^{T} \boldsymbol{q}_{ijt}^{\ddagger *} y_{ijt}^{\ddagger *}\right] \tag{10}$$

4. Asymptotic Distribution of the PDOLS-3D Estimator

Taking into account that elements in $\hat{\boldsymbol{\beta}}_{NMT}$ have different rates of convergence, we can rewrite (10) as

$$\mathbf{G}_{NMT}(\hat{\boldsymbol{\beta}}_{NMT} - \boldsymbol{\beta}) = \left[\mathbf{M}_{NMT}\right]^{-1} \boldsymbol{m}_{NMT}$$

Where

$$\mathbf{G}_{NMT} = \begin{bmatrix} \sqrt{NM}T\mathbf{I}_k & \mathbf{0}' & \mathbf{0}' \\ \mathbf{0} & \sqrt{M}T^{\frac{3}{2}}\mathbf{I}_N & \mathbf{0}' \\ \mathbf{0} & \mathbf{0} & \sqrt{N}T^{\frac{3}{2}}\mathbf{I}_M \end{bmatrix},$$

$$\begin{aligned} \mathbf{M}_{NMT} &= \left[\mathbf{G}_{NMT}^{-1} \sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{t=1}^{T} (\boldsymbol{q}_{ijt}^{\ddagger *}\boldsymbol{q}_{ijt}^{\ddagger *'})\mathbf{G}_{NMT}^{-1}\right] \\ &= \begin{bmatrix} \mathbf{M}_{11,NMT} & \mathbf{M}'_{21,NMT} & \mathbf{M}'_{31,NMT} \\ \mathbf{M}_{21,NMT} & \mathbf{M}_{22,NMT} & \mathbf{M}'_{32,NMT} \\ \mathbf{M}_{31,NMT} & \mathbf{M}_{32,NMT} & \mathbf{M}_{33,NMT} \end{bmatrix}, \end{aligned}$$

$$\mathbf{M}_{11,NMT} = \frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\left[\frac{1}{T^2}\sum_{t=1}^{T} \boldsymbol{x}_{ijt}^{\ddagger *}\boldsymbol{x}_{ijt}^{\ddagger *'}\right],$$

$$\mathbf{M}'_{21,NMT} = \left[\frac{1}{\sqrt{N}MT^{\frac{5}{2}}}\sum_{j=1}^{M}\sum_{t=1}^{T} \widetilde{t}\boldsymbol{x}_{1jt}^{\ddagger *} \quad \cdots \quad \frac{1}{\sqrt{N}MT^{\frac{5}{2}}}\sum_{j=1}^{M}\sum_{t=1}^{T} \widetilde{t}\boldsymbol{x}_{Njt}^{\ddagger *}\right],$$

$$\mathbf{M}'_{31,NMT} = \left[\frac{1}{N\sqrt{M}T^{\frac{5}{2}}}\sum_{i=1}^{N}\sum_{t=1}^{T} \widetilde{t}\boldsymbol{x}_{i1t}^{\ddagger *} \quad \cdots \quad \frac{1}{N\sqrt{M}T^{\frac{5}{2}}}\sum_{i=1}^{N}\sum_{t=1}^{T} \widetilde{t}\boldsymbol{x}_{iMt}^{\ddagger *}\right],$$

$$\mathbf{M}_{22,NMT} = \left[\frac{1}{T^3}\sum_{t=1}^{T}\widetilde{t}^{\,2}\right]\mathbf{I}_N,$$

$$\mathbf{M}'_{32,NMT} = \left[\frac{1}{\sqrt{NM}T^3}\sum_{t=1}^{T}\widetilde{t}^{\,2}\right](\mathbf{1})_{N\times M};$$

where $(\mathbf{1})_{N\times M}$ is a matrix of ones,

$$\mathbf{M}_{33,NMT} = \left[\frac{1}{T^3}\sum_{t=1}^{T}\widetilde{t}^{\,2}\right]\mathbf{I}_M,$$

$$\begin{aligned}\boldsymbol{m}_{NMT} &= \mathbf{G}_{NMT}^{-1}\sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{t=1}^{T}\boldsymbol{q}_{ijt}^{\ddagger *}u_{ijt}^{*}\\ &= \begin{bmatrix}\boldsymbol{m}_{1,NMT}\\ \boldsymbol{m}_{2,NMT}\\ \boldsymbol{m}_{3,NMT}\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{NM}}\sum\limits_{i=1}^{N}\sum\limits_{j=1}^{M}\left[\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{x}_{ijt}^{\ddagger *}u_{ijt}^{\ddagger *}\right]\\ \frac{1}{\sqrt{M}T^{\frac{3}{2}}}\sum\limits_{j=1}^{M}\sum\limits_{t=1}^{T}\widetilde{t}u_{1jt}^{*}\\ \vdots\\ \frac{1}{\sqrt{M}T^{\frac{3}{2}}}\sum\limits_{j=1}^{M}\sum\limits_{t=1}^{T}\widetilde{t}u_{Njt}^{*}\\ \frac{1}{\sqrt{N}T^{\frac{3}{2}}}\sum\limits_{i=1}^{N}\sum\limits_{t=1}^{T}\widetilde{t}u_{i1t}^{*}\\ \vdots\\ \frac{1}{\sqrt{N}T^{\frac{3}{2}}}\sum\limits_{i=1}^{N}\sum\limits_{t=1}^{T}\widetilde{t}u_{iMt}^{*}\end{bmatrix}\end{aligned}$$

The asymptotic distribution of the PDOLS-3D estimator is presented in proposition 1 part (ii). The following lemmas are required to prove this proposition. The proofs of the lemmas follow from simple extensions of the results of Mark & Sul (2002). Nevertheless, they are presented in Appendix A, B and C [2]

Following the results of Mark & Sul (2003), a linear hypothesis of the form $\boldsymbol{R\gamma} = \boldsymbol{r}$ can be tested using regular Wald statistics. Let, $\boldsymbol{R}$ a $r \times k$ known matrix and $\mathbf{r}$ a $r \times 1$ known vector.

[2]Following the results of Phillips & Moon (1999) and White (2001) and under the assumptions $\frac{N}{T} \to 0$, $\frac{M}{T} \to 0$ and $\frac{N}{M} \to 1$, we obtain similar asymptotic results when considering joint convergence in the three dimensions ($N \to \infty$, $M \to \infty$ and. $T \to \infty$) instead of sequential convergence.

Lemma 1. *For each i and j as $T \to \infty$,*

(i) $\frac{1}{T^2}\sum_{t=1}^{T} \boldsymbol{x}_{ijt}^{\ddagger *}\boldsymbol{x}_{ijt}^{\ddagger *\prime} - \frac{1}{T^2}\sum_{t=1}^{T} \boldsymbol{x}_{ijt}^{*}\boldsymbol{x}_{ijt}^{*\prime} \xrightarrow{p} \boldsymbol{0}$.

(ii) $\frac{1}{T^{5/2}}\sum_{t=1}^{T} \widetilde{t}\boldsymbol{x}_{ijt}^{\ddagger *} - \frac{1}{T^{5/2}}\sum_{t=1}^{T} t\boldsymbol{x}_{ijt}^{*} \xrightarrow{p} \boldsymbol{0}$.

(iii) $\frac{1}{T}\sum_{t=1}^{T} \boldsymbol{x}_{ijt}^{\ddagger *}u_{ijt}^{*} - \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{x}_{ijt}^{*}u_{ijt}^{*} \xrightarrow{p} \boldsymbol{0}$.

This lemma demonstrates the equivalence in probability of the projected series in the z_{ijt} space and the series which are not projected. This gives an asymptotic justification for ignoring the fact that we are using projection errors instead of the original observations.

Lemma 2. *As $T \to \infty$ and then $N \to \infty$, $M \to \infty$,*

(i) $\boldsymbol{M}_{11,NMT} - \frac{1}{6NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\boldsymbol{\Omega}_{vv,ij} \xrightarrow{p} \boldsymbol{0}$. *Where $\boldsymbol{\Omega}_{vv,ij}$ is associated with the covariance matrix of $\boldsymbol{v}_{ijt}$.*

(ii) $\boldsymbol{M}_{21,NMT}' \xrightarrow{p} \boldsymbol{0}$

(iii) $\boldsymbol{M}_{31,NMT}' \xrightarrow{p} \boldsymbol{0}$

(iv) $\boldsymbol{M}_{22,NMT} \xrightarrow{p} \frac{1}{12}\boldsymbol{I}_N$

(v) $\boldsymbol{M}_{32,NMT}' \xrightarrow{p} \boldsymbol{0}$

(vi) $\boldsymbol{M}_{33,NMT} \xrightarrow{p} \frac{1}{12}\boldsymbol{I}_M$

This lemma shows the convergence of each element in the $\mathbf{M}_{NMT}$ matrix.

Lemma 3. *(i) For N and M fixed, with $T \to \infty$, $\boldsymbol{m}_{1,NMT} \xrightarrow{p} \boldsymbol{m}_{1,NM}$ where*

$$\boldsymbol{m}_{1,NM} = \left[\frac{NM-1}{NM}\right]\frac{1}{\sqrt{NM}}\sum_{i=1}^{N}\sum_{j=1}^{M}\sqrt{\Omega_{uu,ij}}\int \tilde{\boldsymbol{B}}_{vij}dW_{uij} - \frac{1}{NM}O_{NM}(1)$$

(ii) As $T \to \infty$ then $N \to \infty$, $M \to \infty$, $\boldsymbol{V}_{NM}^{-\frac{1}{2}}\boldsymbol{m}_{1,NM} \xrightarrow{D} N(0,\boldsymbol{I})$ where $\boldsymbol{V}_{NM} = \frac{1}{6NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\Omega_{uu,ij}\boldsymbol{\Omega}_{vv,ij}$ and $\Omega_{uu,ij}$ is associated with the variance of u_{ijt}.

(iii) As $T \to \infty$ and $N \to \infty$, $M \to \infty$, $\boldsymbol{m}_{1,NMT}$ is independent of $\boldsymbol{m}_{2,NMT}$ and $\boldsymbol{m}_{3,NMT}$

This lemma shows the convergence in distribution of $\boldsymbol{m}_{NM}$.

Proposition 1. *For the PDOLS-3D estimator in (2), as $T \to \infty$ and then $N \to \infty$, $M \to \infty$,* *(i)* $\sqrt{NM}T(\hat{\boldsymbol{\gamma}}_{NMT} - \boldsymbol{\gamma})$ *is independent of* $\sqrt{M}T^{\frac{3}{2}}(\hat{\boldsymbol{\lambda}}_N - \boldsymbol{\lambda}_N)$ *and* $\sqrt{N}T^{\frac{3}{2}}(\hat{\boldsymbol{\lambda}}_M - \boldsymbol{\lambda}_M)$.

(ii) $\boldsymbol{C}_{NM}^{-\frac{1}{2}}\sqrt{NM}T(\hat{\boldsymbol{\gamma}}_{NMT} - \boldsymbol{\gamma}) \overset{A}{\sim} N(\boldsymbol{0}, \boldsymbol{I}_K)$.

where

$$\boldsymbol{C}_{NM} = \left(\boldsymbol{C}_{NM}^{\frac{1}{2}}\right)\left(\boldsymbol{C}_{NM}^{\frac{1}{2}}\right)' = \overline{\boldsymbol{M}}_{11,NM}^{-1}\overline{\boldsymbol{V}}_{11,NM}\overline{\boldsymbol{M}}_{11,NM}^{-1}$$

$$\overline{\boldsymbol{M}}_{11,NM} = \frac{1}{6NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\boldsymbol{\Omega}_{vv,ij}$$

$$\overline{\boldsymbol{V}}_{11,NM} = \frac{1}{6NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\Omega_{uu,ij}\boldsymbol{\Omega}_{vv,ij}$$

(iii) $\widehat{\boldsymbol{D}}_{NMT} - \boldsymbol{C}_{NM} \overset{p}{\to} 0$.

where

$$\widehat{\boldsymbol{D}}_{NMT} = \boldsymbol{M}_{11,NMT}^{-1}\widehat{\boldsymbol{V}}_{11,NMT}\boldsymbol{M}_{11,NMT}^{-1}$$

$$\boldsymbol{M}_{11,NMT} = \frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\left[\frac{1}{T^2}\sum_{t=1}^{T}\boldsymbol{x}_{ijt}^{\ddagger*}\boldsymbol{x}_{ijt}^{\ddagger*\prime}\right]$$

$$\widehat{\boldsymbol{V}}_{11,NMT} = \frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\sqrt{\widehat{\Omega}_{uu,ij}}\left[\frac{1}{T^2}\sum_{t=1}^{T}\boldsymbol{x}_{ijt}^{\ddagger*}\boldsymbol{x}_{ijt}^{\ddagger*\prime}\right]$$

and $\widehat{\Omega}_{uu,ij}$ *is a consistent estimator of* $\Omega_{uu,ij}$.

This proposition presents the sequential limit distribution of the PDOLS-3D estimator. The proof of part (i) follows from Lemma 2 and Lemma 3.(iii), the proof of part (ii) follows from Lemma 2 and Lemma 3.(ii). The proof of part (iii) is straightforward.

5. Monte Carlo Experiments

This section summarizes the Monte Carlo experiments used for evaluating some small sample properties of the PDOLS-3D estimator derived in Section 3. The design of the experiment follows the structure presented in Mark & Sul (2003). The Data Generating Process (DGP) includes two regressors in the cointegration relation, and is defined as follows:

$$\begin{aligned} y_{ijt} &= \alpha_i + \alpha_j + \gamma_1 x_{1,ijt} + \gamma_2 x_{2,ijt} + \mu_{ijt} \\ \Delta x_{1,ijt} &= \nu_{1,ijt} \\ \Delta x_{2,ijt} &= \nu_{2,ijt} \end{aligned}$$

The short run dynamics are given by:

$$\boldsymbol{\omega}_{ijt} = \mathbf{A}_{ij}\boldsymbol{\omega}_{ijt-1} + \boldsymbol{\epsilon}_{ijt}$$
$$\boldsymbol{\epsilon}_{ijt} = \sqrt{\phi}\boldsymbol{\theta}_t + \sqrt{1-\phi}\boldsymbol{e}_{ijt}$$

with $\boldsymbol{\omega}_{ijt} = (\mu_{ijt}, \nu_{1,ijt}, \nu_{2,ijt})'$, $\boldsymbol{\epsilon}_{ijt} = (\epsilon_{1,ijt}, \epsilon_{2,ijt}, \epsilon_{3,ijt})'$, $\boldsymbol{e}_{ijt} = (e_{1,ijt}, e_{2,ijt}, e_{3,ijt})'$, $\boldsymbol{e}_{ijt} \sim N_3\left[\mathbf{0}, \operatorname{diag}\left(\sigma^2_{1,ij}, \quad \sigma^2_{2,ij}, \sigma^2_{3,ij}\right)\right]$, $\boldsymbol{\theta}_t = (\theta_{1t}, \theta_{2t}, \theta_{3t})'$, $\boldsymbol{\theta}_t \sim N_3[\mathbf{0}, \operatorname{diag}\left(\sigma^2_{\theta_1}, \sigma^2_{\theta_2}, \sigma^2_{\theta_3}\right)]$, for $i = 1, \ldots, N$, $j = 1, \ldots, M$, where $\operatorname{diag}(a_1, \ldots, a_n)$ is an $n \times n$ diagonal matrix which diagonal elements are $a_1, \ldots, a_n$; and $\mathbf{A}_{ij}$ is a 3×3 matrix whose hr-th element is $A_{hr,ij}$ for $h, r = 1, 2, 3$.

The design of the DGP is also related to the empirical work on the Colombian factor productivity by Iregui, Melo & Ramírez (2007), where the regressors $x_{1,ijt}$ and $x_{2,ijt}$ represent the level of capital stock and labor force, respectively, for the i-th industrial sector and j-th metropolitan area in year t. As can be seen in the specification of the DGP, both regressors are driftless $I(1)$ processes. The equilibrium error is modeled to allow for a general form of Cross-Sectional Dependence (CSD), where the CSD is induced by θ_{1t}, while ϕ controls the degree of Cross-Sectional Dependence. The parameters θ_{2t} and θ_{3t} cause cross-sectional endogeneity between the regressors and the equilibrium error.

The cointegration vector is simulated as $(\gamma_1, \gamma_2) = (0.15, 0.85)$. The values $A_{11,ij}$ and σ_{hij}, for each i and j, are extracted from a uniform distribution, and are kept constant throughout the experiment. Different levels of persistence in the short-run dynamics are obtained by varying the limits of the uniform distribution from which the elements of $\mathbf{A}_{ij}$ are drawn. Three levels of persistence are considered: low, $A_{11,ij} \sim U_{[0.3,0.5]}$, medium, $A_{11,ij} \sim U_{[0.5,0.7]}$, and high, $A_{11,ij} \sim U_{[0.7,0.9]}$. Additionally, different values of ϕ are used in the simulations: $\phi = 0$ for no CSD, $\phi = 0.3$ for low CSD, and $\phi = 0.7$ for high CSD.

Other parameters used in the simulation are as follows: $A_{21,ij} \sim U_{[\text{-}0.05,0.5]}$, $A_{12,ij} \sim U_{[\text{-}0.05,0.5]}$, $A_{13,ij} \sim U_{[\text{-}0.05,0.5]}$, $A_{31,ij} \sim U_{[\text{-}0.05,0.5]}$, $A_{23,ij} \sim U_{[\text{-}0.05,0.5]}$, $A_{32,ij} \sim U_{[\text{-}0.05,0.5]}$, $A_{22,ij} \sim U_{[0.0,0.4]}$, $A_{33,ij} \sim U_{[0.0,0.4]}$, $\sigma^2_{1,ij} \sim U_{[1\times10^{-3},53\times10^{-3}]}$, $\sigma^2_{2,ij} \sim U_{[0.25\times10^{-3},1.34\times10^{-3}]}$, $\sigma^2_{3,ij} \sim U_{[2.3\times10^{-3},57\times10^{-3}]}$; and $\sigma^2_{\theta_1} = 1.8$, $\sigma^2_{\theta_2} = 0.645$, $\sigma^2_{\theta_3} = 2.0$ for $i = 1, \ldots, N$, $j = 1, \ldots, M$. The prewhitened quadratic spectral methodology proposed by Sul, Phillips & Choi (2005) was used for estimating the long-run variances $\Omega_{uu,ij}$. The values of the individual-specific fixed effects of the first two dimensions, α_i and α_j, are taken from the PDOLS-3D estimation with the data in Iregui et al. (2007). The simulation structure includes 1,000 samples of size $N = 9$, $M = 18$, and $T = 50$ or $T = 100$ or $T = 150$. The number of leads and lags of $\Delta\boldsymbol{x}_{ijt}$ included in the PDOLS-3D estimation is taken as two. Then, three cases are evaluated:

Case 1: No CSD ($\phi = 0$), with low, medium and high persistence levels.

Case 2: Homogeneous CSD ($A_{11,ij} = A_{11,lk}$, for $i, l = 1, \ldots, N, j, k = 1, \ldots, M$), with low and high CSD degrees ($\phi = 0.3$, $\phi = 0.7$); and with low, medium and high persistence levels.

Case 3: Heterogeneous CSD ($A_{11,ij} \neq A_{11,lk}$, for $i, l = 1, \dots, N$, $j, k = 1, \dots, M$), with low and high CSD degrees ($\phi = 0.3$, $\phi = 0.7$); and with low, medium and high persistence levels.

Tables 1 to 9 report the results of the simulation experiments described in cases 1 to 3 for $N = 18$ and $M = 9$. Tables 1, 4 and 7 present the simulations with $T = 50$, Tables 2, 5 and 8 for $T = 100$, and Tables 3, 6 and 9 for $T = 150$.

Table 1: Effective size of PDOLS-3D tests for Case 1, with $T = 50$.

Persistence	Test	Nominal size 5%	Nominal size 10%
Low	$H_0 : \gamma_1 = 0.15$	0.033	0.084
	$H_0 : \gamma_2 = 0.85$	0.049	0.096
Medium	$H_0 : \gamma_1 = 0.15$	0.030	0.076
	$H_0 : \gamma_2 = 0.85$	0.038	0.090
High	$H_0 : \gamma_1 = 0.15$	0.013	0.034
	$H_0 : \gamma_2 = 0.85$	0.015	0.036

The effective size results for Case 1 with 5% and 10% nominal-sized tests and $T = 50$ are presented in Table 1. Under low levels of persistence, the tests' effective sizes are fairly accurate; nevertheless, the results for $H_0 : \gamma_2 = 0.85$ are slightly better than those for $H_0 : \gamma_1 = 0.15$. For medium levels of persistence, there is a loss of size accuracy of the tests relative to low persistence levels, in nominal sizes of both 10% and 5%. For high levels of persistence, the test sizes for both γ_1 and γ_2 are notably smaller than their nominal sizes.

Table 2 shows the results obtained for Case 1 with $T = 100$. Under medium and high levels of persistence, the effective sizes are closer to the nominal sizes than they were when $T = 50$. When $T = 100$ the tests for low levels of persistence are not as accurate as those for $T = 50$. However, the results are not very different. The results for Case 1 with $T = 150$, presented in Table 3, are very similar to the ones obtained for $T = 100$.

Table 2: Effective size of PDOLS-3D tests for Case 1, with $T = 100$.

Persistence	Test	Nominal size 5%	Nominal size 10%
Low	$H_0 : \gamma_1 = 0.15$	0.033	0.082
	$H_0 : \gamma_2 = 0.85$	0.058	0.115
Medium	$H_0 : \gamma_1 = 0.15$	0.035	0.081
	$H_0 : \gamma_2 = 0.85$	0.053	0.108
High	$H_0 : \gamma_1 = 0.15$	0.032	0.067
	$H_0 : \gamma_2 = 0.85$	0.038	0.080

The size results of PDOLS-3D tests for Case 2, with $T = 50$ are shown in Table 4. When the CSD degree is low, the test for γ_2 is accurate at low levels of persistence, and effective size becomes smaller when persistence increases to medium and high levels. For γ_1, the test is mis-sized when the persistence is low, and its effective size decreases when the persistence reaches medium and high levels.

Table 3: Effective size of PDOLS-3D tests for Case 1, with $T = 150$.

Persistence	Test	Nominal size 5%	Nominal size 10%
Low	$H_0 : \gamma_1 = 0.15$	0.034	0.080
	$H_0 : \gamma_2 = 0.85$	0.052	0.097
Medium	$H_0 : \gamma_1 = 0.15$	0.033	0.082
	$H_0 : \gamma_2 = 0.85$	0.050	0.098
High	$H_0 : \gamma_1 = 0.15$	0.028	0.071
	$H_0 : \gamma_2 = 0.85$	0.039	0.084

Table 4: Effective size of PDOLS-3D tests for Case 2, with $T = 50$.

Homogeneous CSD	Persistence	Test	Nominal size 5%	Nominal size 10%
Low	Low	$H_0 : \gamma_1 = 0.15$	0.034	0.065
		$H_0 : \gamma_2 = 0.85$	0.045	0.089
	Medium	$H_0 : \gamma_1 = 0.15$	0.034	0.062
		$H_0 : \gamma_2 = 0.85$	0.042	0.082
	High	$H_0 : \gamma_1 = 0.15$	0.025	0.057
		$H_0 : \gamma_2 = 0.85$	0.027	0.059
High	Low	$H_0 : \gamma_1 = 0.15$	0.112	0.150
		$H_0 : \gamma_2 = 0.85$	0.177	0.230
	Medium	$H_0 : \gamma_1 = 0.15$	0.088	0.154
		$H_0 : \gamma_2 = 0.85$	0.121	0.197
	High	$H_0 : \gamma_1 = 0.15$	0.065	0.114
		$H_0 : \gamma_2 = 0.85$	0.082	0.138

In Case 2 under high CSD for the simulations presented in Table 4, the tests are, in general, not as well sized as they were in low CSD. For example, for γ_2 the effective sizes in low and medium levels of persistence are not as accurate as under low CSD. Even though the tests are highly mis-sized for some cases, the test accuracy improves when the level of persistence increases.

Table 5 presents the test sizes for case 2 with $T = 100$. Under low CSD, the test results generally improve with respect to those presented in Table 4, specially for γ_1. Under high CSD, the increase in the time dimension improves the accuracy of the tests under low, medium, and some cases of high persistence levels. The simulations for $T = 150$ produce even better results, as is showed in Table 6.

The results for Case 3 with $T = 50$ are reported in Table 7. Size results are closer to the nominal levels in the presence of low CSD than in the presence of high CSD, for low and medium persistence levels. Additionally, as in the previous cases described in Tables 1 and 4, size decreases when persistence approaches higher levels. This leads to small and mis-sized tests for γ_1 in high persistence levels. It is also important to note that, in most of the cases, the results of Case 3 are not very different from those obtained in Case 2.

Table 5: Effective size of PDOLS-3D tests for Case 2, with $T = 100$.

Homogeneous CSD	Persistence	Test	Nominal size 5%	10%
Low	Low	$H_0 : \gamma_1 = 0.15$	0.042	0.088
		$H_0 : \gamma_2 = 0.85$	0.046	0.093
	Medium	$H_0 : \gamma_1 = 0.15$	0.038	0.082
		$H_0 : \gamma_2 = 0.85$	0.043	0.092
	High	$H_0 : \gamma_1 = 0.15$	0.028	0.072
		$H_0 : \gamma_2 = 0.85$	0.038	0.075
High	Low	$H_0 : \gamma_1 = 0.15$	0.100	0.162
		$H_0 : \gamma_2 = 0.85$	0.105	0.168
	Medium	$H_0 : \gamma_1 = 0.15$	0.090	0.149
		$H_0 : \gamma_2 = 0.85$	0.093	0.156
	High	$H_0 : \gamma_1 = 0.15$	0.074	0.130
		$H_0 : \gamma_2 = 0.85$	0.079	0.133

Table 6: Effective size of PDOLS-3D tests for Case 2, with $T = 150$.

Homogeneous CSD	Persistence	Test	Nominal size 5%	10%
Low	Low	$H_0 : \gamma_1 = 0.15$	0.048	0.098
		$H_0 : \gamma_2 = 0.85$	0.041	0.088
	Medium	$H_0 : \gamma_1 = 0.15$	0.050	0.094
		$H_0 : \gamma_2 = 0.85$	0.042	0.084
	High	$H_0 : \gamma_1 = 0.15$	0.045	0.090
		$H_0 : \gamma_2 = 0.85$	0.038	0.082
High	Low	$H_0 : \gamma_1 = 0.15$	0.081	0.146
		$H_0 : \gamma_2 = 0.85$	0.090	0.145
	Medium	$H_0 : \gamma_1 = 0.15$	0.072	0.139
		$H_0 : \gamma_2 = 0.85$	0.075	0.137
	High	$H_0 : \gamma_1 = 0.15$	0.062	0.118
		$H_0 : \gamma_2 = 0.85$	0.066	0.121

Size results for Case 3 and $T = 100$ are shown in Table 8. When the CSD degree is low, there are gains in accuracy for the γ_1 tests in all persistence levels compared with simulations of Table 7. However, there are some cases with no improvement for γ_2. These gains are also obtained under high CSD for most of the cases. For $T = 150$ (Table 9), size distortions are, in general, smaller.

In conclusion, there are four relevant observations related to the simulation experiments. First, nominal and effective sizes are, in general, close enough. Second, persistence levels and size are negatively related; as the level of persistence is increased, effective size systematically decreases. Third, the results of the effective size in Cases 2 and 3 are relatively similar, which indicates that subtracting the cross-sectional average is an effective way to control CSD, even in the presence of heterogeneous CSD. Finally, as expected, increasing the time dimension, in general, improves the accuracy of the tests.

Table 7: Effective size of PDOLS-3D tests for Case 3, with $T = 50$.

Heterogeneous CSD	Persistence	Test	Nominal size 5%	Nominal size 10%
Low	Low	$H_0 : \gamma_1 = 0.15$	0.036	0.077
		$H_0 : \gamma_2 = 0.85$	0.060	0.113
	Medium	$H_0 : \gamma_1 = 0.15$	0.031	0.070
		$H_0 : \gamma_2 = 0.85$	0.051	0.096
	High	$H_0 : \gamma_1 = 0.15$	0.011	0.029
		$H_0 : \gamma_2 = 0.85$	0.014	0.038
High	Low	$H_0 : \gamma_1 = 0.15$	0.165	0.230
		$H_0 : \gamma_2 = 0.85$	0.140	0.210
	Medium	$H_0 : \gamma_1 = 0.15$	0.129	0.204
		$H_0 : \gamma_2 = 0.85$	0.139	0.194
	High	$H_0 : \gamma_1 = 0.15$	0.030	0.065
		$H_0 : \gamma_2 = 0.85$	0.096	0.148

Table 8: Effective size of PDOLS-3D tests for Case 3, with $T = 100$.

Heterogeneous CSD	Persistence	Test	Nominal size 5%	Nominal size 10%
Low	Low	$H_0 : \gamma_1 = 0.15$	0.035	0.076
		$H_0 : \gamma_2 = 0.85$	0.070	0.136
	Medium	$H_0 : \gamma_1 = 0.15$	0.038	0.075
		$H_0 : \gamma_2 = 0.85$	0.066	0.121
	High	$H_0 : \gamma_1 = 0.15$	0.029	0.068
		$H_0 : \gamma_2 = 0.85$	0.052	0.082
High	Low	$H_0 : \gamma_1 = 0.15$	0.109	0.179
		$H_0 : \gamma_2 = 0.85$	0.104	0.184
	Medium	$H_0 : \gamma_1 = 0.15$	0.095	0.149
		$H_0 : \gamma_2 = 0.85$	0.075	0.159
	High	$H_0 : \gamma_1 = 0.15$	0.053	0.091
		$H_0 : \gamma_2 = 0.85$	0.117	0.185

Table 9: Effective size of PDOLS-3D tests for Case 3, with $T = 150$.

Heterogeneous CSD	Persistence	Test	Nominal size 5%	Nominal size 10%
Low	Low	$H_0 : \gamma_1 = 0.15$	0.039	0.078
		$H_0 : \gamma_2 = 0.85$	0.062	0.114
	Medium	$H_0 : \gamma_1 = 0.15$	0.040	0.076
		$H_0 : \gamma_2 = 0.85$	0.055	0.114
	High	$H_0 : \gamma_1 = 0.15$	0.030	0.074
		$H_0 : \gamma_2 = 0.85$	0.039	0.098
High	Low	$H_0 : \gamma_1 = 0.15$	0.075	0.144
		$H_0 : \gamma_2 = 0.85$	0.104	0.184
	Medium	$H_0 : \gamma_1 = 0.15$	0.085	0.144
		$H_0 : \gamma_2 = 0.85$	0.095	0.184
	High	$H_0 : \gamma_1 = 0.15$	0.036	0.093
		$H_0 : \gamma_2 = 0.85$	0.093	0.159

6. Empirical Application

As an empirical exercise, we estimate capital and labor elasticities associated with the total factor productivity for the Colombian industry. This exercise is useful, since productivity is a variable that reflects how efficiently an economy uses its resources to produce goods and services and helps to determine the distribution of value added between capital and labor.

Assuming a Cobb-Douglas production function, we obtain the following equation for value added:

$$Y_{ijt} = A_{ij} K_{ijt}^{\alpha} L_{ijt}^{\beta} \tag{11}$$

where Y is value added, K is capital stock, L is labor, A corresponds to total factor productivity, α and β are the elasticities of capital and labor, respectively, and $\alpha + \beta = 1$. The subscripts i, j and t represent metropolitan areas, industry sectors and time, respectively.

Taking logarithms on both sides of equation (11) we get:

$$\ln Y_{ijt} = \ln A_{ij} + \alpha \ln K_{ijt} + \beta \ln L_{ijt} \tag{12}$$

To estimate equation (12) we employ the dataset used in Iregui et al. (2007). They use data from the annual manufacturing industry survey (EAM) of the national administrative department of statistics (DANE). The value added is calculated as the difference between gross output and intermediate inputs, where the latter corresponds to the value of domestic and foreign consumed raw materials and the value of purchased electricity (kw/h). Labor is defined as the number of employees. And finally, capital stock was calculated using the perpetual inventory method for gross investment[3].

This data includes annual information for 9 metropolitan areas and 18 industrial sectors (three-digits CIIU) from 1975 to 2000. The metropolitan areas considered are: Bogotá, Cali, Medellín, Manizales, Barranquilla, Bucaramanga, Pereira and Cartagena and the rest of the country.

The estimated parameters of (12) are presented in Table 10. As indicated in section 3, these estimations are obtained controlling for individual-specific fixed effects in the metropolitan areas and industrial sector dimensions, individual-specific time trends in those dimensions and cross-sectional dependence. These results show that the elasticities of capital and labor are 0.24 and 0.76, respectively. These elasticities are similar to those found in Colombian literature. For example, a technical document from Secretaría de Hacienda Distrital (2003) estimated coefficients of 0.27 and 0.72 for capital and labor, respectively; and Eslava, Haltiwanger, Kugler & Kugler (2004) found elasticities of 0.32 for capital and 0.74 for labor for the period 1982-1998.

[3] This data is in 1994 Colombian pesos.

Table 10: PDOLS-3D estimation results.

Parameter	Estimation	Standar Error
α	0.24	0.09
β	0.76	0.07

7. Final Remarks

This paper extends the asymptotic results of the dynamic ordinary least squares cointegration vector estimator of Mark & Sul (2003) to a three dimensional panel (PDOLS-3D). This method allows for individual heterogeneity using different short-run dynamics, individual-specific fixed effects and individual specific time trends. Also some degree of cross-sectional dependence is considered by the use of time-specific effects. A convenient feature of this method is that it permits the construction of asymptotically-valid test statistics for hypothesis testing.

The proposed estimators have also acceptable finite sample properties. Throughout the Monte Carlo experiments it was found that the effective sizes of the PDOLS-3D t-tests are relatively close to the nominal sizes, for different persistence levels of the series, and different forms and degrees of cross-sectional dependence.

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Appendix A. Proof of Lemma 1

Proof. Following the definitions presented in Section 3,

$$\boldsymbol{x}^{\ddagger *}_{ijt} = \left(\boldsymbol{x}^{\ddagger}_{ijt} - \frac{1}{T}\sum_{s=1}^{T}\boldsymbol{x}^{\ddagger}_{ijs}\right) - \left(\frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\boldsymbol{x}^{\ddagger}_{ijt} - \frac{1}{NMT}\sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{s=1}^{T}\boldsymbol{x}^{\ddagger}_{ijs}\right)$$

where $\boldsymbol{x}^{\ddagger}_{ijt} = \boldsymbol{x}_{ijt} - \Phi_{ij}\boldsymbol{z}_{ijt}$, is the linear projection of each $\boldsymbol{x}_{ijt}$ into $\boldsymbol{z}_{ijt}$, with Φ_{ij} a matrix of projections coefficients, then

$$\begin{aligned}
\boldsymbol{x}^{\ddagger *}_{ijt} = & \left[\boldsymbol{x}_{ijt} - \Phi_{ij}\boldsymbol{z}_{ijt} - \frac{1}{T}\sum_{s=1}^{T}(\boldsymbol{x}_{ijs} - \Phi_{ij}\boldsymbol{z}_{ijt})\right] \\
& - \left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}(\boldsymbol{x}_{nmt} - \Phi_{nm}\boldsymbol{z}_{nmt}) - \frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}(\boldsymbol{x}_{nms} - \Phi_{nm}\boldsymbol{z}_{nms})\right] \\
= & \left[\boldsymbol{x}_{ijt} - \frac{1}{T}\sum_{s=1}^{T}\boldsymbol{x}_{ijs} - \frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\boldsymbol{x}_{nmt} + \frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\boldsymbol{x}_{nms}\right] \\
& - \Phi_{ij}\left[\boldsymbol{z}_{ijt} - \frac{1}{T}\sum_{s=1}^{T}\boldsymbol{z}_{ijt}\right] + \frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\Phi_{nm}\left(\boldsymbol{z}_{nmt} - \frac{1}{T}\sum_{s=1}^{T}\boldsymbol{z}_{nms}\right) \\
= & \, \boldsymbol{x}^{*}_{ijt} - \left[\Phi_{ij}\widetilde{\boldsymbol{z}}_{ijt} - \frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\Phi_{nm}\widetilde{\boldsymbol{z}}_{nmt}\right]
\end{aligned} \tag{13}$$

(i) Using (13), we obtain the following expression

$$\begin{aligned}
\frac{1}{T^2}\sum_{t=1}^{T}\boldsymbol{x}^{\ddagger *}_{ijt}\boldsymbol{x}^{\ddagger *\prime}_{ijt} = & \frac{1}{T^2}\sum_{t=1}^{T}\left[\boldsymbol{x}^{*}_{ijt} - \left(\Phi_{ij}\widetilde{\boldsymbol{z}}_{ijt} - \frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\Phi_{nm}\widetilde{\boldsymbol{z}}_{nmt}\right)\right] \\
& \left[\boldsymbol{x}^{*\prime}_{ijt} - \left(\widetilde{\boldsymbol{z}}^{\prime}_{ijt}\Phi^{\prime}_{ij} - \frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\widetilde{\boldsymbol{z}}^{\prime}_{nmt}\Phi^{\prime}_{nm}\right)\right] \\
= & \frac{1}{T^2}\sum_{t=1}^{T}\boldsymbol{x}^{*}_{ijt}\boldsymbol{x}^{*\prime}_{ijt} - \frac{1}{T^2}\sum_{t=1}^{T}\left(\Phi_{ij}\widetilde{\boldsymbol{z}}_{ijt} - \frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\Phi_{nm}\widetilde{\boldsymbol{z}}_{nmt}\right)\boldsymbol{x}^{*\prime}_{ijt} \\
& - \frac{1}{T^2}\sum_{t=1}^{T}\boldsymbol{x}^{*}_{ijt}\left(\widetilde{\boldsymbol{z}}^{\prime}_{ijt}\Phi^{\prime}_{ij} - \frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\widetilde{\boldsymbol{z}}^{\prime}_{nmt}\Phi^{\prime}_{nm}\right) \\
& + \frac{1}{T^2}\sum_{t=1}^{T}\left(\Phi_{ij}\widetilde{\boldsymbol{z}}_{ijt} - \frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\Phi_{nm}\widetilde{\boldsymbol{z}}_{nmt}\right) \\
& \left(\widetilde{\boldsymbol{z}}^{\prime}_{ijt}\Phi^{\prime}_{ij} - \frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\widetilde{\boldsymbol{z}}^{\prime}_{nmt}\Phi^{\prime}_{nm}\right) \\
= & \frac{1}{T^2}\sum_{t=1}^{T}\boldsymbol{x}^{*}_{ijt}\boldsymbol{x}^{*\prime}_{ijt} - \frac{1}{T^2}O_p(T) - \frac{1}{T^2}O_p(T) + \frac{1}{T}O_p(T^{\frac{1}{2}})
\end{aligned}$$

then

$$\frac{1}{T^2}\sum_{t=1}^{T}\boldsymbol{x}^{\ddagger *}_{ijt}\boldsymbol{x}^{\ddagger *\prime}_{ijt} - \frac{1}{T^2}\sum_{t=1}^{T}\boldsymbol{x}^{*}_{ijt}\boldsymbol{x}^{*\prime}_{ijt} = \frac{1}{T}O_p(T^{\frac{1}{2}}) - \frac{2}{T^2}O_p(T) \xrightarrow{p} 0$$

(ii) Based on (13), we also find

$$\begin{aligned}
\frac{1}{T^{5/2}}\sum_{t=1}^{T}\widetilde{t}\boldsymbol{x}_{ijt}^{\ddagger *} &= \frac{1}{T^{5/2}}\sum_{t=1}^{T}\widetilde{t}\left[\boldsymbol{x}_{ijt}^{*}-\left(\Phi_{ij}\widetilde{\boldsymbol{z}}_{ijt}-\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\Phi_{nm}\widetilde{\boldsymbol{z}}_{nmt}\right)\right]\\
&= \frac{1}{T^{5/2}}\sum_{t=1}^{T}\widetilde{t}\boldsymbol{x}_{ijt}^{*}-\frac{1}{T^{5/2}}\sum_{t=1}^{T}O_p(T^{\frac{3}{2}})\\
&= \frac{1}{T^{5/2}}\sum_{t=1}^{T}\left(t-\frac{(T+1)}{2}\right)\boldsymbol{x}_{ijt}^{*}-\frac{1}{T^{5/2}}\sum_{t=1}^{T}O_p(T^{\frac{3}{2}})\\
&= \frac{1}{T^{5/2}}\sum_{t=1}^{T}t\boldsymbol{x}_{ijt}^{*}-\frac{(T+1)}{2T^{3/2}}\left(\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{x}_{ijt}^{*}\right)-\frac{1}{T^{5/2}}\sum_{t=1}^{T}O_p(T^{\frac{3}{2}})
\end{aligned}$$

where

$$\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{x}_{ijt}^{*} =& \frac{1}{T}\sum_{t=1}^{T}\left(\boldsymbol{x}_{ijt}-\frac{1}{T}\sum_{s=1}^{T}\boldsymbol{x}_{ijs}-\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\boldsymbol{x}_{nmt}+\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\boldsymbol{x}_{nms}\right)\\
=& \frac{1}{T}\sum_{t=1}^{T}\boldsymbol{x}_{ijt}-\frac{1}{T^2}T\sum_{s=1}^{T}\boldsymbol{x}_{ijs}-\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{t=1}^{T}\boldsymbol{x}_{nmt}\\
&+\frac{1}{T}T\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\boldsymbol{x}_{nms}\\
=& 0
\end{aligned}$$

then

$$\begin{aligned}
\frac{1}{T^{5/2}}\sum_{t=1}^{T}\widetilde{t}\boldsymbol{x}_{ijt}^{\ddagger *}-\frac{1}{T^{5/2}}\sum_{t=1}^{T}t\boldsymbol{x}_{ijt}^{*} &= -\frac{1}{T^{5/2}}\sum_{t=1}^{T}O_p(T^{\frac{3}{2}})\\
&= (-1)\sum_{t=1}^{T}\frac{O_p(T^{\frac{3}{2}})}{T^{5/2}}\\
&\xrightarrow{p}(-1)\sum_{t=1}^{T}0=0
\end{aligned}$$

(iii) Again, from equation (13) we get

$$\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{x}_{ijt}^{\ddagger *}u_{ijt}^{*} &= \frac{1}{T}\sum_{t=1}^{T}\boldsymbol{x}_{ijt}^{*}u_{ijt}^{*}-\frac{1}{T}\sum_{t=1}^{T}\left[\Phi_{ij}\widetilde{\boldsymbol{z}}_{ijt}-\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\Phi_{nm}\widetilde{\boldsymbol{z}}_{nmt}\right]u_{ijt}^{*}\\
&= \frac{1}{T}\sum_{t=1}^{T}\boldsymbol{x}_{ijt}^{*}u_{ijt}^{*}-\frac{1}{T}\left[\sum_{t=1}^{T}\Phi_{ij}\widetilde{\boldsymbol{z}}_{ijt}u_{ijt}^{*}-\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\left(\sum_{t=1}^{T}\Phi_{nm}\widetilde{\boldsymbol{z}}_{nmt}u_{ijt}^{*}\right)\right]\\
&= \frac{1}{T}\sum_{t=1}^{T}\boldsymbol{x}_{ijt}^{*}u_{ijt}^{*}-\frac{1}{T}O_p(T^{\frac{1}{2}})
\end{aligned}$$

then

$$\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{x}_{ijt}^{\ddagger *}u_{ijt}^{*}-\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{x}_{ijt}^{*}u_{ijt}^{*}=-\frac{1}{T}O_p(T^{\frac{1}{2}})\xrightarrow{p}0 \qquad \square$$

Appendix B. Proof of Lemma 2

Proof. (i) First, we need to analyze the term

$$\frac{1}{T^2}\sum_{t=1}^{T}\boldsymbol{x}^*_{ijt}\boldsymbol{x}^{*\prime}_{ijt} = \frac{1}{T^2}\sum_{t=1}^{T}\left(\boldsymbol{x}_{ijt} - \frac{1}{T}\sum_{s=1}^{T}\boldsymbol{x}_{ijs} - \frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\boldsymbol{x}_{nmt} + \frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\boldsymbol{x}_{nms}\right)$$

$$\left(\boldsymbol{x'}_{ijt} - \frac{1}{T}\sum_{s=1}^{T}\boldsymbol{x'}_{ijs} - \frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\boldsymbol{x'}_{nmt} + \frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\boldsymbol{x'}_{nms}\right)$$

$$= \underbrace{\frac{1}{T^2}\sum_{t=1}^{T}\boldsymbol{x}_{ijt}\boldsymbol{x}'_{ijt}}_{(a)} + \underbrace{\frac{1}{T}\left[\frac{1}{T}\sum_{s=1}^{T}\boldsymbol{x}_{ijs}\right]\left[\frac{1}{T}\sum_{s=1}^{T}\boldsymbol{x}'_{ijs}\right]}_{(b)}$$

$$+ \underbrace{\frac{1}{T^2}\sum_{t=1}^{T}\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\boldsymbol{x}_{nmt}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\boldsymbol{x}'_{nmt}\right]}_{(c)}$$

$$+ \underbrace{\frac{1}{T}\left[\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\boldsymbol{x}_{nms}\right]\left[\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\boldsymbol{x}'_{nms}\right]}_{(d)}$$

$$+ \underbrace{\frac{1}{T^2}\sum_{t=1}^{T}\left[\boldsymbol{x}_{ijt}\left(\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\boldsymbol{x}'_{nms}\right) + \left(\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\boldsymbol{x}_{nms}\right)\boldsymbol{x}'_{ijt}\right]}_{(e)}$$

$$+ \underbrace{\frac{1}{T^2}\sum_{t=1}^{T}\left[\left(\frac{1}{T}\sum_{s=1}^{T}\boldsymbol{x}_{ijs}\right)\left(\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\boldsymbol{x}'_{nmt}\right) + \left(\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\boldsymbol{x}_{nmt}\right)\left(\frac{1}{T}\sum_{s=1}^{T}\boldsymbol{x}'_{ijs}\right)\right]}_{(f)}$$

$$- \underbrace{\frac{1}{T^2}\sum_{t=1}^{T}\left[\boldsymbol{x}_{ijt}\left(\frac{1}{T}\sum_{s=1}^{T}\boldsymbol{x}'_{ijs}\right) + \left(\frac{1}{T}\sum_{s=1}^{T}\boldsymbol{x}_{ijs}\right)\boldsymbol{x}'_{ijt}\right]}_{(g)}$$

$$- \underbrace{\frac{1}{T^2}\sum_{t=1}^{T}\left[\boldsymbol{x}_{ijt}\left(\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\boldsymbol{x}'_{nmt}\right) + \left(\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\boldsymbol{x}_{nmt}\right)\boldsymbol{x}'_{ijt}\right]}_{(h)}$$

$$- \underbrace{\frac{1}{T^2}\sum_{t=1}^{T}\left[\left(\frac{1}{T}\sum_{s=1}^{T}\boldsymbol{x}_{ijs}\right)\left(\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\boldsymbol{x}'_{nms}\right) + \left(\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\boldsymbol{x}_{nms}\right)\left(\frac{1}{T}\sum_{s=1}^{T}\boldsymbol{x}'_{ijs}\right)\right]}_{(i)}$$

$$- \underbrace{\frac{1}{T^2}\sum_{t=1}^{T}\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\boldsymbol{x}_{nmt}\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\boldsymbol{x}'_{nms} + \frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\boldsymbol{x}_{nms}\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\boldsymbol{x}'_{nmt}\right]}_{(j)}$$

as $T \to \infty$, we have the following results for the ten factors of the previous expression

(a) $\frac{1}{T^2}\sum_{t=1}^{T}\boldsymbol{x}_{ijt}\boldsymbol{x}'_{ijt} \xrightarrow{d} \int \boldsymbol{B}_{vij}\boldsymbol{B}'_{vij}$

(b) $\frac{1}{T}\left[\frac{1}{T}\sum_{s=1}^{T}\boldsymbol{x}_{ijs}\right]\left[\frac{1}{T}\sum_{s=1}^{T}\boldsymbol{x}'_{ijs}\right] \xrightarrow{d} \left(\int \boldsymbol{B}_{vij}\right)\left(\int \boldsymbol{B}'_{vij}\right)$ (Hamilton 1994, propositions 18.1g and 17.3f)

(c) $\frac{1}{T^2}\sum_{t=1}^{T}\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\boldsymbol{x}_{nmt}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\boldsymbol{x}'_{nmt}\right]$
$\xrightarrow{d} \frac{1}{N^2M^2}\sum_{n=1}^{N}\sum_{m=1}^{M}\int \boldsymbol{B}_{vnm}\boldsymbol{B}'_{vnm}$

(d) $\frac{1}{T}\left[\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\boldsymbol{x}_{nms}\right]\left[\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\boldsymbol{x}'_{nms}\right]$
$=\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\left(\frac{1}{T^{3/2}}\sum_{s=1}^{T}\boldsymbol{x}_{nms}\right)\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\left(\frac{1}{T^{3/2}}\sum_{s=1}^{T}\boldsymbol{x}'_{nms}\right)\right]$
$\xrightarrow{d}\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\int \boldsymbol{B}_{vnm}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\int \boldsymbol{B}'_{vnm}\right]$ (Hamilton 1994, proposition 18.1g)

(e) $\frac{1}{T^2}\sum_{t=1}^{T}\left[\boldsymbol{x}_{ijt}\left(\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\boldsymbol{x}'_{nms}\right)\right.$
$\left.+\left(\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\boldsymbol{x}_{nms}\right)\boldsymbol{x}'_{ijt}\right]$
$=\left[\frac{1}{T^{3/2}}\sum_{t=1}^{T}\boldsymbol{x}_{ijt}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\left(\frac{1}{T^{3/2}}\sum_{s=1}^{T}\boldsymbol{x}'_{nms}\right)\right]$
$+\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\left(\frac{1}{T^{3/2}}\sum_{s=1}^{T}\boldsymbol{x}_{nms}\right)\right]\left[\frac{1}{T^{3/2}}\sum_{s=1}^{T}\boldsymbol{x}'_{ijt}\right]$
$\xrightarrow{d}\left[\int \boldsymbol{B}_{vij}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\int \boldsymbol{B}'_{vnm}\right]+\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\int \boldsymbol{B}_{vnm}\right]\left[\int \boldsymbol{B}'_{vij}\right]$
(Hamilton 1994, proposition 16.1g)

(f) $\frac{1}{T^2}\sum_{t=1}^{T}\left[\left(\frac{1}{T}\sum_{s=1}^{T}\boldsymbol{x}_{ijs}\right)\left(\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\boldsymbol{x}'_{nmt}\right)\right.$
$\left.+\left(\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\boldsymbol{x}_{nmt}\right)\left(\frac{1}{T}\sum_{s=1}^{T}\boldsymbol{x}'_{ijs}\right)\right]$
$=\left[\frac{1}{T^{3/2}}\sum_{s=1}^{T}\boldsymbol{x}_{ijs}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\left(\frac{1}{T^{3/2}}\sum_{t=1}^{T}\boldsymbol{x}'_{nmt}\right)\right]$
$+\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\left(\frac{1}{T^{3/2}}\sum_{t=1}^{T}\boldsymbol{x}_{nmt}\right)\right]\left[\frac{1}{T^{3/2}}\sum_{s=1}^{T}\boldsymbol{x}'_{ijs}\right]$
$\xrightarrow{d}\left[\int \boldsymbol{B}_{vij}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\int \boldsymbol{B}'_{vnm}\right]+\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\int \boldsymbol{B}_{vnm}\right]\left[\int \boldsymbol{B}'_{vij}\right]$

(g) $\frac{1}{T^2}\sum_{t=1}^{T}\left[\boldsymbol{x}_{ijt}\left(\frac{1}{T}\sum_{s=1}^{T}\boldsymbol{x}'_{ijs}\right)+\left(\frac{1}{T}\sum_{s=1}^{T}\boldsymbol{x}_{ijs}\right)\boldsymbol{x}'_{ijt}\right]$
$=\left[\frac{1}{T^{3/2}}\sum_{t=1}^{T}\boldsymbol{x}_{ijt}\right]\left[\frac{1}{T^{3/2}}\sum_{s=1}^{T}\boldsymbol{x}'_{ijs}\right]+\left[\frac{1}{T^{3/2}}\sum_{s=1}^{T}\boldsymbol{x}_{ijs}\right]\left[\frac{1}{T^{3/2}}\sum_{t=1}^{T}\boldsymbol{x}'_{ijt}\right]$
$\xrightarrow{d} 2\int \boldsymbol{B}_{vij}\int \boldsymbol{B}'_{vij}$

(h) $\frac{1}{T^2}\sum_{t=1}^{T}\left[\boldsymbol{x}_{ijt}\left(\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\boldsymbol{x}'_{nmt}\right)+\left(\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\boldsymbol{x}_{nmt}\right)\boldsymbol{x}'_{ijt}\right]$
$=\frac{1}{NM}\left[\frac{1}{T^2}\sum_{t=1}^{T}\left(\sum_{n=1}^{N}\sum_{m=1}^{M}\boldsymbol{x}_{ijt}\boldsymbol{x}'_{nmt}\right)\right]$
$+\frac{1}{NM}\left[\frac{1}{T^2}\sum_{t=1}^{T}\left(\sum_{n=1}^{N}\sum_{m=1}^{M}\boldsymbol{x}_{nmt}\boldsymbol{x}'_{ijt}\right)\right]$
$=\frac{1}{NM}\left[\frac{1}{T^2}\sum_{t=1}^{T}\boldsymbol{x}_{ijt}\boldsymbol{x}'_{ijt}\right]+\frac{1}{NM}\left[\frac{1}{T^2}\sum_{t=1}^{T}\boldsymbol{x}_{ijt}\boldsymbol{x}'_{ijt}\right]$
$\xrightarrow{d}\frac{1}{NM}\int \boldsymbol{B}_{vij}\boldsymbol{B}'_{vij}+\frac{1}{NM}\int \boldsymbol{B}_{vij}\boldsymbol{B}'_{vij}$

(i) $\frac{1}{T^2}\sum_{t=1}^{T}\left[\left(\frac{1}{T}\sum_{s=1}^{T}\boldsymbol{x}_{ijs}\right)\left(\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\boldsymbol{x}'_{nms}\right)\right.$
$\left.+\left(\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\boldsymbol{x}_{nms}\right)\left(\frac{1}{T}\sum_{s=1}^{T}\boldsymbol{x}'_{ijs}\right)\right]$
$=\left[\frac{1}{T^{3/2}}\sum_{s=1}^{T}\boldsymbol{x}_{ijs}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\left(\frac{1}{T^{3/2}}\sum_{s=1}^{T}\boldsymbol{x}'_{nms}\right)\right]$

$+\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\left(\frac{1}{T^{3/2}}\sum_{s=1}^{T}\boldsymbol{x}_{nms}\right)\right]\left[\frac{1}{T^{3/2}}\sum_{s=1}^{T}\boldsymbol{x}'_{ijs}\right]$
$\xrightarrow{d}\left[\int\boldsymbol{B}_{vij}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\int\boldsymbol{B}'_{vnm}\right]+\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\int\boldsymbol{B}_{vnm}\right]\left[\int\boldsymbol{B}'_{vij}\right]$
(Hamilton 1994, proposition 17.3f)

(j) $\frac{1}{T^2}\sum_{t=1}^{T}\left[\left(\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\boldsymbol{x}_{nmt}\right)\left(\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\boldsymbol{x}'_{nms}\right)\right.$
$\left.+\left(\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\boldsymbol{x}_{nms}\right)\left(\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\boldsymbol{x}'_{nmt}\right)\right]$
$=\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\left(\frac{1}{T^{3/2}}\sum_{t=1}^{T}\boldsymbol{x}_{nmt}\right)\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\left(\frac{1}{T^{3/2}}\sum_{s=1}^{T}\boldsymbol{x}'_{nms}\right)\right]$
$+\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\left(\frac{1}{T^{3/2}}\sum_{s=1}^{T}\boldsymbol{x}_{nms}\right)\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\left(\frac{1}{T^{3/2}}\sum_{t=1}^{T}\boldsymbol{x}'_{nmt}\right)\right]$
$\xrightarrow{d}2\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\int\boldsymbol{B}_{vnm}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\int\boldsymbol{B}'_{vnm}\right]$

Using the definition of $M_{11,NMT}$, Lemma 1 part (i), and the previous results of the terms of $\frac{1}{T^2}\sum\limits_{t=1}^{T}\boldsymbol{x}^{*}_{ijt}\boldsymbol{x}^{*'}_{ijt}$, for fixed N and M as $T\to\infty$, then

$$M_{11,NMT}\xrightarrow{d}M_{11,NM}$$

where

$$\begin{aligned}
M_{11,NMT} &= \frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\left(\frac{1}{T^2}\sum_{t=1}^{T}\boldsymbol{x}^{\ddagger*}_{ijt}\boldsymbol{x}^{\ddagger*'}_{ijt}\right)\\
&\xrightarrow{p}\frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}[(a)+(b)+(c)+(d)+(e)+(f)-(g)-(h)-(i)-(j)]\\
&=\frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\{[(a)+(c)-(h)]+[(b)-(g)]+[((d)-(j))\\
&\quad+((e)+(f)-(i))]\}\\
M_{11,NM} &= \frac{NM-1}{NM}\frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\int\boldsymbol{B}_{vij}\boldsymbol{B}'_{vij}-\frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\int\boldsymbol{B}_{vij}\int\boldsymbol{B}'_{vij}\\
&\quad+\left[\frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\int\boldsymbol{B}_{vij}\right]\left[\frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\int\boldsymbol{B}'_{vij}\right]
\end{aligned}$$

As $N\to\infty$ and $M\to\infty$ we have the following result

- $\frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\int\boldsymbol{B}_{vij}\boldsymbol{B}'_{vij}-\frac{1}{2NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\Omega_{vv,ij}\xrightarrow{p}0$
- $\frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\int\boldsymbol{B}_{vij}\int\boldsymbol{B}'_{vij}-\frac{1}{3NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\Omega_{vv,ij}\xrightarrow{p}0$
- $\left(\frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\int\boldsymbol{B}_{vij}\right)\left(\frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\int\boldsymbol{B}'_{vij}\right)\xrightarrow{p}0$

then

$$M_{11,NM} - \frac{1}{6NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\Omega_{vv,ij} \xrightarrow{p} 0$$

(ii) Analyzing the $i-th$ column of $\mathbf{M}'_{21,NMT}$, defined in section 4, and using the result of lemma 1 part (ii)

$$\frac{1}{M\sqrt{N}T^{5/2}}\sum_{j=1}^{M}\sum_{t=1}^{T} t\boldsymbol{x}^*_{ijt} = \frac{1}{M}\sum_{j=1}^{M}\underbrace{\left(\frac{1}{\sqrt{N}T^{5/2}}\sum_{t=1}^{T} t\boldsymbol{x}^*_{ijt}\right)}_{(k)}$$

Examining (k) for fixed j

$$\begin{aligned}
\frac{1}{\sqrt{N}T^{5/2}}\sum_{t=1}^{T} t\boldsymbol{x}^*_{ijt} &= \frac{1}{\sqrt{N}T^{5/2}}\sum_{t=1}^{T} t\left(\boldsymbol{x}_{ijt} - \frac{1}{T}\sum_{s=1}^{T}\boldsymbol{x}_{ijs} - \frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\boldsymbol{x}_{nmt} + \frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\boldsymbol{x}_{nms}\right)\\
&= \frac{1}{\sqrt{N}T^{5/2}}\sum_{t=1}^{T} t\boldsymbol{x}_{ijt} - \frac{1}{\sqrt{N}T^{7/2}}\sum_{t=1}^{T} t\sum_{s=1}^{T}\boldsymbol{x}_{ijs} - \frac{1}{MN^{3/2}T^{5/2}}\sum_{t=1}^{T} t\sum_{n=1}^{N}\sum_{m=1}^{M}\boldsymbol{x}_{nmt}\\
&\quad + \frac{1}{N^{3/2}MT^{7/2}}\sum_{t=1}^{T} t\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\boldsymbol{x}_{nms}\\
&= \frac{1}{\sqrt{N}}\left(\frac{1}{T^{5/2}}\sum_{t=1}^{T} t\boldsymbol{x}_{ijt}\right) - \frac{1}{\sqrt{N}}\left(\frac{1}{T^2}\sum_{t=1}^{T} t\right)\left(\frac{1}{T^{3/2}}\sum_{s=1}^{T}\boldsymbol{x}_{ijs}\right)\\
&\quad - \frac{1}{MN^{3/2}}\sum_{n=1}^{N}\sum_{m=1}^{M}\left(\frac{1}{T^{5/2}}\sum_{t=1}^{T} t\boldsymbol{x}_{nmt}\right)\\
&\quad + \left(\frac{1}{T^2}\sum_{t=1}^{T} t\right)\left(\frac{1}{N^{3/2}M}\sum_{n=1}^{N}\sum_{m=1}^{M}\left(\frac{1}{T^{3/2}}\sum_{s=1}^{T}\boldsymbol{x}_{nms}\right)\right)\\
&\xrightarrow{d} \frac{1}{\sqrt{N}}\int r\boldsymbol{B}_{vij} - \frac{1}{2\sqrt{N}}\int \boldsymbol{B}_{vij} - \frac{1}{MN^{3/2}}\sum_{n=1}^{N}\sum_{m=1}^{M}\int r\boldsymbol{B}_{vnm}\\
&\quad + \frac{1}{2}\left(\frac{1}{N^{3/2}M}\sum_{n=1}^{N}\sum_{m=1}^{M}\int \boldsymbol{B}_{vnm}\right)\\
&= \frac{1}{\sqrt{N}}\left[\int r\boldsymbol{B}_{vij} - \frac{1}{2}\int \boldsymbol{B}_{vij}\right] - \frac{1}{N^{3/2}M}\sum_{n=1}^{N}\sum_{m=1}^{M}\left[\int r\boldsymbol{B}_{vnm} - \frac{1}{2}\int \boldsymbol{B}_{vnm}\right]
\end{aligned}$$

then

$$\mathbf{M}'_{21,NMT} = \left[\frac{1}{\sqrt{N}MT^{\frac{5}{2}}}\sum_{j=1}^{M}\sum_{t=1}^{T}\widetilde{t}\boldsymbol{x}^{\ddagger *}_{1jt},\cdots,\frac{1}{\sqrt{N}MT^{\frac{5}{2}}}\sum_{j=1}^{M}\sum_{t=1}^{T}\widetilde{t}\boldsymbol{x}^{\ddagger *}_{Njt}\right] \xrightarrow{d} \mathbf{M}'_{21,NM}$$

where $\left[\mathbf{M}'_{21,NM}\right]_i$ is the *i-th* column of the matrix $\mathbf{M}'_{21,NM}$ and

$$\left[\mathbf{M}'_{21,NM}\right]_i = \frac{1}{M}\sum_{j=1}^{M}\frac{1}{\sqrt{N}}\left[\left(\int r\boldsymbol{B}_{vij} - \frac{1}{2}\int\boldsymbol{B}_{vij}\right) - \frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\left(\int r\boldsymbol{B}_{vnm} - \frac{1}{2}\int\boldsymbol{B}_{vnm}\right)\right]$$

As $N\to\infty$, $\left[\mathbf{M}'_{21,NM}\right]_i \xrightarrow{p} 0$ for all i and fixed M, then

$$\mathbf{M}'_{21,NM} \xrightarrow{p} 0$$

(iii) Similar to the proof of lemma 2 part (ii).

(iv) From previous definitions, $\mathbf{M}_{22,NMT} = \left(\frac{1}{T^3}\sum_{t=1}^{T}\widetilde{t}^{\,2}\right)\mathbf{I}_N$, with

$$\begin{aligned}
\frac{1}{T^3}\sum_{t=1}^{T}\widetilde{t}^{\,2} &= \frac{1}{T^3}\sum_{t=1}^{T}\left[t-\frac{(T+1)}{2}\right]^2\\
&= \frac{1}{T^3}\sum_{t=1}^{T}\left[t^2-\frac{2t(T+1)}{2}+\frac{(T+1)^2}{4}\right]\\
&= \frac{1}{T^3}\sum_{t=1}^{T}t^2-\frac{(T+1)}{T^3}\sum_{t=1}^{T}t+\frac{(T+1)^2}{4T^2}\\
&= \frac{(T+1)(2T+1)}{6T^2}-\frac{(T+1)^2}{2T^2}+\frac{(T+1)^2}{4T^2}\\
&\xrightarrow[T\to\infty]{} \frac{1}{3}-\frac{1}{2}+\frac{1}{4}=\frac{1}{12}
\end{aligned}$$

then

$$\mathbf{M}_{22,NMT} \xrightarrow{p} \frac{1}{12}\mathbf{I}_N$$

(v) From section 4.

$$\begin{aligned}
\mathbf{M}'_{32,NMT} &= \left[\frac{1}{\sqrt{NM}T^3}\sum_{t=1}^{T}\widetilde{t}^{\,2}\right](1)_{N\times M}\\
&\xrightarrow[T\to\infty]{} \frac{1}{\sqrt{NM}}\frac{1}{12}(1)_{N\times M}\\
&\xrightarrow{p} 0 \text{ as } N\to\infty \text{ and } M\to\infty
\end{aligned}$$

(vi) Using lemma 2 part (iv), the following result is obtained

$$\mathbf{M}_{33,NMT} \xrightarrow{p} \frac{1}{12}\mathbf{I}_M$$

□

Appendix C. Proof of Lemma 3

Proof. (i) First, we analyze the following term

$$\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{x}^{*}_{ijt}u^{*}_{ijt}=\frac{1}{T}\sum_{t=1}^{T}\left(\boldsymbol{x}_{ijt}-\frac{1}{T}\sum_{s=1}^{T}\boldsymbol{x}_{ijs}-\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\boldsymbol{x}_{nmt}+\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\boldsymbol{x}_{nms}\right)$$
$$\left(u_{ijt}-\frac{1}{T}\sum_{s=1}^{T}u_{ijs}-\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}u_{nmt}+\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}u_{nms}\right)$$
$$=\underbrace{\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{x}_{ijt}u_{ijt}}_{(a)}+\underbrace{\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\boldsymbol{x}_{nmt}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}u_{nmt}\right]}_{(b)}$$
$$+\underbrace{\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{T}\sum_{s=1}^{T}\boldsymbol{x}_{ijs}\right]\left[\frac{1}{T}\sum_{s=1}^{T}u_{ijs}\right]}_{(c)}$$
$$+\underbrace{\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\boldsymbol{x}_{nms}\right]\left[\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}u_{nms}\right]}_{(d)}-\underbrace{\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{x}_{ijt}\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}u_{nmt}\right]}_{(e)}$$
$$-\underbrace{\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{x}_{ijt}\left[\frac{1}{T}\sum_{s=1}^{T}u_{ijs}\right]}_{(f)}+\underbrace{\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{x}_{ijt}\left[\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}u_{nms}\right]}_{(g)}$$
$$-\underbrace{\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\boldsymbol{x}_{nmt}\right]u_{ijt}}_{(h)}+\underbrace{\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\boldsymbol{x}_{nmt}\right]\left[\frac{1}{T}\sum_{s=1}^{T}u_{ijs}\right]}_{(i)}$$
$$-\underbrace{\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\boldsymbol{x}_{nmt}\right]\left[\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}u_{nms}\right]}_{(j)}-\underbrace{\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{T}\sum_{s=1}^{T}\boldsymbol{x}_{ijs}\right]u_{ijt}}_{(k)}$$
$$+\underbrace{\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{T}\sum_{s=1}^{T}\boldsymbol{x}_{ijs}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}u_{nmt}\right]}_{(l)}$$
$$-\underbrace{\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{T}\sum_{s=1}^{T}\boldsymbol{x}_{ijs}\right]\left[\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}u_{nms}\right]}_{(m)}$$
$$+\underbrace{\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\boldsymbol{x}_{nms}\right]u_{ijt}}_{(n)}$$
$$-\underbrace{\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\boldsymbol{x}_{nms}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}u_{nmt}\right]}_{(o)}$$
$$-\underbrace{\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\boldsymbol{x}_{nms}\right]\left[\frac{1}{T}\sum_{s=1}^{T}u_{ijs}\right]}_{(p)}$$

with

(a) $\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{x}_{ijt}u_{ijt} \xrightarrow{d} \sqrt{\Omega_{uu,ij}}\int \boldsymbol{B}_{vij}dW_{uij}$

(b) $\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\boldsymbol{x}_{nmt}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}u_{nmt}\right]$
$\xrightarrow{d} \frac{1}{(NM)^2}\sum_{n=1}^{N}\sum_{m=1}^{M}\sqrt{\Omega_{uu,nm}}\int \boldsymbol{B}_{vnm}dW_{unm}$

(c) $\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{T}\sum_{s=1}^{T}\boldsymbol{x}_{ijs}\right]\left[\frac{1}{T}\sum_{s=1}^{T}u_{ijs}\right] \xrightarrow{d} [\int \boldsymbol{B}_{vij}]\sqrt{\Omega_{uu,ij}}W_{uij}(1)$

(d) $\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\boldsymbol{x}_{nms}\right]\left[\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}u_{nms}\right]$
$\xrightarrow{d} \left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\int \boldsymbol{B}_{vnm}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\sqrt{\Omega_{uu,nm}}W_{unm}(1)\right]$

(e) $\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{x}_{ijt}\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}u_{nmt}\right] \xrightarrow{d} \frac{1}{NM}\sqrt{\Omega_{uu,ij}}\int \boldsymbol{B}_{vij}dW_{uij}$

(f) $\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{x}_{ijt}\left[\frac{1}{T}\sum_{s=1}^{T}u_{ijs}\right] \xrightarrow{d} [\int \boldsymbol{B}_{vij}]\sqrt{\Omega_{uu,ij}}W_{uij}(1)$

(g) $\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{x}_{ijt}\left[\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}u_{nms}\right]$
$\xrightarrow{d} [\int \boldsymbol{B}_{vij}]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\sqrt{\Omega_{uu,nm}}W_{unm}(1)\right]$

(h) $\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\boldsymbol{x}_{nmt}\right]u_{ijt} \xrightarrow{d} \frac{1}{NM}\sqrt{\Omega_{uu,ij}}\int \boldsymbol{B}_{vij}dW_{uij}$

(i) $\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\boldsymbol{x}_{nmt}\right]\left[\frac{1}{T}\sum_{s=1}^{T}u_{ijs}\right]$
$\xrightarrow{d} \left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\int \boldsymbol{B}_{vnm}\right]\sqrt{\Omega_{uu,ij}}W_{uij}(1)$

(j) $\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\boldsymbol{x}_{nmt}\right]\left[\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}u_{nms}\right]$
$\xrightarrow{d} \left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\int \boldsymbol{B}_{vnm}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\sqrt{\Omega_{uu,nm}}W_{unm}(1)\right]$

(k) $\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{T}\sum_{s=1}^{T}\boldsymbol{x}_{ijs}\right]u_{ijt} \xrightarrow{d} [\int \boldsymbol{B}_{vij}]\sqrt{\Omega_{uu,ij}}W_{uij}(1)$

(l) $\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{T}\sum_{s=1}^{T}\boldsymbol{x}_{ijs}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}u_{nmt}\right]$
$\xrightarrow{d} [\int \boldsymbol{B}_{vij}]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\sqrt{\Omega_{uu,nm}}W_{unm}(1)\right]$

(m) $\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{T}\sum_{s=1}^{T}\boldsymbol{x}_{ijs}\right]\left[\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}u_{nms}\right]$
$\xrightarrow{d} [\int \boldsymbol{B}_{vij}]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\sqrt{\Omega_{uu,nm}}W_{unm}(1)\right]$

(n) $\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\boldsymbol{x}_{nms}\right]u_{ijt}$
$\xrightarrow{d} \left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\int \boldsymbol{B}_{vnm}\right]\sqrt{\Omega_{uu,ij}}W_{uij}(1)$

(o) $\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\boldsymbol{x}_{nms}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}u_{nmt}\right]$
$\xrightarrow{d}\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\int\boldsymbol{B}_{vnm}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\sqrt{\Omega_{uu,nm}}W_{unm}(1)\right]$

(p) $\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{NMT}\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{s=1}^{T}\boldsymbol{x}_{nms}\right]\left[\frac{1}{T}\sum_{s=1}^{T}u_{ijs}\right]$
$\xrightarrow{d}\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\int\boldsymbol{B}_{vnm}\right]\sqrt{\Omega_{uu,ij}}W_{uij}(1)$

Using the previous results and lemma 1 part (iii), we have

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{x}_{ijt}^{\ddagger*}u_{ijt}^{*}\xrightarrow{d}&\left[\frac{NM-2}{NM}\right]\sqrt{\Omega_{uu,ij}}\int\boldsymbol{B}_{vij}dW_{uij}+\frac{1}{N^2M^2}\sum_{n=1}^{N}\sum_{m=1}^{M}\sqrt{\Omega_{uu,nm}}\int\boldsymbol{B}_{vnm}dW_{unm}\\&-\left[\int\boldsymbol{B}_{vij}\right]\sqrt{\Omega_{uu,ij}}W_{uij}(1)-\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\int\boldsymbol{B}_{vnm}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}\sqrt{\Omega_{uu,nm}}W_{unm}(1)\right]\\&+\left[\int\boldsymbol{B}_{vij}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\sqrt{\Omega_{uu,nm}}W_{unm}(1)\right]+\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\int\boldsymbol{B}_{vnm}\right]\left[\sqrt{\Omega_{uu,ij}}W_{uij}(1)\right]\end{aligned}$$

Summing over i and j and dividing the result by $\sqrt{NM}$, we obtain

$$\begin{aligned}&\left[\frac{NM-2}{NM}\right]\frac{1}{\sqrt{NM}}\sum_{i=1}^{N}\sum_{j=1}^{M}\sqrt{\Omega_{uu,ij}}\int\boldsymbol{B}_{vij}dW_{uij}\\&+\frac{1}{(NM)^{\frac{3}{2}}}\sum_{n=1}^{N}\sum_{m=1}^{M}\sqrt{\Omega_{uu,nm}}\int\boldsymbol{B}_{vnm}dW_{unm}-\frac{1}{\sqrt{NM}}\sum_{i=1}^{N}\sum_{j=1}^{M}\left[\int\boldsymbol{B}_{vij}\sqrt{\Omega_{uu,ij}}W_{uij}(1)\right]\\&-\frac{1}{(NM)^{\frac{3}{2}}}\left[\sum_{n=1}^{N}\sum_{m=1}^{M}\int\boldsymbol{B}_{vnm}\right]\left[\sum_{n=1}^{N}\sum_{m=1}^{M}\sqrt{\Omega_{uu,nm}}W_{unm}(1)\right]\\&+\left[\frac{1}{\sqrt{NM}}\sum_{i=1}^{N}\sum_{j=1}^{M}\int\boldsymbol{B}_{vij}\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\sqrt{\Omega_{uu,nm}}W_{unm}(1)\right]\\&+\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\int\boldsymbol{B}_{vnm}\right]\left[\frac{1}{\sqrt{NM}}\sum_{i=1}^{N}\sum_{j=1}^{M}\sqrt{\Omega_{uu,ij}}W_{uij}(1)\right]\\=&\left[\frac{NM-1}{NM}\right]\frac{1}{\sqrt{NM}}\sum_{i=1}^{N}\sum_{j=1}^{M}\sqrt{\Omega_{uu,ij}}\left(\int\boldsymbol{B}_{vij}dW_{uij}-\int\boldsymbol{B}_{vij}\int dW_{uij}\right)\\&-\frac{1}{(NM)^{3/2}}\sum_{i=1}^{N}\sum_{j=1}^{M}\int B_{vij}\sqrt{\Omega_{uu,ij}}W_{uij}(1)\\&+\frac{1}{(NM)^{3/2}}\sum_{i=1}^{N}\sum_{j=1}^{M}\int B_{vij}\sum_{n=1}^{N}\sum_{m=1}^{M}\sqrt{\Omega_{uu,nm}}W_{unm}(1)\\=&\left[\frac{NM-1}{NM}\right]\frac{1}{\sqrt{NM}}\sum_{i=1}^{N}\sum_{j=1}^{M}\sqrt{\Omega_{uu,ij}}\left(\int\boldsymbol{B}_{vij}dW_{uij}-\int\boldsymbol{B}_{vij}\int dW_{uij}\right)\\&-\frac{1}{(NM)^{3/2}}\sum_{i=1}^{N}\sum_{j=1}^{M}\int B_{vij}\left(\sqrt{\Omega_{uu,ij}}W_{uij}(1)-\sum_{n=1}^{N}\sum_{m=1}^{M}\sqrt{\Omega_{uu,nm}}W_{unm}(1)\right)\end{aligned}$$

Therefore

$$\boldsymbol{m}_{1,NM} = \left[\frac{NM-1}{NM}\right]\frac{1}{\sqrt{NM}}\sum_{i=1}^{N}\sum_{j=1}^{M}\sqrt{\Omega_{uu,ij}}\int \tilde{\boldsymbol{B}}_{vij}dW_{uij} - \frac{1}{NM}O_{NM}(1)$$

where $\widetilde{B}_{vij} = \boldsymbol{B}_{vij} - \int \boldsymbol{B}_{vij}$

(ii) First, we need to establish the asymptotic normality of $\left[\frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\mathbf{U}_{ij}\right]^{-1}$ $\left[\frac{1}{\sqrt{NM}}\sum_{i=1}^{N}\sum_{j=1}^{M}\boldsymbol{e}_{ij}\right]$, where $\mathbf{U}_{ij} = \int \boldsymbol{B}_{vij}\boldsymbol{B}'_{vij}$ and $\boldsymbol{e}_{ij} = \sqrt{\Omega_{uu,ij}}\int \boldsymbol{B}_{vij}dW_{uij}$.

Working with $\{\boldsymbol{e}_{Lij}\}_{i,j=0}^{\infty}$ and $L = NM$, we have

(a) $\bar{\boldsymbol{e}}_{Lij} = \frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\boldsymbol{e}_{Lij}$

(b) $\boldsymbol{\mu}_{Lij} = E(\boldsymbol{e}_{Lij}) = E\{E(\boldsymbol{e}_{Lij}|\mathbf{U}_{Lij})\} = E\{\mathbf{0}\} = 0$

(c) $\bar{\boldsymbol{\mu}}_{Lij} = \frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\boldsymbol{\mu}_{Lij} = 0$

(d) $\mathbf{V}_{Lij} = Var(\boldsymbol{e}_{Lij}) = E(\boldsymbol{e}_{Lij}\boldsymbol{e}'_{Lij}) = E\{E(\boldsymbol{e}_{Lij}\boldsymbol{e}'_{Lij}|\mathbf{U}_{Lij})\} = E\{\Omega_{uu,Lij}\mathbf{U}_{Lij}\} = \frac{1}{6}\Omega_{uu,Lij}\boldsymbol{\Omega}_{vv,Lij}$

(e) $$\begin{aligned} E\|\boldsymbol{e}_{Lij}\|^{2+\delta} &= E\|\sqrt{\Omega_{uu,Lij}}\int \boldsymbol{B}_{v,Lij}dW_{u,Lij}\|^{2+\delta} \\ &= E\|\sqrt{\Omega_{uu,Lij}}\boldsymbol{\Omega}_{vv,Lij}^{1/2}\int \boldsymbol{W}_{v,Lij}dW_{u,Lij}\|^{2+\delta} \\ &\leqslant \|\sqrt{\Omega_{uu,Lij}}\boldsymbol{\Omega}_{vv,Lij}^{1/2}\|^{2+\delta}E\|\int \boldsymbol{W}_{v,Lij}dW_{u,Lij}\|^{2+\delta} < \Delta < \infty \end{aligned}$$

(f) $\mathbf{V}_L = \frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\mathbf{V}_{Lij} = \frac{1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M}$
$Var(\boldsymbol{e}_{Lij}) = \frac{1}{6NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\Omega_{uu,Lij}\boldsymbol{\Omega}_{vv,Lij}$

Since $\boldsymbol{\Omega}_{vv,Lij}$ is positive definite for all i, j, $\mathbf{V}_L$ is $O(1)$ and uniformly positive definite. Using theorem 5.11 of White (2001), it follows that as $T \longrightarrow \infty$, $N \longrightarrow \infty$ and $M \longrightarrow \infty$

$$V_{NM}^{-\frac{1}{2}}\frac{1}{\sqrt{NM}}\sum_{i=1}^{N}\sum_{j=1}^{M}\boldsymbol{e}_{ij} \xrightarrow{d} N(0, \mathbf{I})$$

where $\mathbf{V}_{NM} = Var\left(\frac{1}{\sqrt{NM}}\sum_{i=1}^{N}\sum_{j=1}^{M}\boldsymbol{e}_{ij}\right) = \frac{1}{6NM}\sum_{i=1}^{N}\sum_{j=1}^{M}\Omega_{uu,ij}\boldsymbol{\Omega}_{vv,ij}$.

(iii) Analyzing part of the $i-th$ element of $\boldsymbol{m}_{2,NM}$, for fixed N, M as $T \longrightarrow \infty$

$$
\begin{aligned}
\frac{1}{T^{3/2}}\sum_{t=1}^{T}\widetilde{t}u_{ijt}^{*} &= \frac{1}{T^{3/2}}\sum_{t=1}^{T}tu_{ijt}^{*}\\
=&\frac{1}{T^{3/2}}\sum_{t=1}^{T}tu_{ijt}-\left[\frac{1}{T^{2}}\sum_{t=1}^{T}t\right]\left[\frac{1}{T^{1/2}}\sum_{s=1}^{T}u_{ijs}\right]-\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\left[\frac{1}{T^{3/2}}\sum_{t=1}^{T}tu_{nmt}\right]\\
&+\left[\frac{1}{T^{2}}\sum_{t=1}^{T}t\right]\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\left(\frac{1}{T^{1/2}}\sum_{s=1}^{T}u_{nms}\right)\right]\\
=&\frac{1}{T^{3/2}}\sum_{t=1}^{T}tu_{ijt}-\left(\frac{1}{2}+\frac{1}{2T}\right)\left[\frac{1}{T^{1/2}}\sum_{s=1}^{T}u_{ijs}\right]-\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\left[\frac{1}{T^{3/2}}\sum_{t=1}^{T}tu_{nmt}\right]\\
&+\left(\frac{1}{2}+\frac{1}{2T}\right)\left[\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\left(\frac{1}{T^{1/2}}\sum_{s=1}^{T}u_{nms}\right)\right]\\
\xrightarrow{d}&\sqrt{\Omega_{uu,ij}}\left[W_{uij}(1)-\int_{0}^{1}W_{uij}(r)dr\right]-\frac{1}{2}\sqrt{\Omega_{uu,ij}}W_{uij}(1)\\
&-\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\sqrt{\Omega_{uu,nm}}\left[W_{unm}(1)-\int_{0}^{1}W_{unm}(r)dr\right]+\frac{1}{2}\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\sqrt{\Omega_{uu,nm}}W_{unm}(1)\\
=&\sqrt{\Omega_{uu,ij}}\left(\int rdW_{uij}\right)-\frac{1}{2}\sqrt{\Omega_{uu,ij}}W_{uij}(1)\\
&-\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\sqrt{\Omega_{uu,nm}}\left(\int rdW_{unm}\right)+\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\sqrt{\Omega_{uu,nm}}\frac{1}{2}W_{unm}(1)\\
=&\sqrt{\Omega_{uu,ij}}\left(\int rdW_{uij}-\frac{1}{2}W_{uij}(1)\right)-\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\sqrt{\Omega_{uu,nm}}\left(\int rdW_{unm}-\frac{1}{2}W_{unm}(1)\right)
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{1}{T^{3/2}}\sum_{t=1}^{T}\widetilde{t}u_{ijt}^{*}\xrightarrow{d}\ &\sqrt{\Omega_{uu,ij}}\left(\int rdW_{uij}-\frac{1}{2}W_{uij}(1)\right)\\
&-\frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\sqrt{\Omega_{uu,nm}}\left(\int rdW_{unm}-\frac{1}{2}W_{unm}(1)\right)
\end{aligned}
$$

from Lemma 3 part (i), $\boldsymbol{m}_{1,NMT}\xrightarrow{p}\boldsymbol{m}_{1,NM}$, and

$$
\boldsymbol{m}_{1,NM}=\left[\frac{NM-1}{NM}\right]\frac{1}{\sqrt{NM}}\sum_{i=1}^{N}\sum_{j=1}^{M}\sqrt{\Omega_{uu,ij}}\int\tilde{\boldsymbol{B}}_{vij}dW_{uij}-\frac{1}{NM}O_{NM}(1)
$$

Using these results and an extension of proposition 4 part (a) from Mark & Sul (2002), we obtain that as $T\longrightarrow\infty$, $N\rightarrow\infty$ and $M\rightarrow\infty$, $\boldsymbol{m}_{1,NM}$ is independent of $\boldsymbol{m}_{2,NM}$.

The proof of the asymptotic independence of $\boldsymbol{m}_{1,NM}$ and $\boldsymbol{m}_{3,NM}$ follows in a similar way. □