A probabilistic approach for multiaxial fatigue criteria

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ABSTRACT. Models proposed to study the multiaxial fatigue damage phenomenon generally lack probabilistic interpretation due to their deterministic form. This implies failure compulsory happening at the plane exhibiting the maximum damage value, whereas the remaining planes are disregarded. Nevertheless, the random orientation of the predominant defect evidences the possibility of failure being initiated as a function of the predominant defect presence without requiring, necessarily, maximum values of the damage parameter, which emphasizes the need of introducing probabilistic concepts into the failure prediction analysis. In this paper, a probabilistic model is presented that enables the failure probability to be found for any selected plane orientation by considering the damage gradient as a parameter for both proportional and non-proportional loading. The applicability of the model is elucidated by means of an example. Assuming the cdf for the local failure of the material to be known, the probability of failure is calculated for a cross shaped specimen in which shift between the principal stresses σ_{xx} and σ_{yy} ranges from 0º to 180º.

KEYWORDS. Fatigue Weibull Model; Multiaxial Fatigue; Generalized Local Model.

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INTRODUCTION

Usually, models proposed to study the multiaxial fatigue damage phenomenon are applied in their deterministic form. This implies failure always happening compulsory at the plane exhibiting the maximum damage, whereas the remaining planes are disregarded. Nevertheless, the random orientation of the predominant defect evidences

the possibility of failure being initiated as a function of the predominant defect presence without requiring maximum values of the damage parameter, which emphasizes the need of introducing probabilistic concepts in the failure prediction analysis.
The generalization of the probabilistic fatigue model of Castillo-Canteli [1] proposed by Muniz-Calvente et al. [2] allows the primary Weibull cdf of failure to be derived for any failure parameter, regardless of its distribution. In this way, any multiaxial damage value can be related to a number of cycles for a certain probability of failure. Once this relation is established, the probability of failure for any of the orientation planes can be calculated as a function of the local multiaxial damage value and, by extension, the global probability of failure for the component can be determined from the survival probabilities for all the planes assuming the weakest link principle.
From a purely deterministic viewpoint, two specimens exhibiting the same maximum damage value would yield the same lifetime, what contradicts the inherent scatter of the experimental data, which must be necessarily taken into account. Moreover, a deterministic approach is insensitive to the variation of the damage parameter in planes adjacent to the critical one thus ignoring the angular interval at which failure is likely to happen.
On the contrary, the probabilistic model proposed in this paper enables the failure probability to be found for any plane orientation, distinct from the one related to the maximum damage, by considering the local damage value in each plane.
The main objective of this study is to determine the probability of failure resulting for each plane orientation by considering a suitable damage parameter as the generalized parameter (GP) causing failure. In order to take into account the variation of the GP for the different planes, the extension of the GLM to fatigue problems [2] is considered. The GLM stablishes that the probability of failure for a plane exhibiting a certain value of the generalized parameter (GP) can be obtained by using the primary failure cumulative distribution function (PFCDF):

$$P = 1 - \exp\left[-\left(\frac{(\log(GP) - B)(\log(N) - C) - \lambda}{\delta} \right)^{\beta} \right] \quad (1)$$

where N is the number of cycles and λ, δ, β, B, C are, respectively, the Weibull parameters estimated from the iterative process shown in Fig.1a and explained in the following section.
The applicability of the model is elucidated by means of an example. Assuming the cdf for the local failure of the material to be known, the probability of failure is calculated for a cross shaped specimen in which shift between the principal stresses σ_{xx} and σ_{yy} ranges from 0º to 180º.

PROBABILISTIC MULTIAXIAL FATIGUE MODEL

Fig. 1a illustrates the iterative process applied for deriving the PFCDF that relates damage parameter exhibit at each plane studied to a certain probability of failure. In the following, each step is explained in detail:
Step 1: Performing an experimental program: To perform an experimental program using different biaxial loading ranges and to obtain the fatigue life for each experiment
Step 2: Calculation of multiaxial fatigue parameter: The different biaxial loading ranges selected in the previous step are using to obtain the values of GP for each plane of the specimen. Some examples of the results of this step are found in Fig. 3.
Step 3: Equivalent angle interval for each experiment: The *equivalent angle interval*, $A_{eq,i}$, is defined as the *angle interval* that subject to the maximum GP value occurring at test failure would have the same probability of failure than the real distribution of the GP at failure. It is given by:

$$A_{eq,i} = -\log(1 - P_{\text{int},i}) S_{ref} \left[\frac{\delta_{ref}}{(\log(GP) - B)(\log(N) - C)} \right]^{\beta} \quad (2)$$

where $P_{\text{int},i}$ is the global probability of failure [3]:

$$P_{\mathrm{int},i}=1-\prod_{i=1\dots n}(1-P_{fail,\Delta S_{ij}})=1-\prod_{j=1\dots n}\exp\left[-\frac{\Delta A_j}{A_{ref}}\left(\frac{(\log(GP_{ij})-B)(\log(N_j)-C)-\lambda}{\delta}\right)^{\beta}\right] \quad (3)$$

where ΔA_j is the angle interval assigned to each value of GP_{ij}, which is the damage value for the plane *j* of the specimen *i*. To start the iteration process, an initial estimation of $A_{eq,i}$, close to 40º, must be assumed because it depends on the values of *B*, *C* and the three Weibull parameters, which are still unknown.

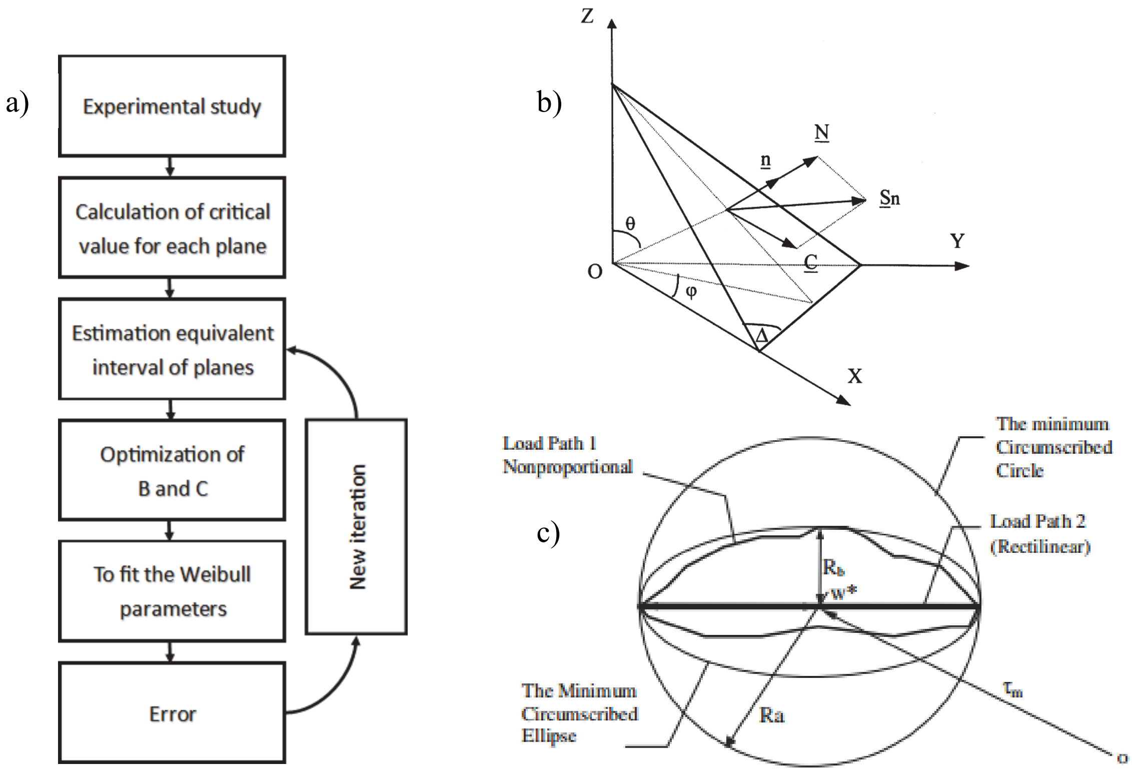

Figure 1: a) Iterative process applied to fit the PFCDF; b) Material plane selected for the projection of the normal and shear stresses [10]; c) Difference between MCC and MCE multiaxial fatigue criteria [10].

Step 4: Estimation of B and C: The estimation of B and C must be obtained by minimizing the least square equation proposed in [1] with respect to B, C and μ_1, μ_2, … μ_t for different sizes:

$$Q=\sum_{i=1}\left[\log N_i-B-\frac{\mu_i}{\log GP_i-C}\right]^2 \quad (4)$$

where μ is the median value for each of the different equivalent *angle intervals* obtained in the previous step, *n* is the sample size and GP_i and N_i are the maximum value of the critical parameter and the number of cycles to failure of the i-th specimen, respectively.
Step 5: Estimation of Weibull parameters: The probability of failure for each of the specimens is obtained using a plotting point position rule [4]:

$$P=\frac{i-0.3}{N+0.4} \quad (5)$$

Finally, the results are obtained by fitting Eq.(1) to a straight line using a probabilistic paper or a Matlab subroutine [5].
Step 6: Convergence of the model: When the variation of the sum of all the parameters becomes less than a certain threshold, ε, the fitting process is considered to be fulfilled.

$$\left|\lambda_i - \lambda_{i-1}\right| + \left|\delta_i - \delta_{i-1}\right| + \left|\beta_i - \beta_{i-1}\right| < \varepsilon \tag{6}$$

Otherwise, the iterative process continues returning to step 3.

EXAMPLE OF APPLICATION

Many multiaxial fatigue limits criteria, such as Sines [6] or Crossland [7] criteria, are based on the calculation of a equivalence shear stress amplitude, $\sqrt{J_{2a}}$, which becomes quite complex to be obtained for general multiaxial loading [8]. Some examples of models to handle the multiaxial fatigue damage phenomenon are the Maximum Circumscribe Circle (MCC) and the Maximum Circumscribe Ellipse (MCE) models, which propose the calculation of the equivalent shear stress amplitude as proposed by Papadopoulos [9] and Freitas et al. [10]. Both multiaxial fatigue criteria are based on the calculation of the curve described by the shear stress in the critical plane during a cycle.
With the aim of calculating the path described by the shear stress, it is necessary to define a material plane (See Fig. 1b), Δ, passing through the selected point and assuming a biaxial loading state, i.e.:

$$\sigma = \begin{bmatrix} \sigma_{xx} & 0 & 0 \\ 0 & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \sigma_{xx,a}\sin(wt) & 0 & 0 \\ 0 & \sigma_{yy,a}\sin(wt+\delta_{yy}) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

In order to evaluate the stress vector T acting on the plane Δ passing through the point considered, a local coordinate system is defined by three unit vectors:

$$\begin{aligned} n &= \left[\sin(\theta)\cos(\varphi) \quad \sin(\theta)\sin(\varphi) \quad \cos(\theta)\right] \\ r &= \left[-\cos(\theta)\cos(\varphi) \quad -\cos(\theta)\sin(\varphi) \quad \sin(\theta)\right] \\ l &= \left[\sin(\varphi) \quad \cos(\varphi) \quad 0\right] \end{aligned}$$

where n is the vector perpendicular to the plane and r and l are vectors in the plane, which define an orthogonal basis with the previous one. θ and φ are the angles between these vectors and the xyz axes. Thus, the stress vector T acting on the plane can be obtained by the Cauchy's theorem:

$$T = n\cdot\sigma = \left[\sigma_{xx}\sin(\theta)\cos(\varphi) \quad \sigma_{yy}\sin(\theta)\sin(\varphi) \quad 0\right]$$

Then, the stress vector could be decomposed in two stresses: a normal stress, σ_{nn}, that changes in magnitude but not in direction during a cycle of loading; and a shear stress, τ, that changes in magnitude and direction along each loading cycle, and can be decomposed in two directions τ_{rr} and τ_{ll}:

$$\begin{aligned} \sigma_{nn} &= n'\cdot T = \sin^2(\theta)\left(\sigma_{xx}\cos^2(\varphi) + \sigma_{yy}\sin^2(\varphi)\right) \\ \tau_{rr} &= r'\cdot T = -\cos(\theta)\sin(\theta)\left(\sigma_{xx}\cos^2(\varphi) + \sigma_{yy}\sin^2(\varphi)\right) \\ \tau_{ll} &= l'\cdot T = \sin(\theta)\cos(\varphi)\sin(\varphi)\left(\sigma_{xx} + \sigma_{yy}\right) \end{aligned}$$

The variation of the shear stress, τ , during a cycle defines a closed curve , ψ , that is different for each plane passing through the selected point. As a consequence the equivalent shear stress amplitude $\sqrt{J_{2a}}$, which is a function of the MCC or MCE that could envelop ψ (See Fig. 1c), is a function of θ and φ . In other words, $\sqrt{J_{2a}}(\theta,\varphi)$.

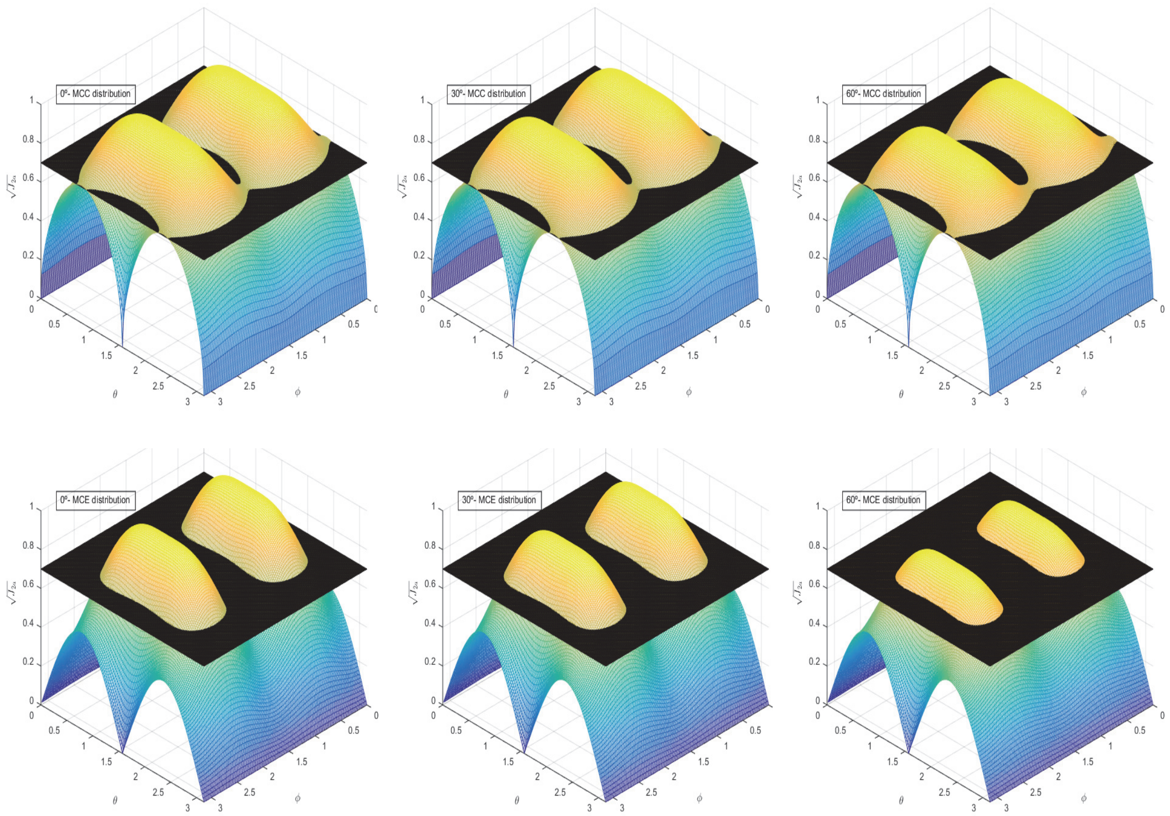

Figure 2: Distribution of $\sqrt{J_{2a}}(\theta,\varphi)$ calculated by the MCC and MCE criteria for $\sigma_{xx,a} = \sigma_{yy,a} = 1$ and $\delta_{yy} = [0 \quad 30 \quad 60]$;

MCM and MCE criteria differ in that the first one is based on the calculation of the minimum radius (R_a) of the circumference circumscribing the shear stress path, $\sqrt{J_{2a}} = R_a$; whilst the second one is based on the combination of the two radios (R_a - R_b) of the minimum ellipse that circumscribes the shear stress path $\sqrt{J_{2a}} = \sqrt{R_a + R_b}$. Fig. 1c shows the difference between the two multiaxial fatigue criteria.

Fig. 2 displays some examples of the $\sqrt{J_{2a}}$ distribution over all planes (all combinations of θ and φ), for different angular offsets $\delta_{yy} = [0 \quad 30 \quad 60]$, assuming unit values of $\sigma_{xx,a}$ and $\sigma_{yy,a}$. As can be seen, there is a difference that depends on the angular offset applied (δ_{yy}).

Imagine that an experimental program proves that the minimum value of $\sqrt{J_{2a}}$ producing failure at a certain number of cycles N happens to be 0.7 (see Fig. 2), that is, any plane subject to a $\sqrt{J_{2a}} > 0.7$ could fail. This methodology allows us to evaluate the local probability of failure for any plane subjected to a value of $\sqrt{J_{2a}}$ during N cycles by applying Eq.1. After that, it is possible to obtain the global probability of failure as the combination of the local probabilities using Eq.3, which allows the risk of failure over all planes to be taken into account.

CONCLUSIONS

The main conclusions of this work are the following:

- Failure for a certain multiaxial fatigue loading must not happen, necessarily, at the plane subject to the maximum value of the multiaxial fatigue criteria (MCE or MCC in this case), but it may occur at other planes subjected to lower values of the critical parameter due to its interaction with the existence of local defects.
- The probabilistic model proposed in this paper enables the failure probability for any plane orientation to be found.
- The applicability of this methodology is not limited to the use of these two criteria (MCE or MCC), but the iterative process can be extended to any other failure criterion regardless of the complexity of its calculation.
- In this work, an analytical solution for the local calculation of the critical parameter is assumed, but as in the GLM, this is not mandatory, so that the calculation of the critical parameter distribution in other complex cases can be found using the finite element method.

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