Revista Colombiana de Estadística - Theoretical Statistics
January 2022, volume 45, issue 1, pp. 193–208
http://doi.org/10.15446/rce.v45n1.97466

Bahadur's Stochastic Comparison of Combining infinitely Independent Tests in Case of Extreme Value Distribution

Comparación estocástica de Bahadur de la combinación de pruebas infinitamente independientes en caso de distribución de valor extremo

Abedel-Qader Al-Masri[a], Noor Al-Momani[b]

Department of Statistics, Science, Yarmouk University, Irbid, Jordan

Abstract

For simple null hypothesis, given any non-parametric combination method which has a monotone increasing acceptance region, there exists a problem for which this method is most powerful against some alternative. Starting from this perspective and recasting each method of combining p-values as a likelihood ratio test, we present theoretical results for some of the standard combiners which provide guidance about how a powerful combiner might be chosen in practice. In this paper we consider the problem of combining n independent tests as $n \to \infty$ for testing a simple hypothesis in case of extreme value distribution (EV(θ,1)). We study the six free-distribution combination test producers namely; Fisher, logistic, sum of p-values, inverse normal, Tippett's method and maximum of p-values. Moreover, we studying the behavior of these tests via the exact Bahadur slope. The limits of the ratios of every pair of these slopes are discussed as the parameter $\theta \to 0$ and $\theta \to \infty$. As $\theta \to 0$, the logistic procedure is better than all other methods, followed in decreasing order by the inverse normal, the sum of p-values, Fisher, maximum of p-values and Tippett's procedure. Whereas, $\theta \to \infty$ the logistic and the sum of p-values procedures are equivalent and better than all other methods, followed in decreasing order by Fisher, the inverse normal, maximum of p-values and Tippett's procedure.

***Key words*:** Bahadur efficiency; Bahadur slope; combining independent tests; extreme value distribution.

[a]MSc. E-mail: almasri68@yu.edu.jo
[b]MSc. E-mail: noor.m@yu.edu.jo

Resumen

Para hipótesis nulas simples, dado cualquier método de combinación no paramétrico que tenga una región de aceptación creciente monótona, existe un problema para el cual este método es más poderoso frente a alguna alternativa. Partiendo de esta perspectiva y reformulando cada método de combinación de valores p como una prueba de razón de verosimilitud, presentamos resultados teóricos para algunos de los combinadores estándar que brindan orientación sobre cómo se podría elegir un combinador poderoso en la práctica. En este artículo consideramos el problema de combinar pruebas independientes de n como $n \to \infty$ para probar una hipótesis simple en el caso de una distribución de valor extremo (EV (θ, 1)). Estudiamos los seis productores de prueba de combinación de distribución gratuita, a saber; Fisher, logística, suma de valores p, normal inversa, método de Tippett y máximo de valores p. Además, estudiamos el comportamiento de estas pruebas a través de la pendiente exacta de Bahadur. Los límites de las razones de cada par de estas pendientes se analizan como el parámetro $\theta \to 0$ y $\theta \to \infty$. Como $\theta \to 0$, la logística El procedimiento es mejor que todos los demás métodos, seguido en orden decreciente por el inverso normal, la suma de valores p, Fisher, el máximo de valores p y el procedimiento de Tippett. Considerando que, $\theta \to \infty$ la logística y la suma de los procedimientos de valores p so equivalentes y mejores que todos los demás métodos, seguidos en orden decreciente por Fisher, la inversa normal, máxima de valores p y procedimiento de Tippett.

***Palabras clave*:** combinación de pruebas independientes; distribución de valor extremo; eficiencia Bahadur; pendiente Bahadur.

1. Introduction

We consider the asymptotic relative efficiency (ARE) of two test procedures in which the probabilities of the two types of error change with increasing sample size n, and with respect to the alternative behavior. Abu-Dayyeh & El-Masri (1994) studied six methods of combining infinitely number of independent tests in case of triangular distribution. These methods are sum of p-values, inverse normal, logistic, Fisher, minimum of p-values and maximum of p-values. They showed that the sum of p-values is the best of all other methods. Abu-Dayyeh et al. (2003) combined infinity number of independent tests for testing simple hypotheses against one-sided alternative for normal and logistic distributions, they used four methods of combining (Fisher, logistic, sum of p-values and inverse normal). Al-Masri (2010) studied six methods of combining independent tests. He showed under conditional shifted Exponential distribution that the inverse normal method is the best among six combination methods. Al-Talib et al. (2020) considered combining independent tests in case of conditional normal distribution with probability density function $X|\theta \sim N(\gamma\theta)$, $\theta \in [a, \infty], a \geq 0$ when $\theta_1, \theta_2, \ldots$ have a distribution function (DF) F_θ. They concluded that the inverse normal procedure is better than the other procedures. Al-Masri (2021*a*) considered combining n independent tests of simple hypothesis, vs one-tailed alternative as n approaches infinity, in case of Laplace distribution $\mathbb{L}(\gamma, 1)$. He showed that the sum of p-values procedure is better than all other procedures under the null hypothesis, and the inverse normal procedure is better than the other procedures under the alternative hypothesis. Al-Masri & Al-Momani (2021) considered combining n independent tests of simple hypothesis, vs one-tailed alternative as n approaches infinity, in case of log-logistic distribution. They showed that the sum of p-values procedure is better than all other procedures under the null hypothesis and under the alternative hypothesis. Al-Masri (2021*b*) considered the problem of combining n independent tests as $n \to \infty$ for testing a simple hypothesis in case of log-normal distribution. He showed that as $\xi \to 0$, the maximum of p-values is better than all other methods, followed in decreasing order by the inverse normal, logistic, the sum of p-values, Fisher and Tippett's procedure. Also, as $\xi \to \infty$ the worst method the sum of p-values and the other methods remain the same, since they have the same limit.

2. Extreme Value (Gumbel) Distribution

The extreme value Gumbel distribution (EV(θ,1)) is used to model maximums and minimums. For example, it has been used to predict earthquakes, floods and other natural disasters, as well as modeling operational risk in risk management and the life of products that quickly wear out after a certain age.

Extreme value distributions are the limiting distributions for the minimum or the maximum of a very large collection of random observations from the same arbitrary distribution. Extreme value distributions for the minimum are frequently encountered. For example, if a system consists of n identical components in series, and the system fails when the first of these components fails, then system failure times are the minimum of n random component failure times. In extreme value theory, independent of the choice of component model, the system model will approach a Weibull as n becomes large. The same reasoning can also be applied at a component level, if the component failure occurs when the first of many similar competing failure processes reaches a critical level.

The $EV(\theta, 1)$ distribution with location parameter θ, has distribution function (DF) and probability density function (pdf) are given, respectively, by

$$F(x;\theta) = e^{-e^{-(x-\theta)}}, x \in \mathbb{R}, \theta \in \mathbb{R} \tag{1}$$

$$f(x;\theta) = e^{-(x-\theta)-e^{-(x-\theta)}} = -F(x;\theta)\ln F(x;\theta), x \in \mathbb{R}, \theta \in \mathbb{R} \tag{2}$$

3. The Basic Problem

Consider testing the hypothesis

$$H_0^{(i)} : \eta_i = \eta_0^i, \quad \text{vs}, \quad H_1^{(i)} : \eta_i \in \Omega_i - \{\eta_0^i\} \tag{3}$$

such that $H_0^{(i)}$ becomes rejected for large values of some real valued continuous random variable $T^{(i)}$, $i = 1, 2, \ldots, n$. The n hypotheses are combined into one as

$$H_0^{(i)} : (\eta_1, \ldots, \eta_n) = (\eta_0^1, \ldots, \eta_0^n), \text{ vs }, H_1^{(i)} : (\eta_1, \ldots, \eta_n) \in \left\{ \prod_{i=1}^{n} \Omega_i - \{(\eta_0^1, \ldots, \eta_0^n)\} \right\} \quad (4)$$

For $i = 1, 2, \ldots, n$, the p-value of the i-th test is given by

$$P_i(t) = P_{H_0^{(i)}}\left(T^{(i)} > t\right) = 1 - F_{H_0^{(i)}}(t) \quad (5)$$

where $F_{H_0^{(i)}}(t)$ is the DF of $T^{(i)}$ under $H_0^{(i)}$. Note that $P_i \sim U(0,1)$ under $H_0^{(i)}$.

As a special case where $\eta_i = \theta$ and $\eta_0^i = \theta_0$ for $i = 1, \ldots, n$, and assume that $T^{(1)}, \ldots, T^{(n)}$ are independent, then (3) reduces to

$$H_0 : \theta = \theta_0, \quad \text{vs,} \quad H_1 : \theta \in \Omega - \{\theta_0\} \quad (6)$$

It follows that the p-values $P_1, \ldots, P_n$ are also iid rv's that have a $U(0,1)$ distribution under H_0, and under H_1 have a distribution whose support is a subset of the interval $(0,1)$ and is not a $U(0,1)$ distribution. Therefore, if f is the probability density function (pdf) of P, then (6) is equivalent to

$$H_0 : P \sim U(0,1), \quad \text{vs,} \quad H_1 : P \nsim U(0,1) \quad (7)$$

where P has a pdf f with support a subset of the interval $(0,1)$.

This study considers the case: $\eta_i = 0$, $i = 1, \ldots, n$. Also we are assuming that $T^{(1)}, T^{(2)}, \ldots, T^{(n)}$ are independent. Then (6) reduced to

$$H_0 : \theta = 0, \quad \text{vs,} \quad H_1 : \theta > 0 \quad (8)$$

Thus, under H_0, the p-values $P_1, P_2, \ldots, P_n$ are iid rv's distributed with a uniform distribution $U(0,1)$ which is given by (8).

By sufficiency we may assume $n_i = 1$ and $T^{(i)} = X_i$ for $i = 1, \ldots, n$. Then we consider the sequence $\left\{T^{(n)}\right\}$ of independent test statistics that is we will take a random sample $X_1, \ldots, X_n$ of size n and let $n \to \infty$ and compare the four non-parametric methods via exact Bahadur slope (EBS).

The producers will be used in this paper are Fisher, logistic, sum of p-values, inverse normal, Tippett's method and maximum of p-values. These producers are based on p-values of the individual statistics T_i, and reject H_0 if

$$
\begin{aligned}
\Psi_{Fisher} &= -2\sum_{i=1}^{n} \ln(P_i) > \chi^2_{2n,\alpha},\\
\Psi_{logistic} &= -\sum_{i=1}^{n} \ln\left(\frac{P_i}{1-P_i}\right) > b_\alpha,\\
\Psi_{normal} &= -\sum_{i=1}^{n} \Phi^{-1}(P_i) > \sqrt{n}\Phi^{-1}(1-\alpha),\\
\Psi_{sum} &= -\sum_{i=1}^{n} P_i > C_\alpha,\\
\Psi_{max} &= -\max\ P_i < \alpha^{\frac{1}{n}},\\
\Psi_{T} &= -\min\ P_i < 1-(1-\alpha)^{\frac{1}{n}}.
\end{aligned}
$$

where Φ is the DF of standard normal distribution.

4. Definitions

This section lays out some tools basic to Bahadur's stochastic comparison theory as used in this article

Definition 1 (Serfling 2009, Bahadur efficiency and exact Bahadur slope (EBS))**.** Let $X_1, \dots, X_n$ be i.i.d. from a distribution with a probability density function $f(x, \theta)$, and we want to test $H_0 : \theta = \theta_0$ vs $H_1 : \theta \in \Theta - \{\theta_0\}$. Let $\left\{T_n^{(1)}\right\}$ and $\left\{T_n^{(2)}\right\}$ be two sequences of test statistics for testing H_0. Let the significance attained by $T_n^{(i)}$ be $L_n^{(i)} = 1 - F_i\left(T_n^{(i)}\right)$, where $F_i\left(T_n^{(i)}\right) = P_{H_0}\left(T_n^{(i)} \leq t_i\right)$, $i = 1, 2$. Then there exists a positive valued function $C_i(\theta)$ called the exact Bahadur slope of the sequence $\{T_n^{(i)}\}$ such that

$$C_i(\theta) = \lim_{\theta\to\infty} -2n^{-1}\ln\left(L_n^i\right)$$

with probability 1 (w.p.1) under θ and the Bahadur efficiency of $\left\{T_n^{(1)}\right\}$ relative to $\left\{T_n^{(2)}\right\}$ is given by $e_B\left(T_1, T_2\right) = C_1(\theta)/C_2(\theta)$.

Theorem 1 (Serfling 2009, Large deviation theorem)**.** *Let $X_1, X_2, \dots, X_n$ be i.i.d., with distribution F and put $S_n = \sum_{i=1}^n X_i$. Assume existence of the moment generating function (mgf) $M(z) = E_F\left(e^{zX}\right)$, z real, and put $m(t) = \inf_z e^{-z(X-t)} = \inf_z e^{-zt}M(z)$. The behavior of large deviation probabilities $P\left(S_n \geq t_n\right)$, where $t_n \to \infty$ at rates slower than $O(n)$. The case $t_n = tn$, if $-\infty < t \leq EY$, then $P\left(S_n \leq nt\right) \leq [m(t)]^n$, the*

$$-2n^{-1}\ln P_F\left(S_n \geq nt\right) \to -2\ln m(t) \qquad a.s. \quad (F_\theta)\,.$$

Theorem 2 (Bahadur 1960, Bahadur theorem). *Let $\{T_n\}$ be a sequence of test statistics which satisfies the following:*

1. *Under $H_1 : \theta \in \Theta - \{\theta_0\}$:*

$$n^{-\frac{1}{2}} T_n \to b(\theta) \qquad a.s. \qquad (F_\theta),$$

where $b(\theta) \in \mathbb{R}$.

2. *There exists an open interval I containing $\{b(\theta) : \theta \in \Theta - \{\theta_0\}\}$, and a function g continuous on I, such that*

$$\begin{aligned}\lim_n -2n^{-1} \log \sup_{\theta \in \Theta_0} \left[1 - F_{\theta_n}(n^{\frac{1}{2}} t)\right] &= \lim_n -2n^{-1} \log \left[1 - F_{\theta_n}(n^{\frac{1}{2}} t)\right] \\ &= g(t), \qquad t \in I.\end{aligned}$$

If $\{T_n\}$ satisfied (1)-(2), then for $\theta \in \Theta - \{\theta_0\}$

$$-2n^{-1} \log \sup_{\theta \in \Theta_0} [1 - F_{\theta_n}(T_n)] \to C(\theta) \qquad a.s. \qquad (F_\theta).$$

Theorem 3 (Al-Masri 2010). *Let $X_1, \dots, X_n$ be i.i.d. with probability density function $f(x, \theta)$, and we want to test $H_0 : \theta = 0$ vs $H_1 : \theta > 0$. For $j = 1, 2$, let $T_{n,j} = \sum_{i=1}^n f_i(x_i)/\sqrt{n}$ be a sequence of statistics such that H_0 will be rejected for large values of $T_{n,j}$ and let φ_j be the test based on $T_{n,j}$. Assume $\mathbb{E}_\theta(f_i(x)) > 0, \forall \theta \in \Theta$, $\mathbb{E}_0(f_i(x)) = 0$, $\mathrm{Var}(f_i(x)) > 0$ for $j = 1, 2$. Then*
1. If the derivative $b_j'(0)$ is finite for $j = 1, 2$, then

$$\lim_{\theta \to 0} \frac{C_1(\theta)}{C_2(\theta)} = \frac{\mathrm{Var}_{\theta=0}(f_2(x))}{\mathrm{Var}_{\theta=0}(f_1(x))} \left[\frac{b_1'(0)}{b_2'(0)}\right]^2,$$

where $b_i(\theta) = \mathbb{E}_\theta(f_j(x))$, and $C_j(\theta)$ is the EBS of test φ_j at θ.
2. If the derivative $b_j'(0)$ is infinite for $j = 1, 2$, then

$$\lim_{\theta \to 0} \frac{C_1(\theta)}{C_2(\theta)} = \frac{\mathrm{Var}_{\theta=0}(f_2(x))}{\mathrm{Var}_{\theta=0}(f_1(x))} \left[\lim_{\theta \to 0} \frac{b_1'(\theta)}{b_2'(\theta)}\right]^2.$$

Theorem 4 (Serfling 2009). *If $T_n^{(1)}$ and $T_n^{(2)}$ are two test statistics for testing $H_0 : \theta = 0$ vs $H_1 : \theta > 0$ with distribution functions $F_0^{(1)}$ and $F_0^{(2)}$ under H_0, respectively, and that $T_n^{(1)}$ is at least as powerful as $T_n^{(2)}$ at θ for any α, then if φ_j is the test based on $T_n^{(j)}$, $j = 1, 2$, then*

$$C_{\varphi_1}^{(1)}(\theta) \geq C_{\varphi_2}^{(2)}(\theta)$$

.

Corollary 1 (Serfling 2009). *If T_n is the uniformly most powerful test for all α, then it is the best via EBS.*

Theorem 5 (Al-Masri 2010).

$$2t \leq m_S(t) \leq et, \quad \forall : 0 \leq t \leq 0.5,$$

where

$$m_S(t) = \inf_{z>0} e^{-zt} \frac{e^z - 1}{z}.$$

Theorem 6 (Al-Masri 2010). *1.* $m_L(t) \geq 2te^{-t}, \quad \forall t \geq 0,$

2. $m_L(t) \leq te^{1-t}, \quad \forall t \geq 0.852,$

3. $m_L(t) \leq t\left(\frac{t^2}{1+t^2}\right)^3 e^{1-t}, \quad \forall t \geq 4,$
where $m_L(t) = \inf_{z\in(0,1)} e^{-zt}\pi z \ csc(\pi z)$ *and csc is an abbreviation for cosecant function.*

Theorem 7 (Al-Masri 2010). *For* $x > 0$,

$$\phi(x)\left[\frac{1}{x} - \frac{1}{x^3}\right] \leq 1 - \Phi(x) \leq \frac{\phi(x)}{x}.$$

Where ϕ *is the pdf of standard normal distribution.*

Theorem 8 (Al-Masri 2010). *For* $x > 0$,

$$1 - \Phi(x) > \frac{\phi(x)}{x + \sqrt{\frac{\pi}{2}}}.$$

Lemma 1 (Al-Masri 2010). *1.* $m_L(t) \geq \inf_{0<z<1} e^{-zt} = e^{-t}$

2. $m_L(t) \leq \dfrac{e^{-t^2/(t+1)}\left(\frac{\pi t}{t+1}\right)}{sin\left(\frac{\pi t}{t+1}\right)}$

3. $\begin{cases} m_s(t) = \inf_{z>0} \frac{e^{-zt}(1-e^{-z})}{z} \leq \inf_{z>0} \frac{e^{-zt}}{z} \leq -et, & t < 0 \\ m_s(t) \geq -2t, & -\frac{1}{2} \leq t \leq 0. \end{cases}$

5. Deviation of the EBS for $\mathbb{EV}(\theta, 1)$

In this section we will study testing problem (8). We will compare the six methods Fisher, logistic, sum of p-values, the inverse normal, Tippett's method and maximum of p-values via EBS.
Let $X_1, \ldots, X_n$ be iid with probability density function (2) and we want to test (8). Then by (1), the P-value is given by

$$P_n(X_n) = 1 - F^{H_0}(X_n) = 1 - e^{-e^{-x}} \tag{9}$$

The next three lemmas give the EBS for Fisher (C_F), logistic (C_L), inverse normal (C_N), and sum of p-values (C_S), Tippett's method (C_T) and maximum of p-values (C_{max})methods.

Lemma 2. *The exact Bahadurs slope (EBSs) result for the tests, which is given in Section 2, are as follows:*

B1. Fisher method. $C_F(\theta) = b_F(\theta) - 2\ln(b_F(\theta)) + 2\ln(2) - 2$, *where*

$$b_F(\theta) = -2\left(\psi(1) - \psi(e^{\theta}+1)\right),$$

and $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ *is the digamma function.*

B2. Logistic method. $C_L(\theta) = -2\ln(m(b_L(\theta)))$, *where*

$$m_L(t) = \inf_{z\in(0,1)} e^{-zt}\pi z \ csc(\pi z)$$

and

$$b_L(\theta) = \psi(e^{\theta}+1) - e^{-\theta} - \psi(1).$$

B3. Sum of p-values method. $C_S(\theta) = -2\ln(m(b_S(\theta)))$, *where*

$$m_S(t) = \inf_{z>0} e^{-zt}\frac{1-e^{-z}}{z}$$

and

$$b_S(\theta) = -\left(e^{\theta}+1\right)^{-1}.$$

B4. Inverse Normal method. $C_N(\theta) = -2\ln(m(b_N(\theta))) = b_N^2(\theta)$. *Where*

$$b_N(\theta) = -e^{\theta}\,\mathbb{E}_{Beta(e^{\theta}-1,1)}\,\phi\left(\Phi^{-1}(1-W)\right)$$

Proof of B1. By Theorem (2)

$$T_F = -2\sum_{i=1}^{n}\frac{\ln\left[1-e^{-e^{-x}}\right]}{\sqrt{n}}.$$

By the strong law of large number (SLLN)

$$\frac{T_F}{\sqrt{n}} \xrightarrow{\text{w.p.1}} b_F(\theta) = -2\,\mathbb{E}^{H_1}\ln\left[1-e^{-e^{-x}}\right]$$

then

$$b_F(\theta) = -2\int_{\mathbb{R}}\ln\left[1-e^{-e^{-x}}\right]e^{-(x-\theta)-e^{-(x-\theta)}}\,dx.$$

Now, let $U = e^{-(X-\theta)}$, and $Z = 1-e^{-e^{-\theta}U}$, then

$$\begin{aligned}\int_{\mathbb{R}}\ln\left[1-e^{-e^{-x}}\right]e^{-(x-\theta)-e^{-(x-\theta)}}\,dx &= e^{\theta}\int_0^1 \ln(z)(1-z)^{e^{\theta}-1}\,dz \\ &= \mathbb{E}_{Beta(1,e^{\theta})}\ln Z = \psi(1)-\psi(e^{\theta}+1).\end{aligned}$$

Then $b_F(\theta) = -2\left(\psi(1)-\psi(e^{\theta}+1)\right)$.

Now under H_0, then by Theorem 1, we have $m_S(t) = \inf_{z>0} e^{-zt} M_S(z)$, where $M_S(z) = \mathbb{E}_F(e^{zX})$. Under $H_0 : -\left(1 - e^{-e^{-x}}\right) \sim U(-1, 0)$, so $M_S(z) = \frac{1-e^{-z}}{z}$, by part (2) of Theorem 2 we complete the proof, that is

$$\begin{aligned} C_F(\theta) = & -2\ln(m_F(b_F(\theta))) = -2\ln\left(\frac{b_F(\theta)}{2} e^{1-\frac{b_F(\theta)}{2}}\right) \\ & = b_F(\theta) - 2\ln(b_F(\theta)) + 2\ln(2) - 2. \end{aligned}$$

□

Proof of B2.

$$T_L = -\sum_{i=1}^{n} \frac{\ln\left[\frac{1-e^{-e^{-x}}}{e^{-e^{-x}}}\right]}{\sqrt{n}}.$$

By the strong law of large number (SLLN)

$$\frac{T_L}{\sqrt{n}} \xrightarrow{\text{w.p.1}} b_L(\theta) = -\mathbb{E}^{H_1} \ln\left[\frac{1-e^{-e^{-x}}}{e^{-e^{-x}}}\right]$$

then

$$\begin{aligned} b_L(\theta) = & -\int_{\mathbb{R}} \ln\left[\frac{1-e^{-e^{-x}}}{e^{-e^{-x}}}\right] e^{-(x-\theta)-e^{-(x-\theta)}}\, dx \\ & = \int_{\mathbb{R}} \ln\left[1-e^{-e^{-x}}\right] e^{-(x-\theta)-e^{-(x-\theta)}}\, dx - \int_{\mathbb{R}} e^{-x} e^{-(x-\theta)-e^{-(x-\theta)}}\, dx. \end{aligned}$$

Now,

$$\int_{\mathbb{R}} e^{-x} e^{-(x-\theta)-e^{-(x-\theta)}}\, dx = e^{-\theta},$$

and from Proof (B1), $\int_{\mathbb{R}} \ln\left[1-e^{-e^{-x}}\right] e^{-(x-\theta)-e^{-(x-\theta)}}\, dx = \psi(1) - \psi(e^{\theta}+1)$. Then

$$b_L(\theta) = \psi(e^{\theta}+1) - e^{-\theta} - \psi(1)$$

□

Proof of B3.

$$T_S = -\sum_{i=1}^{n} \frac{1-e^{-e^{-x}}}{\sqrt{n}}.$$

By the strong law of large number (SLLN)

$$\frac{T_S}{\sqrt{n}} \xrightarrow{\text{w.p.1}} b_S(\theta) = -\mathbb{E}^{H_1}\left(1-e^{-e^{-x}}\right)$$

then

$$b_S(\theta) = -\int_{\mathbb{R}} \left(1-e^{-e^{-x}}\right) e^{-(x-\theta)-e^{-(x-\theta)}}\, dx = -\left(e^{\theta}+1\right)^{-1}.$$

Now, by Theorem 1, we have $m_S(t) = \inf_{z>0} e^{-zt} M_S(z)$, where $M_S(z) = \mathbb{E}_F(e^{zX})$. Under $H_0 : -\left(1 - e^{-e^{-x}}\right) \sim U(-1, 0)$, so $M_S(z) = \frac{1-e^{-z}}{z}$, by part (2) of Theorem 2 we complete the proof, that is $C_S(\theta) = -2\ln(m_S(b_S(\theta)))$. □

Proof of B4.

$$T_N = -\sum_{i=1}^{n} \frac{\Phi^{-1}\left(1 - e^{-e^{-x}}\right)}{\sqrt{n}}.$$

By the strong law of large number (SLLN)

$$\frac{T_N}{\sqrt{n}} \xrightarrow{\text{w.p.1}} b_N(\theta) = -\,\mathbb{E}^{H_1}\,\Phi^{-1}\left(1 - e^{-e^{-x}}\right),$$

$$b_N(\theta) = -e^{\theta} \int_{\mathbb{R}} \Phi^{-1}\left(1 - e^{-e^{-x}}\right) e^{-x-e^{-(x-\theta)}}\, dx,$$

then put $U = \Phi^{-1}\left(1 - e^{-e^{-x}}\right)$ to get

$$b_N(\theta) = -e^{\theta} \int_{\mathbb{R}} u\phi(u)\left(1 - \Phi(u)\right)^{e^{\theta}-1} du = e^{\theta} \int_{\mathbb{R}} \frac{d\phi(u)}{du} \left(1 - \Phi(u)\right)^{e^{\theta}-1} du,$$

where $-u\phi(u) = \dfrac{d}{du}\phi(u)$.
Now, by using integration by parts and put $W = 1 - \Phi(U)$ to get

$$\begin{aligned} b_N(\theta) &= -\,e^{\theta}\left(e^{\theta} - 1\right) \int_0^1 w^{e^{\theta}-2} \phi\left(\Phi^{-1}(1-w)\right) dw \\ &= -\,e^{\theta}\, \mathbb{E}_{Beta(e^{\theta}-1,1)}\, \phi\left(\Phi^{-1}(1-W)\right) \end{aligned}$$

where $\phi^2\left(\Phi^{-1}(1-w)\right) = \frac{1}{\sqrt{2\pi}}\phi\left(\dfrac{\Phi^{-1}(1-w)}{\sqrt{2}}\right)$.
Now, by Theorem 1, we have $m_N(t) = \inf_{z>0} e^{-zt} M_N(z)$, where $M_N(z) = \mathbb{E}_F(e^{zX})$. Under $H_0 : -\left(1 - e^{-e^{-x}}\right) \sim N(0, 1)$, so $M_N(z) = e^{z^2/2}$, by part (2) of Theorem 2, $C_N(\theta) = -2\ln(m_N(b_N(\theta))) = b_N^2(\theta)$. □

Theorem 9 (Abu-Dayyeh & El-Masri 1994)**.** *Let $U_1, U_2, \ldots$ be i.i.d. like U with probability density function f and suppose that we want to test $H_0 : U_i \sim U(0,1)$ vs $H_1 : U_i \sim f$ on $(0,1)$ but not $U(0,1)$. Then $C_{\max}(f) = -2\ln\left(ess.sup_f(u)\right)$ where $ess.Sup_f(u) = sup\{u : f(u) > 0\}$ w.p.1 under f.*

Lemma 3.

$$C_{max}(\theta) = 0.$$

Proof. By Theorem (9) $C_{\max}(f) = -2\ln\left(ess.sup_f(u)\right)$ where $ess.Sup_f(u) = Sup\{u : f(u) > 0\}$ w.p.1 under θ.

For $f(x) = e^{-(x-\theta)-e^{-(x-\theta)}}$, $x, \theta \in \mathbb{R}$, let $Y = 1 - e^{-e^{-x}}$, then $Y \sim Beta\left(e^{\theta}, 1\right)$. Then $ess.sup_f(u) = 1$.

Therefore, $C_{\max}(\theta) = -2\ln(1) = 0$. □

Theorem 10 (Abu-Dayyeh & El-Masri 1994). *If* $\Delta(\ln\Delta)^2 f(\Delta) \to 0$ *as* $\Delta \to 0$, *then* $C_T(f) = 0$.

Lemma 4.

$$C_T(\theta) = 0.$$

Proof. By Theorem (10)

$$\lim_{\Delta\to 0}\Delta(\ln\Delta)^2 f(\Delta) = \lim_{\Delta\to 0}\Delta(\ln\Delta)^2 e^{-(\Delta-\theta)-e^{-(\Delta-\theta)}} = e^{\theta+e^{\theta}}\lim_{\Delta\to 0}\Delta(\ln\Delta)^2.$$

Clearly, by L'Hopital rule twice, $\lim_{\Delta\to 0}\Delta(\ln\Delta)^2 = 0$ which implies $C_T(\theta) = 0$. □

5.1. Comparison of the EBSs when $\theta \to 0$

Now, we will compare the EBSs that obtained in Section (4). We will find the limit of the ratio of the EBSs of any two methods when $\theta \to 0$.

Corollary 2. *The limits of ratios of different tests are as follows:*

C1. $\dfrac{C_T(\theta)}{C_{\mathfrak{D}}(\theta)} = \dfrac{C_{\max}(\theta)}{C_{\mathfrak{D}}(\theta)} = 0$, *where* $C_{\mathfrak{D}}(\theta) \in \{C_F(\theta), C_L(\theta), C_S(\theta), C_N(\theta)\}$.

C2. $e_B(T_S, T_F) \to 1.80314$

C3. $e_B(T_L, T_F) \to 1.97729$

C4. $e_B(T_N, T_F) \to 1.96121$

C5. $e_B(T_L, T_N) \to 1.0082$

C6. $e_B(T_N, T_S) \to 1.08764$

C7. $e_B(T_L, T_S) \to 1.09656$

Proof of C2.

$$b_F(\theta) = -2\left(\psi(1) - \psi(e^{\theta}+1)\right).$$

Therefore

$$b'_F(\theta) = 2e^{\theta}\psi_1(1+e^{\theta}),$$

where $\psi_1(x) = \frac{d}{dx}\psi(x)$ is the trigamma function.

$$\lim_{\theta\to 0} b'_F(\theta) = 2\left(\frac{\pi^2}{6} - 1\right) < \infty.$$

Also

$$b_S(\theta) = -\left(e^{\theta}+1\right)^{-1},$$

then

$$\lim_{\theta \to 0} b'_S(\theta) = \lim_{\theta \to 0} \tfrac{1}{2} \left(\cosh(\theta) + 1\right)^{-1} = \tfrac{1}{4} < \infty.$$

Now under $H_0 : h_F(x) = -2\ln\left[1 - e^{-e^{-x}}\right] \sim \chi^2_2$ and $h_S(x) = -\left(1 - e^{-e^{-x}}\right) \sim U(-1, 0)$, so $\mathrm{Var}_{\theta=0}(h_F(x)) = 4$ and $\mathrm{Var}_{\theta=0}(h_S(x)) = \frac{1}{12}$, also, $\dfrac{b'_S(0)}{b'_F(0)} = \left(\dfrac{8\pi^2}{6} - 8\right)^{-1}$. By applying Theorem 3 we can get $\lim_{\theta \to 0} \dfrac{C_S(\theta)}{C_F(\theta)} = \dfrac{27}{\left(\pi^2 - 6\right)^2} = 1.80314$. Similarly we can prove the other parts.

□

5.2. The Limiting ratio of the EBS for different tests when $\theta \to \infty$

Now, we compare the limit of the ratio of the EBSs of any two methods when $\theta \to \infty$.

Corollary 3. *The limits of ratios for different tests are as follows:*
D1. $e_B\left(T_L, T_F\right) \to 1$

D2. $e_B\left(T_S, T_F\right) \to 1$

D3. $e_B\left(T_N, T_S\right) \to 0$

D4. $\lim_{\theta \to \infty} \left\{C_F(\theta) - C_L(\theta)\right\} \leq 0$

D5. $e_B\left(T_N, T_F\right) \to 0, e_B\left(T_N, T_L\right) \to 0, e_B\left(T_L, T_S\right) \to 1.$

Proof of D1. By Lemma 1 part (1) $C_L(\theta) \leq 2b_L(\theta)$. So

$$\frac{C_L(\theta)}{C_F(\theta)} \leq \frac{2b_L(\theta)}{b_F(\theta) - 2\ln(b_F(\theta)) + 2\ln(2) - 2}.$$

It is sufficient to obtain $\lim_{\theta \to \infty} \dfrac{2b_L(\theta)}{b_F(\theta)}$.

Therefore,

$$\lim_{\theta \to \infty} \frac{2b_L(\theta)}{b_F(\theta)} = -\lim_{\theta \to \infty} \frac{\psi(e^\theta + 1) - e^{-\theta} - \psi(1)}{\psi(1) - \psi(e^\theta + 1)} = 1.$$

Then,

$$\lim_{\theta \to \infty} \frac{C_L(\theta)}{C_F(\theta)} \leq 1.$$

Also, by Theorem 6 part (2), we have $C_L(\theta) \geq 2b_L(\theta) - 2\ln(b_L(\theta)) - 2$. So

$$\lim_{\theta\to\infty} \frac{C_L(\theta)}{C_F(\theta)} \geq \lim_{\theta\to\infty} \frac{2b_L(\theta) - 2\ln(b_L(\theta)) - 2}{b_F(\theta) - 2\ln(b_F(\theta)) + 2\ln(2) - 2}.$$

It is sufficient to obtain the limit of $\lim_{\theta\to\infty} \dfrac{2b_L(\theta)}{b_F(\theta)}$.

Therefore,

$$\lim_{\theta\to\infty} \frac{2b_L(\theta)}{b_F(\theta)} = -\lim_{\theta\to\infty} \frac{\psi(e^\theta+1) - e^{-\theta} - \psi(1)}{\psi(1) - \psi(e^\theta+1)} = 1.$$

Then,

$$\lim_{\theta\to\infty} \frac{C_L(\theta)}{C_F(\theta)} \geq 1$$

By pinching theorem, we have $\lim_{\theta\to\infty} \dfrac{C_L(\theta)}{C_F(\theta)} = 1$. □

Proof of D2. By Lemma 1 part (3) $C_S(\theta) \leq -2\ln(2) - 2\ln(-b_S(\theta))$. So

$$\lim_{\theta\to\infty} \frac{C_S(\theta)}{C_F(\theta)} \leq \lim_{\theta\to\infty} \frac{-2\ln(2) - 2\ln(-b_S(\theta))}{b_F(\theta) - 2\ln(b_F(\theta)) + 2\ln(2) - 2}.$$

It is sufficient to obtain the limit of $\lim_{\theta\to\infty} \dfrac{-2\ln(-b_S(\theta))}{b_F(\theta)}$.

Then

$$\lim_{\theta\to\infty} \frac{-2\ln(-b_S(\theta))}{b_F(\theta)} = -\lim_{\theta\to\infty} \frac{\ln\left(1+e^\theta\right)}{\psi(1) - \psi(e^\theta+1)} = -\lim_{\theta\to\infty} \frac{\ln\left(1+e^\theta\right)}{\psi(1) - \psi(e^\theta) - e^{-\theta}}.$$

Now, by using Gauss's integral for asymptotic expansion of ψ

$$\psi(z) = \ln z - \frac{1}{2z} - \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t-1}\right) e^{-tz}\, dt,$$

we get

$$\psi(1+e^\theta) = \ln\left(1+e^\theta\right) - \frac{1}{2\left(1+e^\theta\right)} - \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t-1}\right) e^{-t\left(1+e^\theta\right)}\, dt \ln\left(1+e^\theta\right) \text{ as } \theta\to\infty.$$

Therefore,

$$\lim_{\theta\to\infty} \frac{-2\ln(-b_S(\theta))}{b_F(\theta)} = -\lim_{\theta\to\infty} \frac{\ln\left(1+e^\theta\right)}{\psi(1) - \ln\left(1+e^\theta\right)} = 1.$$

So

$$\lim_{\theta\to\infty} \frac{C_S(\theta)}{C_F(\theta)} \leq 1.$$

Also, by Theorem Lemma 1 part (3), we have $C_S(\theta) \geq -2 - 2\ln(-b_S(\theta))$. So, in the same manner, we get

$$\lim_{\theta\to\infty} \frac{C_S(\theta)}{C_F(\theta)} \geq 1.$$

By pinching theorem, we have $\lim\limits_{\theta\to\infty} \dfrac{C_S(\theta)}{C_F(\theta)} = 1$. □

Proof of D3. From B4 we have

$$C_N(\theta) = e^{2\theta}\left[\mathbb{E}_{Beta(e^\theta-1,1)}\,\phi\left(\Phi^{-1}(1-W)\right)\right]^2.$$

By Lemma 1 part (3) $C_S(\theta) \geq -2 - 2\ln(-b_S(\theta))$. So

$$\begin{aligned}\lim_{\theta\to\infty} \frac{C_N(\theta)}{C_S(\theta)} &\leq \lim_{\theta\to\infty} \frac{e^{2\theta}\left[\mathbb{E}_{Beta(e^\theta-1,1)}\,\phi\left(\Phi^{-1}(1-W)\right)\right]^2}{-2-2\ln(-b_S(\theta))}\\ &= \lim_{\theta\to\infty} \frac{e^{2\theta}\left[\mathbb{E}_{Beta(e^\theta-1,1)}\,\phi\left(\Phi^{-1}(1-W)\right)\right]^2}{-2+\ln\left(1+e^\theta\right)}.\end{aligned}$$

Now by using reflection symmetry, then $W \sim Beta\left(e^\theta-1,1\right)$ then $1-W \sim Beta\left(1,e^\theta-1\right)$.

Now we will find the limiting distribution for $H_\theta = e^\theta W_\theta$ when $e^\theta \to \infty$.

$$\begin{aligned}G_{H_\theta}\left(h_\theta\right) &= P_\theta\left[H_\theta \leq h_\theta\right] = P_\theta\left[W_\theta \leq e^{-\theta}h_\theta\right] = F_{W_\theta}\left(e^{-\theta}h_\theta\right)\\ &= \left(e^\theta-1\right)\int_0^{e^{-\theta}h_\theta}\left(1-w_\theta\right)^{e^\theta-2}dw_\theta = 1-\left[1-\frac{h_\theta}{e^\theta}\right]^{e^\theta-1}.\end{aligned}$$

$$\lim_{e^\theta\to\infty} G_{H_\theta}\left(h_\theta\right) = 1 - \frac{\lim_{e^\theta\to\infty}\left[1-\frac{h_\theta}{e^\theta}\right]^{e^\theta}}{\lim_{e^\theta\to\infty}\left[1-\frac{h_\theta}{e^\theta}\right]} = 1-e^{-h}.$$

Then, $\lim\limits_{e^\theta\to\infty} e^\theta\,\mathrm{Beta}(1,e^\theta-1) = \mathrm{Exponential}(1)$. Then,

$$\lim_{\theta\to\infty} \frac{C_N(\theta)}{C_S(\theta)} \leq \frac{\left[\mathbb{E}_{\mathrm{Exp}(1)}\,\phi\left(\Phi^{-1}(W)\right)\right]^2}{\lim_{\theta\to\infty}\left\{-2+\ln\left(1+e^\theta\right)\right\}} = 0.$$

Then

$$\lim_{\theta\to\infty} \frac{C_N(\theta)}{C_S(\theta)} = 0.$$

□

Proof of D4. By Theorem 6 (2), we have

$$\begin{aligned}C_F(\theta) - C_L(\theta) &\leq b_F(\theta) - 2\ln b_F(\theta) + 2\ln(2) + 2\ln b_L(\theta) - 2b_L(\theta)\\ &= b_F(\theta) - 2b_L(\theta) + 2\ln\left(\frac{b_L(\theta)}{b_F(\theta)}\right) + 2\ln(2).\end{aligned}$$

Now,

$$b_F(\theta) - 2b_L(\theta) = 2e^{-\theta}.$$

Also,

$$\lim_{\theta\to\infty} \frac{b_L(\theta)}{b_F(\theta)} = -\lim_{\theta\to\infty} \frac{\psi(e^\theta + 1) - e^{-\theta} - \psi(1)}{2\left(\psi(1) - \psi(e^\theta + 1)\right)} = \frac{1}{2}.$$

Then,

$$\begin{aligned}\lim_{\theta\to\infty} \left(C_F(\theta) - C_L(\theta)\right) \leq & \lim_{\theta\to\infty} \left(b_F(\theta) - 2\ln b_F(\theta)\right) + 2\lim_{\theta\to\infty} \ln\left(\frac{b_L(\theta)}{b_F(\theta)}\right) \\ & + 2\ln(2) = 0 - 2\ln(2) + 2\ln(2) = 0.\end{aligned}$$

So, $C_F(\theta) \leq C_L(\theta)$ for large θ □

Proof of D5. By using D1-D3 □

5.3. Comparison of the EBS for the Four Combination Procedures

From the relations in section (4.1) we conclude that locally as $\theta \to 0$, the logistic procedure is better than all other procedures since it has the highest EBS, followed in decreasing order by the inverse normal, sum of p-values procedure and the Fisher's procedure. The worst are the Tippett's and the maximum of p-values procedures, i.e.,

$$C_L(\theta) > C_N(\theta) > C_S(\theta) > C_F(\theta) > C_T(\theta) = C_{\max}(\theta).$$

Whereas, from result of Section (4.2) as $\theta \to \infty$ the worst methods are Tippett's and the maximum of p-values, the logistic and sum of p-values methods remain the same, they are better than all other procedures since it has the highest EBS, followed in decreasing order by Fisher's and the inverse normal procedures, i.e.,

$$C_L(\theta) = C_S(\theta) > C_F(\theta) > C_N(\theta) > C_T(\theta) = C_{\max}(\theta).$$

Acknowledgement

We are very grateful to the Editor and the reviewer for useful comments and suggestions on an early version of this paper.

[Received: July 2021 — Accepted: December 2021]

References

Abu-Dayyeh, W., Al-Momani, M. & Muttlak, H. (2003), 'Exact bahadur slope for combining independent tests for normal and logistic distributions', *Applied Mathematics and Computation* **135**(2-3), 345–360.

Abu-Dayyeh, W. & El-Masri, A. (1994), 'Combining independent tests of triangular distribution', *Statistics & Probability Letters* **21**(3), 195–202.

Al-Masri, A. (2010), 'Combining independent tests of conditional shifted exponential distribution', *Journal of Modern Applied Statistical Methods* **9**(1), 221–226.

Al-Masri, A. (2021*a*), 'Exact bahadur slope for combining independent tests in case of laplace distribution', *Jordan Journal of Mathematics and Statistics* **14**(1), 163–176.

Al-Masri, A. (2021*b*), 'On combining independent tests in case of log-normal distribution', *American Journal of Mathematical and Management Sciences* . https://doi.org/10.1080/01966324.2021.1997676

Al-Masri, A. & Al-Momani, N. (2021), 'On combining independent tests in case of log-logistic distribution', *Electronic Journal of Applied Statistical Analysis* **14**(1), 217–229.

Al-Talib, M., Al-Kadiri, M. & Al-Masri, A. (2020), 'On combining independent tests in case of conditional normal distribution', *Communications in Statistics - Theory and Methods* **49**(23), 5627–5638.

Bahadur, R. (1960), 'Stochastic comparison of tests', *The Annals of Mathematical Statistics* **31**(2), 276–295.

Serfling, R. (2009), *Approximation theorems of mathematical statistics*, Vol. 162, John Wiley & Sons.