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THE SPACE $D[0,1]^2$

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ABSTRACT. It is well known that the space $D[0,1]^2$ with the metric defined by $k(x,y) = \inf\{\epsilon \in \mathbb{R}^+ : \exists \lambda \in \Lambda^2,\ \sup_t |x(t) - y(\lambda(t))| < \epsilon \quad \text{and} \quad \sup_t |\lambda(t) - t| < \epsilon\}$ is a separable space (Bickel- Wichura 1971) but this is not a topological complete space like they claim, in this notes I give a counterexample of this fact and I define a metric that makes this space a separable and complete metric space.

0. INTRODUCTION

In $[0,1]^2$ we can define a partial order relation by $(s,t) < (s',t')$ iff $s < s'$ and $t < t'$; in a natural way we can also define the open, closed and closed from left open from right intervals.

Let Λ^2 be the set of all mappings $\lambda : [0,1]^2 \longrightarrow [0,1]^2$ such that $\lambda(t_1,t_2) = (\lambda_1(t_1), \lambda_2(t_2))$, where λ_i is a strictly increasing function from $[0,1]$ onto $[0,1]$ with $\lambda_i(0) = 0$, $\lambda_i(1) = 1$ and λ_i and $(\lambda_i)^{-1}$ satisfice a Lipschitz condition of order 1, that is

$$|\lambda_i(t) - \lambda_i(s)| \leq M_i|t-s|$$

$$|(\lambda_i)^{-1}(t) - (\lambda_i)^{-1}(s)| \leq M_i'|t-s|$$

for $i = 1,2$. Each mapping in Λ^2 is called a deformation of $[0,1]^2$. For $\lambda \in \Lambda^2$ we define two measures of the amount of the deformation

$$(1) \qquad ||\lambda|| = \sup_{t\in[0,1]^2} ||\lambda(t) - t||$$

$$(2) \qquad d(\lambda) = d_1(\lambda) + d_2(\lambda)$$

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where

$$d_i(\lambda) = \sup_{r \neq s} \left| \log \frac{\lambda_i(r) - \lambda_i(s)}{r - s} \right| \qquad \text{for} \quad i = 1, 2$$

From these definitions it is clear that $\|\lambda^{-1}\| = \|\lambda\|$ and $d(\lambda_{-1}) = d(\lambda)$, moreover if $\lambda, \rho \in \Lambda^2$ then $\|\lambda \circ \rho\| \leq \|\lambda\| + \|\rho\|$ and $d(\lambda \circ \rho) \leq d(\lambda) + d(\rho)$.

Remark. When $s > -1$, $\log(1+s) \leq s$ and if $0 < s < 1/2$, $s < \log(1+2s)$, then for $a, b, \epsilon \in \mathbb{R}^+$, $0 < \epsilon < 1/4$, $0 < b < 1$ we have: if $|\log(a/b)| < \epsilon$, then $\log(1-2\epsilon) < \log(1-\epsilon) \leq -\epsilon < \log(a/b) < \epsilon \leq \log(1+2\epsilon)$, that is $-2\epsilon < a/b - 1 < 2\epsilon$, and so $-2\epsilon < a - b < 2\epsilon$ i.e. $|a-b| < 2\epsilon$.

Lemma 1.

Let $\rho \in \Lambda^2$, if $d(\rho) < \epsilon$, then $\|\rho\| < 4\epsilon$. That means: when $\lim_{n\to\infty} d(\rho_n) = 0$ then $\lim_{n\to\infty} \rho_n(r) = r$ uniformly.

Proof. If $d(\rho) < \epsilon$, then $d_i(\rho) < \epsilon$, $i = 1, 2$. Since $\rho(0,0) = (0,0)$, then

$$\forall r \in [0,1], \; \left| \log \frac{\rho_i(r)}{r} \right| < \epsilon \quad \text{for} \quad i = 1, 2$$

and from the remark we have, since $\left|\log \frac{\rho_i(r)}{r}\right| < \epsilon$, then $\forall r \in [0,1]$, $|\rho_i(r) - r| < 2\epsilon$ and finally

$$\forall (s,t) \in [0,1]^2, \quad \|\rho(s,t) - (s,t)\| \leq |\rho_1(s) - s| + |\rho_2(t) - t| < 4\epsilon$$

It is also clear that this relation between these two measures holds only for $0 < \epsilon < 1/2$. □

Note that we can find deformations that send segments parallel to the axes into segments parallel to the axes, these deformations can be defined by taking the components linear in each subinterval of a partition of $[0,1]$.

1. THE SKOROHOD SPACE $D[0,1]^2$

Let $x : [0,1]^2 \longrightarrow \mathbb{R}$ be a function, for each $(s_o, t_o) \in [0,1]^2$ we can consider the following limits when they exist:

$$x(s_o^+, t_o^+) = \lim_{\substack{s \to s_o^+ \\ t \to t_o^+}} x(s,t) \qquad x(s_o^+, t_o^-) = \lim_{\substack{s \to s_o^+ \\ t \to t_o^-}} x(s,t)$$

$$x(s_o^-, t_o^+) = \lim_{\substack{s \to s_o^- \\ t \to t_o^+}} x(s,t) \qquad x(s_o^-, t_o^-) = \lim_{\substack{s \to s_o^- \\ t \to t_o^-}} x(s,t)$$

We call this limits the quadrantal limits.

Definition 2. Let a bounded function $x : [0,1]^2 \longrightarrow \mathbb{R}$, we say that the function is a regular path (cad-lag) if:

i) for this function the four quadrantal limits exist in each point in $(0,1)^2$

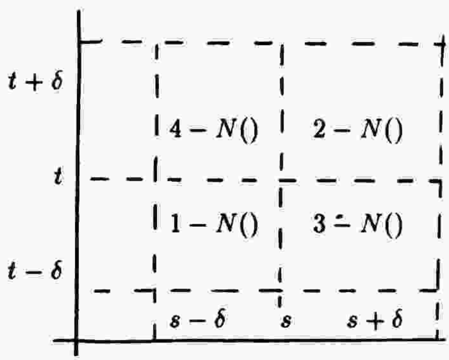
ii) for $(s,t) \in (0,1)^2$, $x(s,t) = x(s^+,t^+)$, that means x is $++$ continuous.

iii) For the boundary points there exist the corresponding quadrantal limits and the function is $++$, $+-$, $-+$ or $--$ continuous depending of the location of the point on the boundary.

We denote by $D[0,1]^2$ the set of regular paths.

For $r = (s,t) \in (0,1)^2$ we can consider for each limit a corresponding admissible neighborhood as in the following figure.

Figure 1



$1 - N(\delta_r) = ((s-\delta, t-\delta),(s,t))$

$2 - N(\delta_r) = ((s,t),(s+\delta, t+\delta))$

$3 - N(\delta_r) = ((s,t-\delta),(s+\delta, t))$

$4 - N(\delta_r) = ((s-\delta, t),(s,t+\delta))$

For boundary points there is one or two of these neighborhoods. We denote these admissible neighborhoods by $i - N(\delta_r)$, for $i = 1,2,3,4$ and they are admissible neighborhoods for the limits $--$, $++$, $+-$, $-+$ respectively.

Let $x \in D[0,1]^2$ and $T \subseteq [0,1]^2$, we define the oscilation of x in T by

$$w_x(T) = \sup\{|x(r) - x(s)| : r, s \in T\}$$

When $0 < \delta < 1$, put

$$w_x(\delta) = \sup_{(r,s)\in[(0,0),(1-\delta,1-\delta)]} w_x\left([(r,s),(r+\delta, s+\delta)]\right)$$

A continuous function on $[0,1]^2$ is uniformly continuous, so that with the above definition we can caracterize the continuous functions in $[0,1]^2$ (x is continuous if and only if $\lim_{\delta \to 0} w_x(\delta) = 0$). If $x \in D[0,1]^2$, in general is not continuous but for this function it follows:

Lemma 3. *For each $x \in D[0,1]^2$ and $\epsilon > 0$, $\epsilon \in \mathbb{R}^+$, there exist real numbers $s_o, s_1, \cdots, s_n$ and $t_0, t_1, \cdots, t_m$ such that $0 = s_0 < s_1 < \cdots < s_n$ and $0 = t_0 < t_1 < \cdots < t_m$*

$$w_x\left([(s_{i-1}, t_{j-1}), (s_i, t_j))\right) < \epsilon, \quad i = 1, 2, \cdots, n; \quad j = 1, 2, \cdots, m$$

Proof. Let $\epsilon > 0$, for $r = (r_1, r_2) \in [(0,0), (1,1))$, there exists a $2-N(\delta_r)$ neighborhood admissible for the ++ limit; for $r = (1,t)$, there exists $4 - N(\delta_r)$ neighborhood admissible for the −+ limit; for $r = (s,1)$ there exists a $3 - N(\delta_r)$ neighborhood admissible for the +− limit; for $0 < s, t < 1$ and for $r = (1,1)$ there exists a $1 - N(\delta_r)$ neighborhood admissible for the −− limit, so that if $u = (u_1, u_2)$ belongs to one of these neighborhoods, $|x(r) - x(s)| < \epsilon/4$.

The collection of all such neighborhoods is an open covering of $[0,1]^2$, since $[0,1]^2$ is a compact set, a finite number of these neighborhoods cover $[0,1]^2$, these finite number of admissible neighborhoods determine a finite number of relative open rectangles, so that when s and u belong to one of them, then $|x(s) - x(u)| < \epsilon/2$, since the function is ++ continuous, so for each of these rectangles $R_{i,j} = ((s_{i-1}, t_{j-1}), (s_i, t_j))$ we have

$$w_x\left([(s_{i-1}, t_{j-1}), (s_i, t_j))\right) < \epsilon/2 < \epsilon$$

for $i = 1, 2, \cdots, n; j = 1, 2, \cdots, m$ □

From lemma it follows inmediatly that for $\epsilon > 0$ there exist at most a finite number of horizontal and vertical segments, where the jumps of their points

$$\{|x(s,t) - x(s^-,t)|,\ |x(s,t^-) - x(s^-,t^-)|, |x(s,t) - x(s,t^-)|,\ |x(s^-,t) - x(s^-,t^-)|\}$$

exceed ϵ. In particular, the set of discontinuities of x is at most a countable union of horizontal and vertical segments in $[0,1]^2$. We have also that x is bounded.

Same as in Billingsley (1968) we introduce a new modulus that characterizes $D[0,1]^2$. For $0 < \delta < 1$ let

$$w'_x(\delta) = \inf_{\{(s_i,t_j)\}} \max_{\substack{0<i<n \\ 0<j<m}} w_x\left[(s_{i-1}, t_{j-1}), (s_i, t_j)\right)$$

where the infimum runs over the sets of points $\{(s_i,t_j) : 0 = s_o < s_1 < \cdots < s_n = 1,$ $0 = t_o < t_1 < \cdots < t_m$ and $s_i - s_{i-1} > \delta,\ t_j - t_{j-1} > \delta,\ i = 1,2,\ldots,n;\ j = 1,2,\ldots,m\}$.

It is easy to see that lemma 3 holds iff $\lim_{\delta\to 0} w'_x(\delta) = 0$ for every x in $D[0,1]^2$. In fact, if the lemma holds, given $\epsilon > 0$, there exists a partition $0 = s_o < s_1 < \cdots < s_n = 1,\quad 0 = t_o < t_1 < \cdots < t_m = 1$ such that

$$w_x\left([(s_{i-1},t_{j-1}),(s_i,t_j))\right) < \epsilon,\ \forall i = 1,2,\cdots,n;\ j = 1,2,\cdots,m$$

then $\max_{\{i,j\}} w_x\left([(s_{i-1},t_{j-1}),(s_i,t_j))\right) < \epsilon$. Taking $\delta' = \min\{s_i - s_{i-1}, t_j - t_{j-1}, i = 1,2,\cdots,n;\ j = 1,2,\cdots,m\}$, if $\delta < \delta'$, $w'_x(\delta) < \epsilon$ that means $\lim_{\delta\to 0} w'_x(\delta) = 0$. On the other side when $w'_x(\delta) = 0$, $\forall\epsilon > 0$, $\exists\delta' > 0$ such that $(\forall\delta < \delta')\ (w'_x(\delta) < \epsilon)$. From the definition of $w'_x(\delta)$ there exists a partition $0 = s_o < s_1 < \cdots < s_n = 1,\quad 0 = t_o < t_1 < \cdots < t_m = 1$ such that $\max_{\{i,j\}} w_x\left([(s_{i-1},t_{j-1}),(s_i,t_j))\right) - \epsilon < w'_x < \epsilon$ with $s_i - s_{i-1} > \delta$ and $t_j - t_{j-1} > \delta$, then $w_x\left([(s_{i-1},t_{j-1}),(s_i,t_j))\right) < 2\epsilon$ for $i = 1,2,\ldots,n;\ j = 1,2,\ldots,m$.

Definition 4. Let $x, y \in D[0,1]^2$, we define two pseudometrics in the set $D[0,1]^2$:

$$k(x,y) = \inf\{\epsilon \in \mathbb{R}^+ : (\exists\rho \in \Lambda^2)(\sup_r |x(r) - y(\rho(r))| < \epsilon \quad \wedge \quad \|\rho\| < \epsilon\}$$

$$k_o(x,y) = \inf\{\epsilon \in \mathbb{R}^+ : (\exists\rho \in \Lambda^2)(\sup_r |x(r) - y(\rho(r))| < \epsilon \quad \wedge \quad d(\rho) < \epsilon\}$$

We will not distinguish between two functions x and y for which $k(x,y) = 0$ or $k_o(x,y) = 0$ and we will work with the metric space of equivalent classes without making a difference in the notation. It is easy to see that these two functions are pseudometrics, we will prove it for one of them.

Lemma 5. *$(D[0,1]^2, k_o)$ is a pseudometric space.*

Proof.

i) $k_o(x,y) \geq 0$, $\forall x,y \in D[0,1]^2$ because the inf in the definition of k_o is taken over a nonempty nonnegative set, othersides $k_o(x,y) < \infty$ because $\sup_t |x(t) - y(t)| \leq \sup_t |x(t)| + \sup_t |y(t)|$ and taking $\lambda(t) = t$ in the definition we have the result.

ii) $k_o(x,y) = k_o(y,x)$ because $d(\rho) = d(\rho^{-1})$ and $\sup_{t\in[0,1]^2} |y(t) - x(\lambda^{-1}(t))| = \sup_{r\in[0,1]^2} |x(r) - y(\lambda(r))|$.

iii) $k_o(x,y) \leq k_o(x,z) + k_o(z,y)$. Let $\epsilon_1 \in \{\epsilon \in \mathbb{R}^+ : (\exists\rho \in \Lambda^2)(\sup_r |x(r) - z(\rho(r))| < \epsilon \wedge d(\rho) < \epsilon\}$ and

$\epsilon_2 \in \{\epsilon \in \mathbb{R}^+ : (\exists \rho \in \Lambda^2)(\sup_r |z(r) - y(\rho(r))| < \epsilon \ \wedge \ d(\rho) < \epsilon\}$. For ϵ_1 and ϵ_2 there exist ρ_1, ρ_2 such that

$$\sup_{r \in [0,1]^2} |x(r) - z((\rho_1(r))| < \epsilon_1 \ \wedge \ d(\rho) < \epsilon_1$$
$$\sup_{r \in [0,1]^2} |z(\rho_1(r)) - y(\rho_2(\rho_1(r)))| < \epsilon_2 \ \wedge \ d(\rho) < \epsilon_2$$

therefore

$$\sup_{r \in [0,1]^2} |x(r) - y(\rho_2(\rho_1(r)))| \leq$$
$$\sup_{r \in [0,1]^2} |x(r) - z(\rho_1(r))| + \sup_{r \in [0,1]^2} |z(\rho_1(r)) - y(\rho_2(\rho_1(r)))| < \epsilon_1 + \epsilon_2$$

Othersides, $d(\rho_2 \circ \rho_1) \leq d(\rho_2) + d(\rho_1) < \epsilon_1 + \epsilon_2$. So given ϵ_1, ϵ_2, there exists $\epsilon_3 = \epsilon_1 + \epsilon_2$ such that

$$\epsilon_3 \in \{\epsilon \in \mathbb{R}^+ : (\exists \rho \in \Lambda^2)(\sup_r |x(r) - z(\rho(r))| < \epsilon \quad \wedge \quad d(\rho) < \epsilon\}$$

and so $k_o(x, y) \leq k_o(x, z) + k_o(z, y)$

□

Theorem 6. $(D[0,1]^2, k)$ *is a separable metric space.*

Proof. Let's take $x \in D[0,1]^2$ and $\epsilon > 0$, there exist elements $0 = s_o < s_1 < \cdots < s_n = 1$ and $0 = t_o < t_1 < \cdots < t_m = 1$ and

$$w_x[(s_{i-1}, t_{j-1}), (s_i, t_j)) < \epsilon/4, \quad i = 1, 2, \cdots, n \quad j = 1, 2, \cdots, m$$

We consider now the segments $L_{s_i} = \{(s_i, t) : 0 < t < 1\}$ and $L_{t_j} = \{(s, t_j) : 0 < s < 1\}$. For $n \in \mathbb{N}$, n large enough, each one of the rectangles $R_{pq} = [(\frac{p}{n}, \frac{q}{n}), (\frac{p+1}{n}, \frac{q+1}{n})]$ $p, q = 0, 1, 2, \cdots, n-1$ intersect at most one of these vertical segments L_{s_i} and at most one of the horizontal segments L_{s_j}.

We define now a deformation $\rho \in \Lambda^2$ that maps $\{\frac{p}{n}\} \times [0,1]$ onto L_{s_i} and $[0,1] \times \{\frac{q}{n}\}$ in L_{t_j}, when $\frac{p-1}{n} < s_i < \frac{p}{n}$, $\frac{q-1}{n} < t_j < \frac{q}{n}$, $p, q = 1, 2, \cdots, n-1$. If there are no segments L_{s_i} or L_{t_j} inside the rectangle, we let the value constant. We can define this deformation linearly by subintervals and by components, so that $||\rho|| < \sqrt{2}/n$.

Define $\overline{x}(s,t) = x(\rho(s,t))$, x has jumps greater than ϵ only on the segments $\{\frac{p}{n}\} \times [0,1]$ and $[0,1] \times \{\frac{q}{n}\}$. $k(x, \overline{x}) < \sqrt{2}/n$. Let $\overline{x}^*(s,t) = \overline{x}([ps]/n, [qt]/n))$, ($[r]$ is the greatest integer less than or equal r), $\overline{x}^*$ takes constant value on the rectangles

$[(\frac{p}{n},\frac{q}{n}),(\frac{p+1}{n},\frac{q+1}{n}))$, $p,q=1,2,\cdots,n-1$ and has discontinuities on the segments $\{\frac{p}{n}\}\times[0,1]$ and $[0,1]\times\{\frac{q}{n}\}$, $p,q=1,2,\cdots,n$.

$$\sup_{(s,t)\in[0,1]^2}|\overline{x}(s,t)-\overline{x}^*(s,t)|\le\sup_{p,q}\sup_{(s,t)\in R_{pq}}|\overline{x}(s,t)-\overline{x}(p/n,q/n)|\le\epsilon/4$$

Taking $\rho(s,t)=(s,t)$, $k(\overline{x},\overline{x}^*)\le\epsilon/4$. Finally let $\{y_1,y_2,y_3,\cdots\}$ be a enumeration for the set of rational numbers, we define $x^*(s,t)=y_m$ when m is the smallest index such that $|\overline{x}^*(s,t)-y_m|<\epsilon/4$, we have then $k(x^*,\overline{x}^*)<\epsilon/4$ and so $k(x,x^*)\le k(x,\overline{x})+k(\overline{x},\overline{x}^*)+k(\overline{x}^*,x^*)<\epsilon/4+\epsilon/4+\sqrt{2}/n$, for $n>2\sqrt{2}/\epsilon$ we have finally that $k(x,x^*)<\epsilon$.

Let $H_n=\{y\in D[0,1]^2:y$ is constant on the rectangles $[(\frac{p-1}{n},\frac{q-1}{n}),(\frac{p}{n},\frac{q}{n}))$, $p,q=1,2,\cdots,n$ and with values in $\mathbb{Q}\}$. H_n is a countable set, if $H=\bigcup_{n=1}^{\infty}H_n$, H is also a countable set and since H is dense in $D[0,1]^2$, $D[0,1]^2$ is a separable metric space. □

II. THE SPACE $(D[0,1]^2,k)$ IS NOT A COMPLET METRIC SPACE

Define $x_n=\chi_{[1/2,1/2+1/n)\times[1/2,1/2+1/n)}$, where

$$\chi_A=\begin{cases}1, & \text{if } x\in A\\ 0, & \text{if } x\notin A\end{cases}$$

If $n,m\in\mathbb{N}$, we can find a deformation ρ_n in Λ^2, that maps the segment $\{\frac{1}{2}+\frac{1}{m}\}\times[0,1]$ onto $\{\frac{1}{2}+\frac{1}{n}\}\times[0,1]$ and the segment $[0,1]\times\{\frac{1}{2}+\frac{1}{m}\}$ onto $[0,1]\times\{\frac{1}{2}+\frac{1}{n}\}$ and let the segments $\{\frac{1}{2}\}\times[0,1]$ and $[0,1]\times\{\frac{1}{2}\}$ constant, as we can see in the figure 2. We define

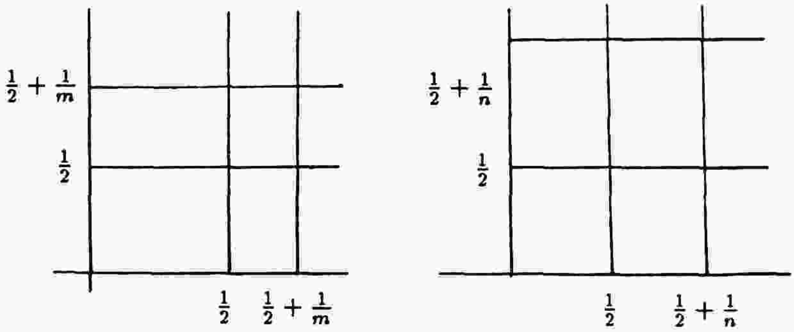
$$(\rho_n)_i(s)=\begin{cases}s, & 0<s<\frac{1}{2}\\ \frac{m}{n}s+\frac{1}{2}(1-\frac{m}{n}), & \frac{1}{2}<s<\frac{1}{2}+\frac{1}{m}\\ \frac{(n-2)m}{(m-2)n}s+2\frac{m-n}{(m-2)n}, & \frac{1}{2}+\frac{1}{m}<s<1\end{cases}$$

for $i=1,2$.

$\|\rho_n\|=2^{1/2}|\frac{1}{m}-\frac{1}{n}|$ and for all $r\in[0,1]^2$, $|x_m(r)-x_n(\rho_n(r))|=0$ so that $k(x_n,x_m)=2^{1/2}|\frac{1}{m}-\frac{1}{n}|$ and the sequence $\{x_n\}$ is a Cauchy sequence, but it does not converge in $D[0,1]^2$ because its limit function is $\chi_{\{(1/2,1/2)\}}$ and this is not a function in $D[0,1]^2$. The sequence is not a Cauchy sequence with respect to the metric k_o, because $k_o(x_n,x_m)>|\log\frac{m}{n}|$, $m,n>3$.

From lemma 1, when $0<k_o(x,y)<1/4$, then $k(x,y)<4k_o(x,y)$. But in general we don't have the other inequality. However we have that $k_o(x,y)$ can be small if $k(x,y)$ and $w'_x(\delta)$ are both small.

Figure 2



Lemma 7. *If* $k(x,y) < \delta^2$, $0 < \delta < 1/4$, *then* $k_o(x,y) < 8\delta + w'_x(\delta)$.

Proof. Let $0 < \delta < 1/4$, there exist points (s_i, t_j) so that $0 = s_o < s_1 < \cdots < s_n = 1$, $0 = t_o < t_1 < \cdots < t_m = 1$, $s_i - s_{i-1} > \delta$, $t_j - t_{j-1} > \delta$ and $w_x\left[(s_{i-1}, t_{j-1}), (s_i, t_j)\right) < w'_x(\delta) + \delta$ for all i, j in the index set.

Following the proof of lemma 2, page 112 in Billingsley (1968), we obtain that there exists a μ in Λ^2, such that

$$\sup_{(s,t)} |x(s,t) - y(\mu(s,t))| = \sup_{(s,t)} |x(\mu^{-1}(s,t)) - y(s,t)| < \delta^2$$

and $\sup_{(s,t)} \|\mu(s,t) - (s,t)\| < \delta^2$.

We can choose now $\lambda \in \Lambda^2$ to agree with μ at the points (s_i, t_j), near μ but with linear component functions and $\mu^{-1} \circ \lambda(s_i, t_j) = (s_i, t_j)$ so that $\mu^{-1} \circ \lambda(s,t)$ and (s,t) are in the same rectangle $\left[(s_{i-1}, t_{j-1}), (s_i, t_j)\right)$ and therefore

$$|x(s,t) - y(\lambda(s,t))| < w'_x(\delta) + \delta + \delta^2 < w'_x(\delta) + 4\delta$$

Now we can see that $d(\lambda) < 8\delta$, in fact, following Billingsley (1968) we have for each component function that

$$\log(1 - 2\delta) < \log \frac{\lambda_i(r) - \lambda_i(r')}{r - r'} < \log(1 + 2\delta)$$

for $i = 1, 2$, since $\delta < 1/4$, $\log(1-2\delta) > -4\delta$ and $\log(1+2\delta) < 4\delta$ and so $d(\lambda) < 8\delta$. □

As in Billingsley (1968) we have also that the metrics $k(x,y)$ and $k_o(x,y)$ are equivalent: inside of each $B(x,\epsilon)$ (sphere respect to k), there is a $B_o(x,\delta)$ (sphere respect to k_o), here the choice of the new radius does not depends on the center x. Now if $0 < \delta < 1/4$ and $8\delta + w'_x(\delta) < \epsilon$, $B(x, \delta^2) \subseteq B_o(x, \epsilon)$, but in this time the new radius depends on the center.

Theorem 8. *The metric space $(D[0,1]^2, k_o)$ is a complet metric space.*

Proof. The proof is analogous like in Billingsley (1968), the difference is only that here we must make the proof for each component. □

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