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Exponentiality Test Based on Progressively Type-II Censored Data Via Extension of Cumulative Tsallis Divergence

Prueba de exponencialidad basada en progresivamente tipo II Datos censurados a través de la extensión de Tsallis acumulados Divergencia

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Abstract

In this article, first new divergences are defined by using Tsallis divergence and a measure of discrepancy between equilibriums associated with two distributions is proposed. Then utilizing the progressively Type-II censored sample, we construct goodness of fit tests for exponentiality based on the estimation of proposed divergences. To investigate the performance of the mentioned tests, Monte Carlo simulations are performed. In order to study the power, the alternatives are considered according to the failure rate function. The powers of the proposed tests are then compared with other existing tests. As regards the last step of the study, in order to explain the use of the proposed tests, three examples are presented.

***Key words*:** Cumulative residual Tsallis divergence; Exponential distribution, Goodness of fit test; Monte Carlo simulation; Tsallis divergence.

Resumen

En este artículo, las primeras nuevas divergencias se definen utilizando la divergencia de Tsallis y se propone una medida de discrepancia entre los equilibrios asociados con dos distribuciones. Luego, utilizando la muestra censurada progresivamente Tipo II, construimos pruebas de bondad de ajuste para exponencialidad basadas en la estimación de divergencias propuestas. Para investigar el desempeño de las pruebas mencionadas, se realizan simulaciones Monte Carlo. Para estudiar la potencia, se consideran las alternativas según la función de tasa de falla. Los poderes de las pruebas propuestas se comparan luego con otras pruebas existentes. En cuanto al último paso del estudio, para explicar el uso de las pruebas propuestas, se presentan tres ejemplos.

***Palabras clave*:** Bondad de ajuste; Distribución exponencial; Divergencia Tsallis; Divergencia Tsallis residual acumulada; Simulación Monte Carlo.

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1. Introduction

Because of the importance of exponential distribution in reliability and lifetime models, many tests with complete samples and some procedures under censored data have been presented in previous studies, attempting to determine the appropriateness of an exponential model for a given dataset. Type-I and Type-II censoring schemes are the most popular ones among the different censoring schemes. One of the disadvantages of these censoring schemes is the impossibility to withdraw units during the experiment, so a generalization of the classical Type-II censoring scheme, known as the progressive Type-II censoring scheme (PCS Type-II), was proposed by analysts to pull back units amid the experiment. PCS has gotten significant consideration in the statistical writing (Balakrishnan & Aggarwala, 2000; Balakrishnan, 2007).

There are several goodness of fit tests available in the literature based on censored data for exponentiality test. Lim & Park (2007) provided a different version of Kullback–Leibler (KL) information with the Type-II censored data for the exponential and normal distributions. Balakrishnan et al. (2007) estimated the joint entropy of a sample with PCS Type-II and proposed a test for exponentiality based on corresponding KL information to the aformentioned joint entropy. Lin et al. (2008) proposed a simple method for testing goodness of fit based on Type-II censored data. Yousefzadeh & Arghami (2008) estimated Shannon entropy by a new estimator of distribution function and used it for testing exponentially and normality. Wang (2008) introduced a test statistic for the exponential distribution and obtained the exact distribution of the test statistic under the null hypothesis. Habibirad et al. (2011) extended the goodness of fit test based on KL information for PCS Type-II data for three distributions. Salinas et al. (2012) proposed goodness of fit tests for the Gumbel distribution with Type-II right censored data. Pakyari & Balakrishnan (2013) developed some goodness of fit tests for the exponential distribution based on Type-I censored samples. Recently; Park & Lim (2015) generalized the cumulative residual KL (CRKL) information to the censored case and used the estimate of modified version of CRKL as a goodness of fit test statistic with the Type-II censored data. Alizadeh Noughabi & Balakrishnan (2015) developed a general goodness of fit test for Type-II censored data by using a new estimate of KL information for Type-II censored data. Also, they considered testing for exponentiality under Type-II censored data as a special case of this general test. Baratpour & Habibirad (2016) constructed two goodness of fit tests based on the CRKL and cumulative KL (CKL) information for testing exponentiality with PCS Type-II data. Alizadeh Noughabi (2017) introduced a general goodness of fit test for PCS Type-II data based on a new estimate of KL information and then used the proposed test statistic for testing exponentially based on PCS Type-II data.

The present paper aims to construct some tests using extentions of Tsallis divergence based on the PCS Type-II sample for exponentiality test. The article is divided to the following sections: Section 2 presents some preliminaries. In Section 3, the new divergences are introduced by using extentions of Tsallis divergence and then test statistics are construcred. The powers of the proposed tests and existing tests are then obtained through Monte Carlo simulations in Section 4. Finally, using some examples, Section 5 illustrates the performance of all the tests.

2. Preliminaries

In this section, some preliminary aspects are provided and in the next section the proposed tests are introduced.

The PCS can be described as follows. Under this general censoring scheme, n units are placed on a life testing experiment and only $m(< n)$ are completely observed until failure. The censoring occurs progressively in m stages. At the time of the first failure (the first stage) $X_{1:m:n}$, R_1 of the remaining $n-1$ surviving units are randomly removed from the experiment. At the second failure (the second stage) $X_{2:m:n}$, R_2 units are randomly removed from the remaining $n-2-R_1$ units, and so on. The procedure is continued until all the remaining surviving $R_m = n-m-R_1-\cdots-R_{m-1}$ units are removed from the experiment at the time of the mth failure (the mth stage) $X_{m:m:n}$. We will denote the m order observed failure times by $X_{1:m:n} < X_{2:m:n} < \cdots < X_{m:m:n}$ and the PCS with the vector $R = (R_1, \ldots, R_m)$, which is fixed previously. If $R = (0, \ldots, 0)$, then no censoring is performed at any of the m stages and corresponds to the complete sample. If $R = (0, \ldots, 0, n-m)$, we obtain the Type-II right censoring.

Let X be a non negative absolutely continuous random variable having cumulative distribution function (cdf) F, and the probability density function (pdf) f. Then the Shannon (1948) information measure is defined as

$$H(X) = -\int_0^\infty f(x)\ln f(x)dx.$$

Rao et al. (2004) introduced a new uncertainty measure, the cumulative residual entropy (CRE), which for a non negative random variable X is defined as follows:

$$CRE(F) = -\int_0^\infty \bar{F}(x)\ln \bar{F}(x)dx.$$

Similar to the CRE, Di Crescenzo & Longobardi (2009) proposed the cumulative entropy (CE) through

$$CE(F) = -\int_0^\infty F(x)\ln F(x)dx.$$

Consider two nonnegative and absolutely continuous random variables X and Y with pdfs f and g, cdfs F and G, respectively. Then, the KL informations as a measure of discrepancy between f and g is given by

$$KL(f:g) = \int_0^\infty f(x)\ln\frac{f(x)}{g(x)}dx,$$

and the Tsallis divergence between f and g is defined as (Tsallis, 1988)

$$D_T(f,g) = \frac{1}{\alpha-1}\left[\int_0^\infty f^\alpha(x)g^{1-\alpha}(x)dx - 1\right], \quad \alpha(\neq 1) > 0. \tag{1}$$

Baratpour & Habibirad (2016) suggested an extension of the KL information to the survival function, which is CRKL, as follows:

$$CRKL(F:G)=\int_0^\infty \bar{F}(x)\ln\frac{\bar{F}(x)}{\bar{G}(x)}dx-[E(X)-E(Y)],$$

where $\bar{F}(x)$ and $\bar{G}(x)$ are the survival functions of random variables X and Y, respectively.

Park et al. (2012) considered another extension to the cumulative distribution, which is called CKL and defined as follows:

$$CKL(F:G)=\int_0^\infty F(x)\ln\frac{F(x)}{G(x)}dx-[E(Y)-E(X)].$$

Let $w(t)$ be a non-negative function, so that $0<E(w(t))<\infty$, then we can define the weighted random variable X^* with density function

$$f^*(t)=\frac{w(t)f_X(t)}{E(w(X))},\quad t\geq 0. \tag{2}$$

The equilibrium distribution results as a special case when $w(t)=\frac{1}{r_X(t)}$, where $r_X(t)=\frac{f_X(t)}{\bar{F}_X(t)}$ is failure rate function of X; then X^* is said the equilibrium random variable associated with X. The pdf of X^* is $f^*(t)=\frac{\bar{F}_X(t)}{E(X)}$ (Gupta & Kirmani, 1990).

Let f^* and g^* be the equilibrium pdfs respectively associated with f and g. Then, we define the Tsallis divergence based on equilibrium distributions as follows:

$$D_T(f^*,g^*)=\frac{1}{\alpha-1}\left[\frac{E^{\alpha-1}(Y)}{E^{\alpha}(X)}\int_0^\infty \bar{F}^{\alpha}(x)\bar{G}^{1-\alpha}(x)dx-1\right],\quad \alpha(\neq 1)>0. \tag{3}$$

Suppose that $x_{1:m:n}<x_{2:m:n}<\cdots<x_{m:m:n}$ are progressively Type-II right censored data with the PCS $R=(R_1,R_2,\ldots,R_m)$ from a continuous distribution function $F(x)$. Based on progressively Type-II right censored data, the cdf estimator can be written as

$$F_{m:n}(x)=\begin{cases}0, & x<x_{1:m:n}\\ \alpha_{i:m:n}, & x_{i:m:n}\leq x<x_{i+1:m:n},\quad i=1,2,\ldots,m-1\\ \alpha_{m:m:n}, & x\geq x_{m:m:n}\end{cases} \tag{4}$$

where $\alpha_{i:m:n}=E(U_{i:m:n})$ is the expected value of the Type-II progressively censored order statistic from the uniform distribution on (0,1), which is given by Balakrishnan & Sandhu (1995) as

$$\alpha_{i:m:n}=1-\prod_{j=m-i+1}^{m}\left\{\frac{j-1+R_{m-j+1}+\ldots+R_m}{j+R_{m-j+1}+\ldots+R_m}\right\}.$$

The testing of interest in this article, is

$$H_0:F(x)=F_0(x)\quad vs\quad H_1:F(x)\neq F_0(x),$$

where $F_0(x) = 1-\exp(-\frac{x}{\theta})$ with $x > 0,\ \theta > 0$, and θ is the unknown parameter. The performance of the proposed tests is, then, compared to exponentiality tests for PCS Type-II data in the literature. These tests are as follows:

- The test statistic proposed by Balakrishnan et al. (2007)

$$T(w,n,m) = -H(w,n,m) + \frac{m}{n}\left[\log\left(\frac{1}{m}\sum_{i=1}^{m}(R_i+1)X_{i:m:n}\right)+1\right],$$

where $H(w,n,m) = \frac{1}{n}\sum_{i=1}^{m}\log\left(\frac{x_{i+w:m:n}-x_{i-w:m:n}}{E(U_{i+w:m:n})-E(U_{i-w:m:n})}\right) - (1-\frac{m}{n})\log(1-\frac{m}{n})$.

- The test statistic proposed by Alizadeh Noughabi (2017)

$$\begin{aligned} TA(w,m,n) = & -\frac{1}{n}\sum_{i=1}^{m}\log\left[\frac{\exp(-X_{i-w:m:n}/\hat{\theta})-\exp(-X_{i+w:m:n}/\hat{\theta})}{E(U_{i+w:m:n})-E(U_{i-w:m:n})}\right] \\ & + \frac{1}{n}\sum_{i=1}^{m}R_i\log\left[\frac{1-m/n}{\exp(-X_{i:m:n}/\hat{\theta})}\right]. \end{aligned}$$

3. Extentions of Tsallis Divergence and Test Statistics

In this section new measures of distance between two distributions that are similar to Tsallis divergences are defined.

Definition 1. Let X and Y be two non negative and absolutely continuous random variables with cdfs F and G and pdfs f and g, respectively. Then cumulative residual Tsallis (CRT) and cumulative Tsallis (CT) divergence between these distributions are respectively as follows:

$$CRT(F:G) = \frac{1}{\alpha-1}\left[\int_0^\infty \bar{F}^{\alpha}(x)\bar{G}^{1-\alpha}(x)dx - \alpha E(X) - (1-\alpha)E(Y)\right], \quad (5)$$

$$\begin{aligned} CT(F:G) = & \frac{1}{\alpha-1}\Bigg[\int_0^\infty F^{\alpha}(x)G^{1-\alpha}(x)dx \\ & -\alpha\int_0^\infty F(x)dx - (1-\alpha)\int_0^\infty G(x)dx\Bigg], \quad 0<\alpha<1. \end{aligned} \quad (6)$$

Lemma 1. *CRT(F:G)$\geq$ 0 and equality holds iff $F = G$.*

Proof. By applying the Hölder inequality, we obtain

$$\int_0^\infty \bar{F}^{\alpha}(x)\bar{G}^{1-\alpha}(x)dx \leq \left(\int_0^\infty \bar{F}(x)dx\right)^{\alpha}\left(\int_0^\infty \bar{G}(x)dx\right)^{1-\alpha}, \qquad 0<\alpha<1, \quad (7)$$

and by using the Young inequality, we get

$$\left(\int_0^\infty \bar{F}(x)dx\right)^\alpha \left(\int_0^\infty \bar{G}(x)dx\right)^{1-\alpha} \leq \alpha \int_0^\infty \bar{F}(x)dx + (1-\alpha)\int_0^\infty \bar{G}(x)dx. \quad (8)$$

Therefore, by (7) and (8) and dividing by $\alpha - 1$, the desired inequality follows. In the Hölder inequality, equality holds iff $\bar{F} = c\bar{G}$ (c is a positive constant) and in the Young inequality, equality holds iff $\int_0^\infty \bar{F}(x)dx = \int_0^\infty \bar{G}(x)dx$. Thus, $c = 1$ and $CRT(F : G) = 0$ iff $F = G$. □

Lemma 2. *CT(F:G)$\geq$ 0 and equality holds iff $F = G$.*

Proof. The proof is similar to the Lemma 1. □

Remark 1. Note that $\lim_{\alpha\to 1} CRT = CRKL$ and $\lim_{\alpha\to 1} CT = CKL$.

In order to construct test statistics, the given properties in Lemma 1 and 2 can be taken as proper motivators.

3.1. Testing Procedures Based on the Extentions of Tsallis Divergence

In this section, by utilizing (4) and estimating new divergences, test statistics are constructed for testing exponentiality with the PCS Type-II data and then some competing tests are considered to be compared with the mentioned tests. Accordingly, letting $F(x) = F_{m:n}(x)$ and $G(x) = F_0(x)$ in (5), we have

$$\begin{aligned} CRT(F_{m:n} : F_0) = {} & \frac{1}{\alpha - 1}\Bigg[\int_0^{x_{m:m:n}} (1 - F_{m:n}(x))^\alpha e^{-\frac{x}{\theta}(1-\alpha)}dx \\ & - \alpha \int_0^{x_{m:m:n}} (1 - F_{m:n}(x))dx - (1-\alpha)\int_0^{x_{m:m:n}} e^{-\frac{x}{\theta}}dx\Bigg], \\ = {} & -\frac{\theta}{(\alpha-1)^2}\Bigg[\sum_{i=0}^{m-1}(1-\alpha_{i:m:n})^\alpha\big(e^{-\frac{x_{i:m:n}}{\theta}(1-\alpha)} - e^{-\frac{x_{i+1:m:n}}{\theta}(1-\alpha)}\big)\Bigg] \\ & - \frac{\alpha}{\alpha-1}\Bigg[\sum_{i=0}^{m-1}(1-\alpha_{i:m:n})(x_{i+1:m:n} - x_{i:m:n})\Bigg] \\ & + \theta(1 - e^{-\frac{x_{m:m:n}}{\theta}}), \end{aligned} \quad (9)$$

where $\alpha_{0:m:n} = x_{0:m:n} = 0$. Dividing (9) by $\int_0^{x_{m:m:n}}(1 - F_{m:n}(x))dx$, we obtain the proposed test as follows:

$$
\begin{aligned}
CRT_{mn} = & -\frac{\hat{\theta}}{(\alpha-1)^2}\left[\frac{\sum\limits_{i=0}^{m-1}(1-\alpha_{i:m:n})^{\alpha}\left(e^{-\frac{x_{i:m:n}}{\hat{\theta}}(1-\alpha)}-e^{-\frac{x_{i+1:m:n}}{\hat{\theta}}(1-\alpha)}\right)}{\sum\limits_{i=0}^{m-1}(1-\alpha_{i:m:n})(x_{i+1:m:n}-x_{i:m:n})}\right] \\
& + \frac{\hat{\theta}(1-e^{-\frac{x_{m:m:n}}{\hat{\theta}}})}{\sum_{i=0}^{m-1}(1-\alpha_{i:m:n})(x_{i+1:m:n}-x_{i:m:n})} - \frac{\alpha}{\alpha-1},
\end{aligned}
\tag{10}
$$

where $\hat{\theta} = \frac{1}{m}\sum_{i=1}^{m}(R_i+1)x_{i:m:n}$ is the maximum likelihood estimate (MLE) based on the PCS Type-II sample.

Similarly, for (6), we have

$$
\begin{aligned}
CT(F_{m:n}:F_0) = & \frac{1}{(\alpha-1)}\left[\sum_{i=1}^{m-1}(\alpha_{i:m:n})^{\alpha}\int_{x_{i:m:n}}^{x_{i+1:m:n}}\left(1-e^{-\frac{x(1-\alpha)}{\theta}}\right)dx\right] \\
& - \frac{\alpha}{\alpha-1}\left[\sum_{i=1}^{m-1}(\alpha_{i:m:n})(x_{i+1:m:n}-x_{i:m:n})\right] \\
& + \int_{0}^{x_{m:m:n}}(1-e^{-\frac{x}{\theta}})dx.
\end{aligned}
\tag{11}
$$

Dividing (11) by $\int_0^{x_{m:m:n}} F_{m:n}(x)dx$, we obtain the proposed test as follows:

$$
\begin{aligned}
CT_{mn} = & \frac{1}{(\alpha-1)}\left[\frac{\sum\limits_{i=1}^{m-1}(\alpha_{i:m:n})^{\alpha}\int_{x_{i:m:n}}^{x_{i+1:m:n}}\left(1-e^{-\frac{x(1-\alpha)}{\hat{\theta}}}\right)dx}{\sum\limits_{i=1}^{m-1}(\alpha_{i:m:n})(x_{i+1:m:n}-x_{i:m:n})}\right] \\
& + \frac{\int_0^{x_{m:m:n}}(1-e^{-\frac{x}{\hat{\theta}}})dx}{\sum\limits_{i=1}^{m-1}(\alpha_{i:m:n})(x_{i+1:m:n}-x_{i:m:n})} - \frac{\alpha}{\alpha-1},
\end{aligned}
\tag{12}
$$

where $\hat{\theta}$ is the MLE of based on the PCS Type-II sample.

3.2. Testing Procedures Based on Equilibrium Distributions for the Tsallis Divergences

Similar to the subsection 3.1, using (3) based on the PCS Type-II data, we obtain the proposed test as follow:

$$\begin{aligned} D^*_{T_{mn}} &= \frac{1}{\alpha-1}\left[\frac{\hat{\theta}^{\alpha-1}\int_0^{x_{m:m:n}}(1-F_{m:n}(x))^{\alpha}e^{-\frac{x}{\hat{\theta}}(1-\alpha)}dx}{\left(\int_0^{x_{m:m:n}}(1-F_{m:n}(x))dx\right)^{\alpha}}-1\right] \\ &= -\frac{\hat{\theta}^{\alpha}}{(\alpha-1)^2}\left[\frac{\sum_{i=0}^{m-1}(1-\alpha_{i:m:n})^{\alpha}\left(e^{-\frac{x_{i:m:n}}{\hat{\theta}}(1-\alpha)}-e^{-\frac{x_{i+1:m:n}}{\hat{\theta}}(1-\alpha)}\right)}{\left(\sum_{i=0}^{m-1}(1-\alpha_{i:m:n})(x_{i+1:m:n}-x_{i:m:n})\right)^{\alpha}}\right] \\ &\quad -\frac{1}{\alpha-1}, \end{aligned} \tag{13}$$

where $\alpha_{0:m:n}=x_{0:m:n}=0$ and $\hat{\theta}$ is the MLE of the PCS Type-II sample.

Note that all the three proposed test statistics are scale-invariant.

4. Simulation Study

For large values of the proposed test statistics, the null hypothesis will be rejected. The power values of the proposed tests depend on two things, the α values and type of failure rate function of alternatives. Thus, the alternatives are selected according to the type of failure rate function as follows:

- Increasing failure rate (IFR): Gamma and Weibull (shape parameter 2),
- Decreasing failure rate (DFR): Gamma and Weibull (shape parameter 0.5),
- Non-monotone failure rate (NFR): Log-normal (shape parameter 0.5), Log-normal (shape parameter 1).

Since the α values have an important role in determining the power of the proposed tests, then the α value that maximizes the power, is considered according to the type of failure rate function. Moreover since the CT_{mn} and CRT_{mn} statistics have not good performance, respectively, for alternatives with IFR and DFR functions, thus the CT_{mn} and CRT_{mn} statistics are recommended for DFR function and IFR function, respectively.

The α value, for $D_{T^*_{mn}}$ and CRT_{mn} statistics, when the alternatives have the IFR function, is suggested to be 2 and 0.01, respectively and for $D^*_{T_{mn}}$ and CT_{mn} statistics, when the alternatives have the DFR function, is suggested to be 0.01. For alternatives with NFR functions, the α value is suggested 0.01 for the proposed statistics. The power values are determined for the 27 censoring schemes used by Pakyari & Balakrishnan (2012). These censoring schemes are given in Table 1. To obtain the power values 50,000 random samples for several sample sizes and PCS, are generated. Results showed that, when the alternative model is the exponential model, significance level of the test is concluded. Also by following Alizadeh Noughabi (2017), the values of w which maximize the power of the TA statistic are chosen.

TABLE 1: Progressive censoring schemes used in the Monte Carlo simulations.

Scheme No.	n	m	$(R_1, \ldots, R_m)$
[1]			$R_1 = 12, R_i = 0$ for $i \neq 1$
[2]		8	$R_8 = 12, R_i = 0$ for $i \neq 8$
[3]			$R_1 = R_8 = 6, R_i = 0$ for $i \neq 1, 8$
[4]			$R_1 = 8, R_i = 0$ for $i \neq 1$
[5]	20	12	$R_{12} = 8, R_i = 0$ for $i \neq 12$
[6]			$R_3 = R_5 = R_7 = R_9 = 2, R_i = 0$ for $i \neq 3, 5, 7, 9$
[7]			$R_1 = 4, R_i = 0$ for $i \neq 1$
[8]		16	$R_{16} = 4, R_i = 0$ for $i \neq 16$
[9]			$R_5 = 4, R_i = 0$ for $i \neq 5$
[10]			$R_1 = 30, R_i = 0$ for $i \neq 1$
[11]		10	$R_{10} = 30, R_i = 0$ for $i \neq 10$
[12]			$R_1 = R_5 = R_{10} = 10, R_i = 0$ for $i \neq 1, 5, 10$
[13]			$R_1 = 20, R_i = 0$ for $i \neq 1$
[14]	40	20	$R_{20} = 20, R_i = 0$ for $i \neq 20$
[15]			$R_i = 1$ for $i = 1, 2, \ldots, 20$
[16]			$R_1 = 10, R_i = 0$ for $i \neq 1$
[17]		30	$R_{30} = 10, R_i = 0$ for $i \neq 30$
[18]			$R_1 = R_{30} = 5, R_i = 0$ for $i \neq 1, 30$
[19]			$R_1 = 40, R_i = 0$ for $i \neq 1$
[20]		20	$R_{20} = 40, R_i = 0$ for $i \neq 20$
[21]			$R_1 = R_{20} = 10, R_{10} = 20, R_i = 0$ for $i \neq 1, 10, 20$
[22]			$R_1 = 20, R_i = 0$ for $i \neq 1$
[23]	60	40	$R_{40} = 20, R_i = 0$ for $i \neq 40$
[24]			$R_{2i-1} = 1, R_{2i} = 0$ for $i = 1, 2, \ldots, 20$
[25]			$R_1 = 10, R_i = 0$ for $i \neq 1$
[26]		50	$R_{50} = 10, R_i = 0$ for $i \neq 50$
[27]			$R_1 = R_{50} = 5, R_i = 0$ for $i \neq 1, 50$

Tables 2-4 present power values of the proposed tests and the existing tests at the significance level 0.10 based on the type of failure rate function. According to these tables, it can be said that the proposed tests are evidency consistent because with increasing sample size, the test power close to 1. Table 2 (for alternatives with IFR functions) indicates that, almost in the most cases, the TA statistic has higher power than other tests. Also, we can see that, the difference of powers of the CRT_{mn} and TA statistics do not differ much. Although the TA statistics have good powers. One of the disadvantages of this statistic is that we should calculate the power values for three different values of window size w, and, for different censoring schemes, there is not a window size w of same value. While if Balakrishnan et al. (2007) had considered w values proposed by Alizadeh Noughabi (2017) for each censorship plan, they would have had higher powers compared to TA.

Table 2: Power of the proposed tests for the alternatives with the IFR function at the significance level 0.10 for several schemes.

	W(2)				G(2)			
Scheme No.	$D^*_{T_{mn}}$	CRT_{mn}	T	TA	$D^*_{T_{mn}}$	CRT_{mn}	T	TA
[1]	0.858	0.893	0.892	**0.896**	0.610	0.614	**0.648**	0.231
[2]	**0.643**	0.661	0.634	0.627	**0.488**	0.457	0.459	0.486
[3]	**0.776**	0.748	0.725	0.712	**0.549**	0.513	0.512	0.454
[4]	0.879	0.938	0.916	**0.949**	0.684	0.613	**0.722**	0.484
[5]	**0.811**	0.803	0.783	0.799	**0.612**	0.533	0.583	0.603
[6]	0.807	0.898	0.891	**0.949**	0.633	0.573	**0.675**	0.226
[7]	0.896	0.959	0.958	**0.976**	0.714	0.658	**0.765**	0.651
[8]	0.900	0.903	**0.922**	0.920	**0.691**	0.658	0.686	0.688
[9]	0.914	0.958	0.970	**0.983**	0.702	0.680	**0.772**	0.633
[10]	0.939	0.942	0.953	**0.965**	0.666	0.756	**0.814**	0.232
[11]	**0.726**	0.716	0.678	0.683	0.594	0.538	0.550	**0.597**
[12]	0.817	**0.841**	0.812	0.815	**0.684**	0.634	0.646	0.527
[13]	0.963	0.988	0.988	**0.995**	0.765	0.791	**0.899**	0.670
[14]	**0.945**	0.933	0.918	0.922	0.794	0.724	0.759	**0.806**
[15]	0.905	0.963	0.970	**0.991**	0.768	0.756	**0.862**	0.465
[16]	0.977	0.996	0.998	**1.000**	0.800	0.827	**0.942**	0.882
[17]	**0.992**	0.994	0.991	0.990	0.892	0.842	0.879	**0.904**
[18]	0.993	**0.997**	0.996	0.995	0.900	0.863	**0.902**	0.901
[19]	0.979	0.989	0.993	**0.998**	0.756	0.850	**0.934**	0.591
[20]	**0.941**	0.930	0.898	0.901	0.825	0.747	0.776	**0.834**
[21]	**0.980**	0.978	0.970	0.974	**0.857**	0.816	0.851	0.738
[22]	0.992	0.999	**1.000**	**1.000**	0.815	0.906	**0.980**	0.945
[23]	0.998	**0.999**	0.998	0.996	0.957	0.927	0.933	**0.965**
[24]	0.964	0.992	0.999	**1.000**	0.773	0.871	**0.960**	0.849
[25]	0.994	0.999	**1.000**	**1.000**	0.831	0.924	**0.991**	0.981
[26]	**1.000**	**1.000**	**1.000**	**1.000**	0.971	0.956	0.973	**0.985**
[27]	**1.000**	**1.000**	**1.000**	**1.000**	0.960	0.961	0.979	**0.984**

For each censoring scheme, the greatest powers are in bold.

Since usually the CRT_{mn} statistic for the scheme $R = (n-m, 0, \ldots, 0)$ shows higher power than the other schemes, so this statistic for early censoring is recommended. In Table 3 for alternatives with DER functions, the CT_{mn} statistic has higher power than $D^*_{T_{mn}}$ and the other existing tests except in the censoring scheme 24. In this table, the scheme $R = (n-m, 0, \ldots, 0)$ generally indicates better power than the other schemes. It can be concluded that early censoring situations seem to possess higher power. Therefore for alternatives with DER functions, the use of CT_{mn} statistic for the case of early censoring is recommended. Table 4 shows that the TA and CRT_{mn} statistics have approximately higher powers than the other tests, but for different censoring schemes a general conclusion cannot be suggested.

TABLE 3: Power of the proposed tests for the alternatives with the DFR function at the significance level 0.10 for several schemes.

	W(0.5)				G(0.5)			
Scheme No.	$D^*_{T_{mn}}$	CT_{mn}	T	TA	$D^*_{T_{mn}}$	CT_{mn}	T	TA
[1]	0.661	**0.717**	0.064	0.303	0.420	**0.467**	0.014	0.161
[2]	0.005	**0.291**	0.069	0.213	0.010	**0.219**	0.047	0.166
[3]	0.002	**0.448**	0.073	0.252	0.005	**0.323**	0.040	0.177
[4]	0.750	**0.806**	0.341	0.476	0.455	**0.511**	0.097	0.230
[5]	0.002	**0.590**	0.272	0.387	0.006	**0.398**	0.134	0.250
[6]	0.752	**0.792**	0.344	0.489	0.496	**0.546**	0.109	0.258
[7]	0.812	**0.869**	0.416	0.605	0.471	**0.544**	0.104	0.291
[8]	0.069	**0.847**	0.402	0.564	0.016	**0.613**	0.147	0.318
[9]	0.830	**0.889**	0.437	0.637	0.509	**0.587**	0.111	0.315
[10]	0.728	**0.787**	0.217	0.442	0.448	**0.508**	0.050	0.227
[11]	0.003	**0.345**	0.152	0.270	0.005	**0.278**	0.112	0.226
[12]	0.003	**0.672**	0.187	0.381	0.005	**0.567**	0.111	0.287
[13]	0.866	**0.919**	0.661	0.748	0.530	**0.615**	0.234	0.395
[14]	0.000	**0.760**	0.543	0.600	0.001	**0.570**	0.336	0.421
[15]	0.628	**0.864**	0.627	0.756	0.336	**0.636**	0.309	0.485
[16]	0.934	**0.971**	0.865	0.892	0.583	**0.695**	0.381	0.520
[17]	0.000	**0.959**	0.829	0.830	0.001	**0.779**	0.487	0.551
[18]	0.415	**0.979**	0.854	0.866	0.079	**0.821**	0.460	0.550
[19]	0.866	**0.922**	0.673	0.765	0.539	**0.630**	0.246	0.429
[20]	0.000	**0.726**	0.524	0.566	0.001	**0.586**	0.375	0.450
[21]	0.000	**0.916**	0.677	0.745	0.001	**0.795**	0.429	0.553
[22]	0.965	**0.990**	0.949	0.960	0.638	**0.769**	0.519	0.639
[23]	0.000	**0.980**	0.928	0.904	0.000	**0.857**	0.670	0.677
[24]	0.898	0.947	0.924	**0.952**	0.572	0.665	0.570	**0.700**
[25]	0.983	**0.997**	0.989	0.985	0.682	**0.821**	0.680	0.711
[26]	0.178	**0.997**	0.986	0.969	0.007	**0.928**	0.783	0.742
[27]	0.918	**0.999**	0.990	0.978	0.402	**0.934**	0.765	0.741

For each censoring scheme, the greatest powers are in bold.

5. Illustrative Examples

In this section, the proposed tests procedure are investigated by three examples. In the first example, the real dataset with $n = 19$ and $m = 8$ is considered, and in the second example, a real dataset with $n = 32$ and $m = 20$ is applied. Finally a real dataset with $n = 20$ and $m = 10$ is used in Example 3. In the examples, the $D^*_{T_{mn}}$ statistic has the same results for the α values equal to 2 and 0.01.

Example 1. Nelson (1982) reported data on times to breakdown of an in-sulating fluid in an accelerated test which was done at different test voltages. From these data, Viveros & Balakrishnan (1994) produced a PCS Type-II sample of size from observations which was recorded at 34 kilovolts.

Table 4: Power of the proposed tests for the alternatives with the NFR function at the significance level 0.10 for several schemes.

	L(0,0.5)					L(0,1)				
S.N	$D^*_{T_{mn}}$	CRT_{mn}	CT_{mn}	T	TA	$D^*_{T_{mn}}$	CRT_{mn}	CT_{mn}	T	TA
[1]	0.254	0.990	0.000	0.995	0.999	0.134	0.280	0.123	0.304	0.364
[2]	0.956	0.993	0.938	0.987	0.987	0.348	0.468	0.287	0.428	0.440
[3]	0.993	0.998	0.453	0.995	0.996	0.410	0.469	0.082	0.437	0.459
[4]	0.008	0.996	0.001	0.996	1.000	0.194	0.269	0.189	0.285	0.353
[5]	0.975	1.000	0.955	0.997	0.999	0.285	0.497	0.185	0.415	0.484
[6]	0.022	0.994	0.000	0.991	0.998	0.164	0.316	0.157	0.283	0.310
[7]	0.001	0.998	0.001	0.997	0.999	0.241	0.250	0.240	0.260	0.351
[8]	0.967	1.000	0.866	1.000	1.000	0.175	0.451	0.084	0.399	0.486
[9]	0.001	0.997	0.001	0.998	1.000	0.236	0.262	0.232	0.269	0.336
[10]	0.615	0.996	0.037	1.000	1.000	0.137	0.283	0.123	0.483	0.581
[11]	0.996	1.000	0.995	0.999	0.999	0.561	0.701	0.492	0.614	0.652
[12]	0.999	1.000	0.295	1.000	1.000	0.513	0.691	0.021	0.635	0.653
[13]	0.016	0.999	0.052	1.000	1.000	0.255	0.225	0.256	0.440	0.588
[14]	1.000	1.000	1.000	1.000	1.000	0.478	0.784	0.379	0.667	0.758
[15]	0.937	1.000	0.000	1.000	1.000	0.201	0.500	0.093	0.520	0.618
[16]	0.001	0.999	0.056	1.000	1.000	0.340	0.181	0.343	0.414	0.588
[17]	1.000	1.000	1.000	1.000	1.000	0.250	0.708	0.146	0.648	0.758
[18]	0.999	1.000	0.985	1.000	1.000	0.170	0.569	0.107	0.619	0.743
[19]	0.251	0.999	0.364	1.000	1.000	0.227	0.213	0.230	0.579	0.736
[20]	1.000	1.000	1.000	1.000	1.000	0.695	0.897	0.620	0.801	0.858
[21]	1.000	1.000	0.698	1.000	1.000	0.509	0.799	0.028	0.779	0.793
[22]	0.002	0.999	0.408	1.000	1.000	0.390	0.137	0.403	0.540	0.764
[23]	1.000	1.000	1.000	1.000	1.000	0.411	0.863	0.323	0.837	0.894
[24]	0.004	0.999	0.002	1.000	1.000	0.243	0.245	0.245	0.496	0.751
[25]	0.001	0.999	0.474	1.000	1.000	0.449	0.113	0.464	0.572	0.773
[26]	1.000	1.000	1.000	1.000	1.000	0.178	0.672	0.153	0.782	0.880
[27]	1.000	1.000	1.000	1.000	1.000	0.106	0.462	0.190	0.762	0.878

These progressively censored data are given in Table 5. Table 6 indicates the critical values and test statistics. Based on Table 6, all of the tests at the significance level 0.10, show that this progressively Type-II right censored sample comes from an exponential distribution.

Example 2. The data of the study by Spineili & Stephens (1987) is considered in this example which consists of the modules of repute of woods. By randomly generating observations from these data, Balakrishnan & Lin (2003) considered PCS Type-II data. The data and the employed PCS are presented in Table 7. Pakyari & Balakrishnan (2012) concluded that for the given PCS Type-II data, the Weibull model is accepted at the significance level 0.05. The critical values and test statistics for the proposed tests and the other tests of this paper, were calculated in Table 8. The results of Table 10 show that, all of the test statistics reject the null hypothesis of exponentiality at the significance level 0.10. It should

TABLE 5: Progressively censored sample generated from the times to breakdown data on insulating fluids tested at 34 kilovolts, given by Viveros & Balakrishnan (1994).

i	1	2	3	4	5	6	7	8
$x_{i:8:19}$	0.19	0.78	0.96	1.31	2.78	4.85	6.50	7.35
R_i	0	0	3	0	3	0	0	5

TABLE 6: Test statistics and critical values of the tests.

	$D^*_{T_{mn}}$	CRT_{mn}	CT_{mn}	T	TA		
					$w=1$	$w=2$	$w=4$
Test statistic	0.8185	0.0225	-0.0100	-0.0906	-0.0135	-0.0598	-0.1227
Critical value	1.1615	0.0706	-0.0099	0.0662	0.2095	0.0975	0.0579

be note that the obtained values for test statistics in Alizadeh Noughabi (2017) in Example 2 are not correct. These values were modified in Table 10.

TABLE 7: Spinelli and Stephens's data and the PCS Type-II.

i	1	2	3	4	5	6	7	8	9	10
$x_{i:20:32}$	43.19	49.44	51.55	56.63	67.27	78.47	86.59	90.63	94.38	98.21
R_i	0	2	0	0	2	0	0	0	0	0
i	11	12	13	14	15	16	17	18	19	20
$x_{i:20:32}$	98.39	99.74	100.22	103.48	105.54	107.13	108.14	108.94	110.81	116.39
R_i	2	2	0	0	0	0	1	1	0	2

TABLE 8: Test statistics and critical values of the tests.

	$D^*_{T_{mn}}$	CRT_{mn}	CT_{mn}	T	TA		
					$w=1$	$w=2$	$w=4$
Test statistic	0.4437	-0.5495	-0.0097	0.5973	0.6709	0.5778	0.5416 1
Critical value	0.2451	-1.8690	-0.0098	-0.0315	0.1583	-0.0381	-0.0518

Example 3. Pakyari & Balakrishnan (2012) randomly generated a PCS Type-II sample of size $m = 10$ from $n = 20$ observations based on the wire connection strength data from Nelson (1982, Table 5.1, p. 111). The data, and the PCS employed are given in Table 9. Nelson (1982) and Pakyari & Balakrishnan (2012) concluded that normal model is strongly supported for describing the wire connection strength data. The test statistics and the critical values are presented in Table 10. Based on Table 10, as expected the exponential model at the significance level 0.10, is supported by none of the test statistics. It should be note that the obtained values for test statistics in Alizadeh Noughabi (2017) in Example 1 and the conclusions drawn from it are not correct. These values were modified in Table 10.

Table 9: Wire connection strength data and the PCS Type-II.

i	1	2	3	4	5	6	7	8	9	10
$x_{i:10:20}$	550	750	950	1150	1150	1150	1350	1450	1550	1850
R_i	0	2	1	0	3	0	0	2	0	2

Table 10: Test statistics and critical values of the tests.

	$D^*_{T_{mn}}$	CRT_{mn}	CT_{mn}	T	TA		
					$w=1$	$w=2$	$w=4$
Test statistic	0.9139	0.1710	-0.0099	0.2537	∞	0.2554	0.2842
Critical value	0.8316	0.1144	-0.0100	0.1858	0.3117	0.2136	0.1770

6. Conclusion

We defined new divergence measures as generalizations of the Tsallis divergence and considered testing exponentiality based on the PCS Type-II sample. In order to compare the power values of proposed tests with the power values of existing tests, we utilized Monte Carlo simulations. The simulation results showed that, for alternative hypothesis with IFR and DER functions, the difference of powers of CRT_{mn} and TA tests are not remarkable. Due to the problems stated for the TA test, the CRT_{mn} test for the censorship scheme $R = (n-m, 0, \dots, 0)$ it is suggested. Moreover, the CT_{mn} test for alternative hypothesis with DER function, for almost all censorship schemes, has higher power than the other existing tests.

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